Classical Mechanics: From Newtonian to Lagrangian Formulation
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Lecture – 10
Central forces – 3

So, in the last class, we have seen that for a motion under central orbit. We can write r dot equal to 2 by m E minus vr L square by m r square to the power half, which might not be seeing easily, but anyway you know this expression. And theta dot equal to ml by mr square.

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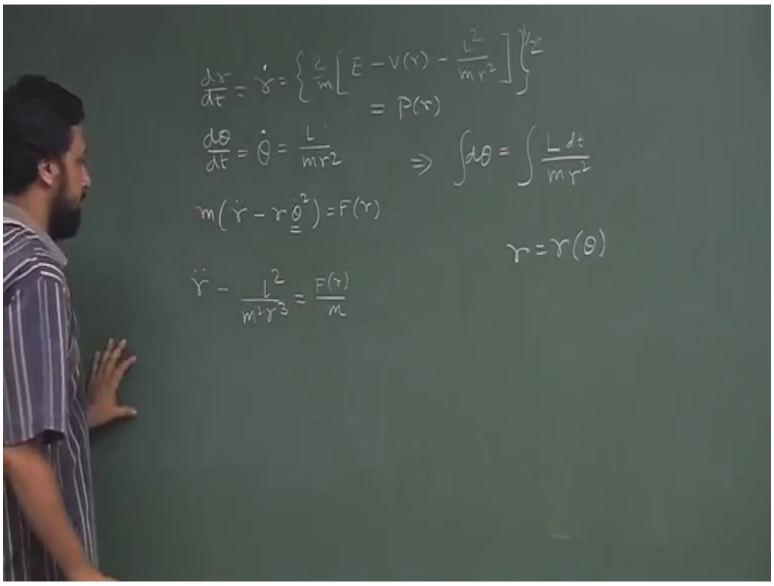

Now this one, we can write this as total derivative of r and t. And from here we can just shift this in order to write 2 by m, E or essentially what we can do is we can instead of. So, instead of writing this full expression we can just replace it, we can just call it some P of r r P being some function now in this particular function and because it has r here and here. So, we can just write this as dr by P of r this integration and dt integration.

Now, performing this integration, will give me r as a function of time. Right now this integration on the sorry this equation for the we for what we have for theta dot is theta dot equal to L by m r square. So, this can be rearranged to get integration d theta equal to L dt by mr square.

So, once we solved, the once we could integrate this part, we can take r as a function of time. And then we can substitute it back in this equation and then we can get theta as a function of time. So, essentially we get r as a function of time, theta as a function of time. And in principle once we can perform these 2 integrals, the problem is totally solved. So, we know that we know r and theta of the entire trajectory as a function of time which essentially it is equivalent of saying that we know the entire trajectory as a function of time. That is the total solution of the problem we are looking for.

Now, there is a technical issue. Solving this equation is hard. Because rather integrating this expression in this particular form is hard. Because Pr first of all it is not a very simple looking expression. Secondly, this vr here is there is a to the power half here. I think now you can see it yes right it is fine. So, V of r is not a known function. So, we do not do not know it might have singular it is a singularity at some point. It could diverge at very low or very high values of r.

So, we do not want to get the we do not want to perform this integral in a sense that it might not be a very easy task to perform especially by hand. So, in order to gain insight, we can also try to get solution of the form r does a function of theta. This will be a time independent form of the orbit. So, this will also give you the orbit, but we does not include the time explicitly. So; that means, it will give you the orbit not as a function of time.

But it will give you the orbit as a functional form of r as a function of theta. Now in order to do that order to get this solution, we need to get an equation, differential equation which includes only r and theta and there is no t dependence. How we can do that? We can do that by start writing recalling that m r double dot minus r theta dot square is equal to Fr. So, this is the standard equation of portion of central force under the central force Fr. Now if we replace theta dot by this expression, this reduces to r double dot minus r L square by m square r to the power 4 is equal to Fr by m or just simplify a bit get rid of this one. So, it will be simply r cubed here.

