Mathematical Methods in Physics-I

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Lecture - 09
Projection Operator

Welcome back students. Let me start with where I stopped in the last class which is the Gram Schmidt Orthonormalization

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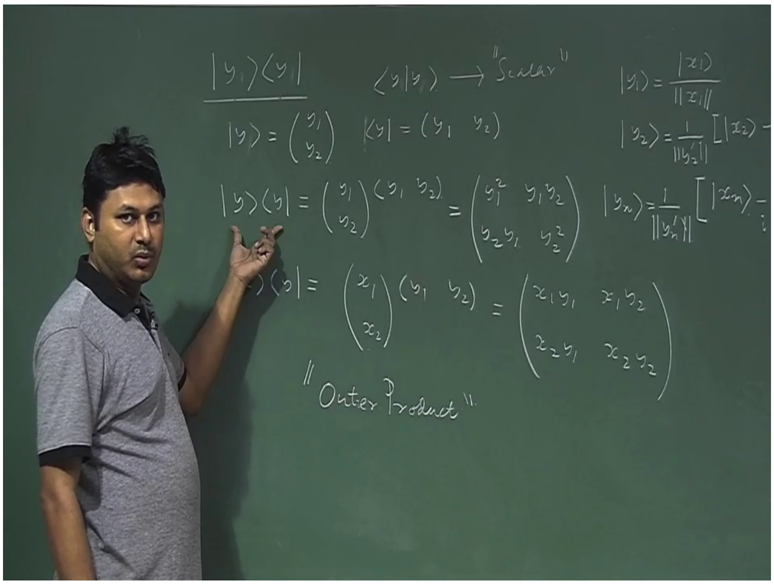

These are the formalism using which I can figure out a set of orthonormal basis by using a set of linearly independent vectors. So, a linearly independent vector is given from using that I am going to find out a set of orthonormal basis. So, if I look it in this way. So, for example, this quantity try to first figure out what is the meaning of this notation. So, one notation I use this; what is the meaning of this notation. So, normally we know how to calculate this, but this I do not know. So, if I write this as this way. So, y 1 say let me just write y to make it simple because I need to use the component.

So, say y 1 and y 2 are the 2 components of the vector y, I need to calculate this; this notation what is the meaning of this notation first I need to find out. So, then y which is in this; this is a bra notation and this is a ket notation you are familiar with this notation, I can I need to write this in this way. So, these things whatever I am writing here is coming from this dual space concept. So, you know about this here since y is a real vector. So, I am not going to write this as a star, normally in principle you should write it as star. So, now, this quantity in the vector notation these and this can form something like this.

So, operation is this and this. So, y 1 square, y 1 y 2, y 2 y 1 