Now, what do we do? We introduce a new set of parameter which we call u as 1 by r.

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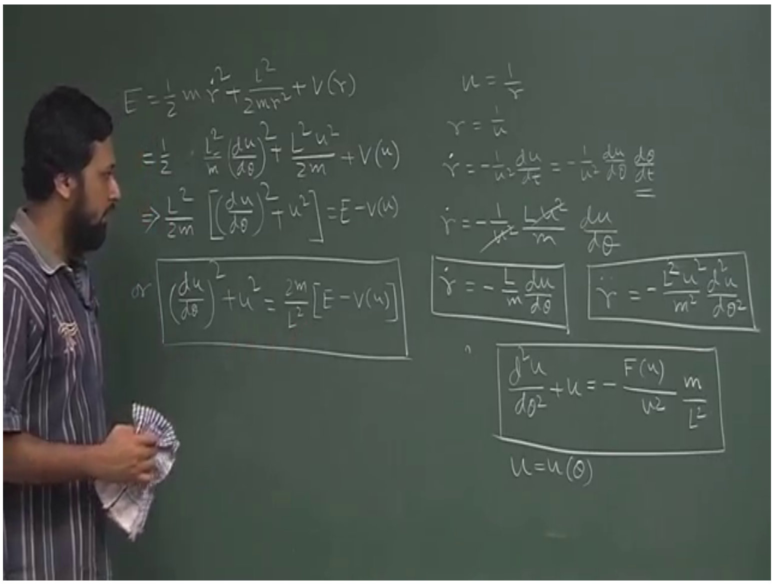

So, it is just the inverse of the radial part. So, if this is the case. Then r is also one over u r dot is minus 1 by u square du dt, we can do a mathematical manipulation on this. And we can write minus 1 by u square tu d theta d theta dt, which is theta dot again. And in this particular notation theta dot which is L by m r square, will be L u square by m right. So, if we substitute this, r dot will be minus 1 by u square. And this is your theta dot d theta dt theta dot.

Which is L u square by m du d theta right. So, you see that you will cancels out leaving behind r dot is equal to minus L by m du d theta. This is one relation. We can do this derivative a second time. I am leaving it to you. So, essentially you have to evaluate r double dot also because this equation has r double dot. So, you have to take another time derivative of this. So, just the same it will be d 2 if you take the time derivative of this you can replace this, as d 2 u d theta 2 and then there is an another d theta dt, which is a theta dot which has to be replaced back. Once again now if you do that, your final expression I am just leaving it to you as an exercise. I think you can do that.

So, your final expression for r double dot will be minus L square u square by m square theta 2, minus L square u square by m square d 2 u d theta 2 that is the expression for r double dot. Now it is time to do the substitution here, take that substitute it here. So, it will be and also you have to substitute r for u E or u for r here. So, this will be minus L square u square by m square d 2 u d theta, 2 minus L square u cubed by m square equal to Fr by m.

So, immediately we see that we can get rid of or whether we can get rid of m square from this side. So, multiply this by m square, which will be just multiplying it or rather it is by m here m square multiply it by m square. So, if you take L square u square common, with a negative sign. And simplify this expression slightly it will be d 2 u d theta 2 plus u equal to. So, this will be m F L square u square minus f. So, this will be F of u Fu, by Fu by u square m by L square. So, F a F L square yeah right. So, it is sorry (Refer Time: 10:03) it is not look nice yeah. So, it is d d 2 u d theta 2, plus u equal to minus Fu by u square m by L square. Now in this equation, we have we can solve this equation to get u as a function of theta which is equivalent of get getting r, as a function of theta right.

So, because we already know that you and r they are just related by u equal to 1 by r, this is the equation we will be using mostly for our description of centre I mean getting in description of central orbit or sometime we will be using it to find out law of force, sometime we will be using it to get the differential equation, when we know the law of force we will take up this examples very soon and we will see. Now before going there we can also get in get an alternative equation which is not a homogeneous equation, but that is a first order equation. If you recall the energy expression was E equal to kinetic energy plus potential energy. So, if you do that you will simply get half m r, r dot square plus L square by 2 m r square plus V of r right. So, if we now substitute r dot with this expression and r with u then energy expression becomes half m L square by m square d u theta d theta whole square plus L square u square by twice m plus b of u right.

Now, rearrangement of this gives you. So, you see one m cancels immediately. And we have we can take L square by 2 m common right. If we take L square by 2 m common, we have du d theta whole square. So, I am just writing in your step here u square is equal to u minus V of u or sorry not this, du d theta du theta whole square plus u square equal to 2 m by L square u minus. So, this is another equation which is not exactly. So, this is the second order homogeneous differential equation. This is the first order in homogeneous differential equation because we have a du dt whole square.

But this is also a this is also usable at some point it is also useful at for certain ex certain cases we will see that. So, now before moving into any further moving any further in the by using this equation and see what happens for different cases, let us take some examples. So, our have we have some classroom problems like we have in our every class.

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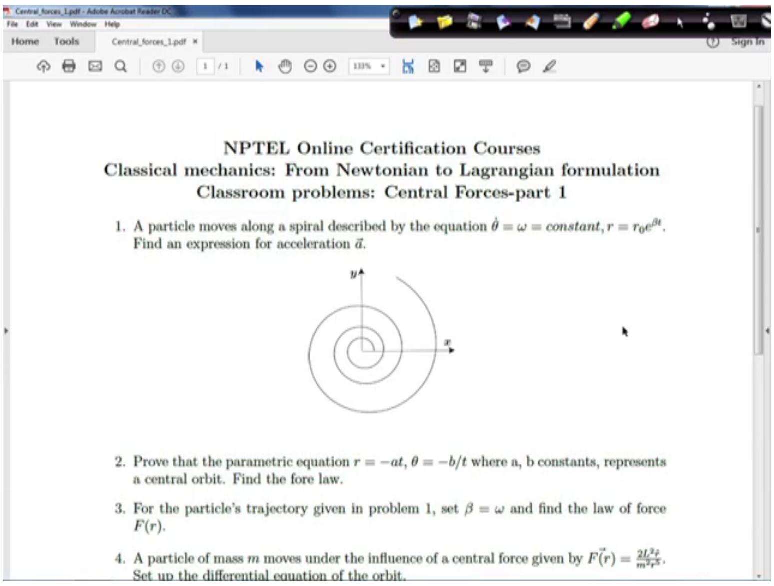

So, the first problem is a particle moves along a spiral described by the equation theta dot equal to omega equal to constant r equal to r 0 E to the power beta t. We have to find an execlar expression for the acceleration a. So, what is given it is given that theta dot is equal to omega which is equal to a constant r equal to r 0 E to the power beta t right bet t.

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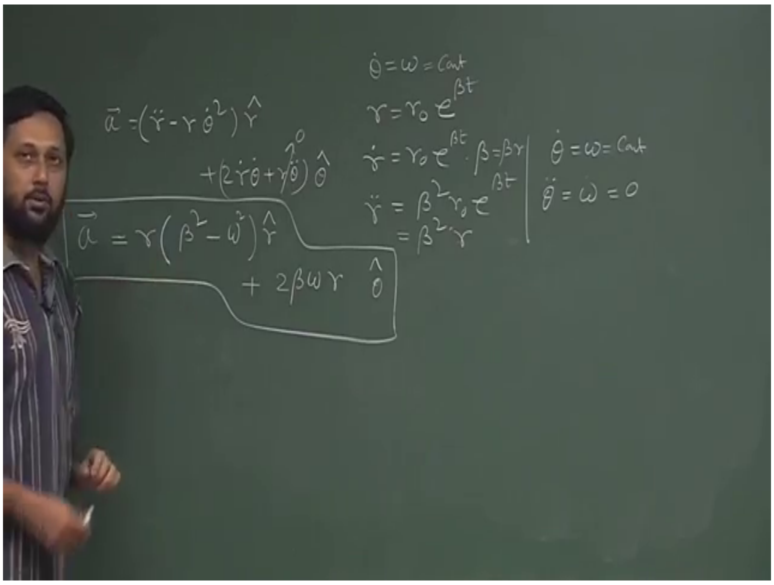

So, if this is r 0 r dot will be r 0 E to the power beta t times beta r, double dot will be beta square r 0 E to the power beta t right. Similarly, theta dot is equal to omega equal to n constant theta double dot is equal to omega dot equals to 0. So, the expression for a in order to write an expression for a we have to keep in mind that it has both radial and transverse component.

So, we have a is equal to r double dot minus r theta dot square r cap, plus 2 r dot theta dot r theta double dot theta cap right now a double dot right. So, if we perform this first I mean of if, we if we expand this first part and put values from sorry no not expand. If we just put values here. So, we get. So, this is essentially beta square beta square r. So, we get r out have beta square minus omega square r cap plus. Now theta double dot is equal to 0. So, this term does not contribute to our final expression r dot is r 0 E to the power beta t times P beta, which is essentially beta times r. So, we have 2 beta or I will just write it down for clarity plus 2 beta r into omega theta dot right.

So, all final we will just rearrange the second term bring the constants omega r theta dot. So, we have this as our answer. So, we have r times beta square minus omega square equal r cap plus 2 beta omega r theta cap. So, this is the final expression for the acceleration a in this particular case good. Now let us move to the second problem which says find the parametric equation r equal to minus 80 theta equal to minus b by t where a and b are constant. We prove that the parametric equation represents the central orbit and also we have to find the force law right. Now in order to find out the weather if it is a central orbit for doing it do let us come to the board again.

So, we have to first prove. Now what is the central orbit central orbit means r square theta dot will be equal to 0. So, if we can prove that for this particular set of parametric equation which is r equal to minus 80 theta equal to minus b by t, we can prove that r square theta dot is equal to 0.

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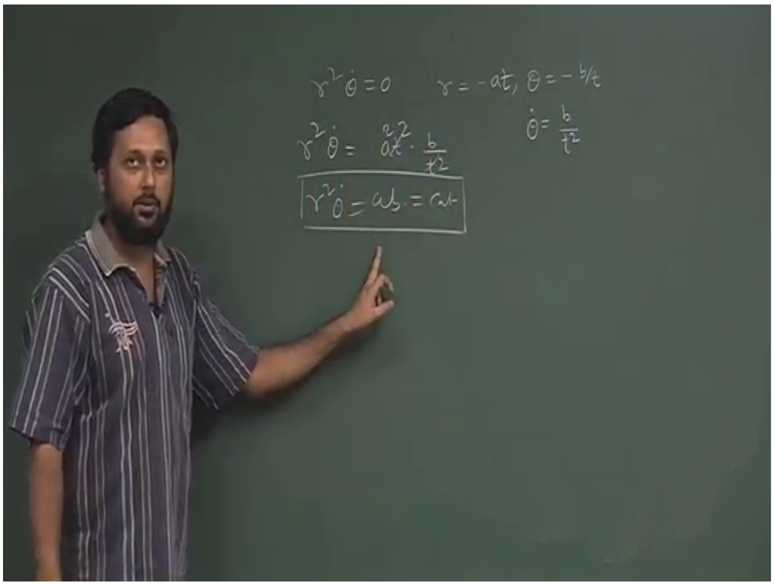

Then that essentially proves that this particular set of parametric equation represents a central orbit. Now theta is equal to minus b by t. So, theta dot is equal to b by t square t being t be a time. So, in this case we can think of t as the time right. So, this is the parameter time is the parameter, in this parametric equation. So, essentially the equation of the orbit is given in terms of a the terms of the parametric equations all we need to do is we need to eliminate. So, in order to we will come to that later.

So, we first estimate theta dot. So, r square theta dot will be nothing, but minus a t s sorry minus will not be there, a square t square into b by t square which will be ab. So, r square theta dot is equal to ab equal to a constant. So, that is our equation. So, we essentially proved that this orbit represented by this particular set of parametric equation is a central orbit. Now the second part is we have to find out right. Now we have to find out the equation of the orbit i t, I am living it on you because what you need to do is we need to eliminate r. And theta I am sorry, from the expression of r and theta you need to eliminate t.

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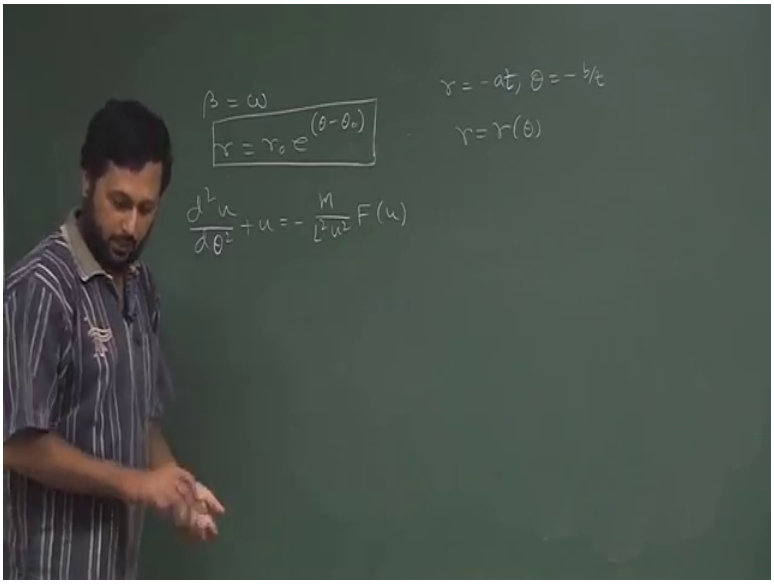

So, essentially you have to get r as a function of theta you can do that. So, I leave it to you and we moves to the x problem. So, next problem is also related to the first problem which says for the particle of that for the particle trajectory given in problem one, which is the first problem here. This spiral said vitae equal to omega and find the law of force. Now this is once again this is also easy what happens is, if we set beta equal to omega in this equation if you recall right r is equal to r 0 E to the power beta t. Now it will be r 0 E to be power omega t and theta dot is equal to omega equal to a constant. Now if we integrate this equation.

We get theta minus theta 0 is equal to omega t. So, we substitute this for omega t. And we immediately see this equation becomes theta minus theta 0. So, this is the equation of the trajectory right. So, in case when theta is equal to omega this is the equation of the trajectory. So, we get r as a function of theta. Now what we need to do is we need to find out the law of force which is F of r. Now recall that d 2 u d theta 2 plus u equal to minus L I always forget this expression m by L square u square F of u. So, what we need to do is we have r as a function of theta. So, we have to construct you first then we have to determine d d 2 u t 2 substitute it here. And we will essentially get an expression for u F of u in terms of theta.

Now, if r is equal to this.

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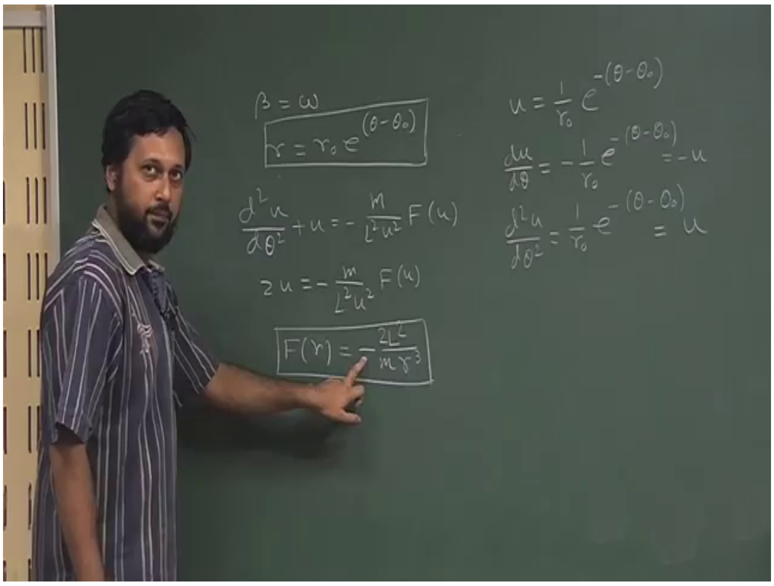

So, u will be equal to 1 by r 0 E to the power minus theta minus theta 0. So, du d theta is equal to minus 1 by r 0 E to the power minus theta minus theta 0 d 2 u d theta 2 is equal to 1 by r 0 to the power minus theta minus theta 0, which is equal to u. So, you see that this is du d theta oh sorry d 2 u d theta 2 is essentially equal to u. And this is equal to minus u which is not very uncommon for exponential functions on a sub subsequent derivative say some time repeat themselves, sometime repeat them with a negative sign this is exactly what is happening here. So, this part is nothing, but to u and the right hand side we have m by L square u square F of u you have to bring this u square to this side.

And rearrangement will tell you that F of u is equal to minus L square by m. So, there will be a 2 here, it will be u cubed we are writing one by r cubed. So, this is the final answer. So, we get a force law which is attractive because there is a minus sign and which propo which is proportional to the inverse of sorts rather one over r cubed. So, the force which gives this type of a trajectory is a inverse cube force attractive. So, just for your curiosity the particular type of spiral which is described in this problem, when beta is equal to omega this particular spiral is called a logarithmic spiral. And this is the very interesting trajectory we probably will come back to it and do some more problems.

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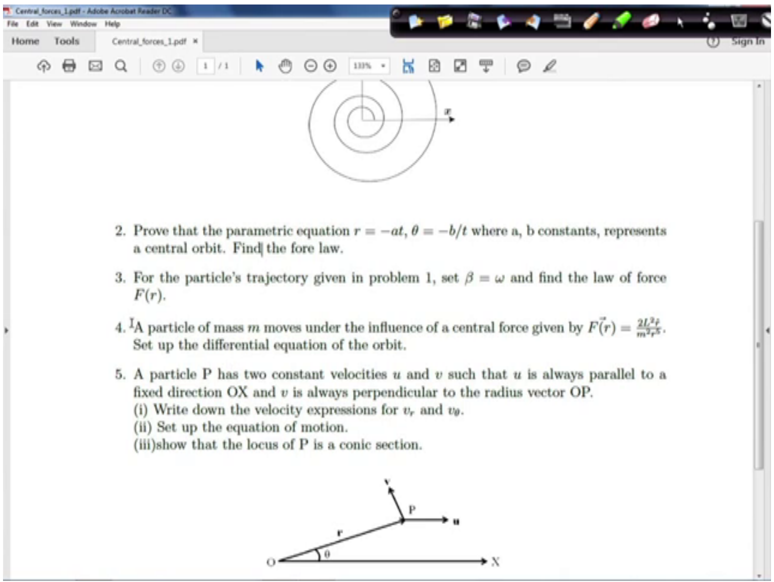

Now, let us move to the fourth problem fourth problem is a particle of mass m moves under the influence of central force given by this. So, this is a repulsive central force because there is no negative sign. And we have r to the power 5 dependence we have to set up the differential equation of this particular orbit, and the differential equation is the same d theta 2 plus u equal to minus m du theta.

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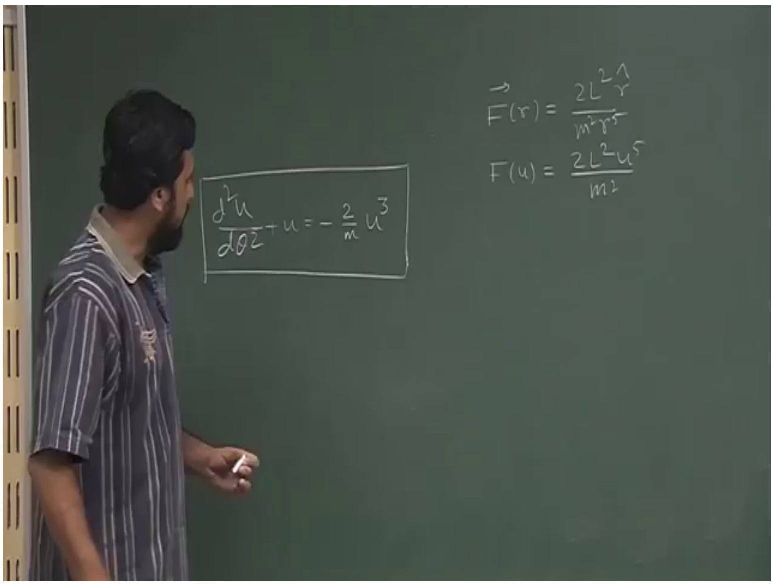

Now, what is given is the force law. So, this side is given F of r is equal to 2 L square r kept by m square r to the power 5 m square r to the power 5 right. So, F of u.

So, we just have to replace r with 1 over u and we immediately see that this gives you E to the power 5 square. When we replace it back here for Fu we get 2 L to the power 5 sorry L square sorry, m square L square gets cancelled out u square is here. So, we get u cubed and also we can get rid of one m here. So, essentially the differential equation becomes minus 2 by m u cubed minus 2 by m u cubed. So, this is our answer. So, all the problems what we have described here, these are very simple problem in reality sometimes we have to solve this differential equation, we will take up some more examples as we progress as we go on in our classes.

But right now, I just wanted to show you how to use this particular form of differential equation or particular form of expression. What we have got for connecting u and theta in order to first we showed how to solve this equation to get the equation of an orbit. And secondly, we sort took a force law and showed you how to set up the equation of motion. Now how to solve this equation that is a different question altogether, but of course, I took easier examples in in order to demonstrate it easily in this classroom oh we will have little more complicated problems. Next class onwards also what we are going to do is in the next class.

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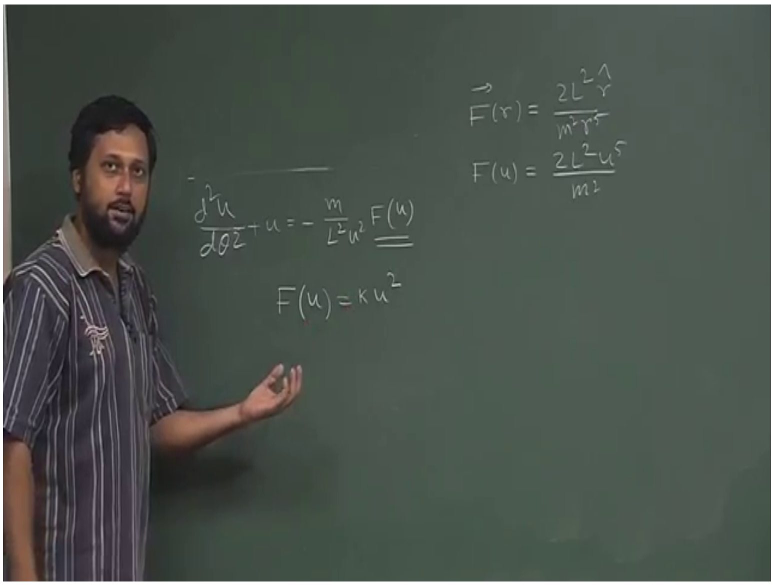

We need to again recall that the original equation has a particular form, which is m L square u square. Sorry I always forget this particular form and I have to look back to my notes. So, I hope you will excuse me.

So, we what we will do is we will try to solve this equation for a very particular type of force which is K u square. So, K u square essentially means Fr is equal to K by r square it there could be a negative sign for attraction. There could be a positive sign for repulsion, but this force law is extremely familiar to all of us, this is the inverse square force field which consists of gravitational force which consist of electrostatic attraction or repulsion. So, this is a very common force which is which is way out which is seen universally you know in the in the world we live. So, we will take this up in the next class, I will see what type of particular aware do we get any very specific information about the orbit by putting this value in this particular equation.

Thank you.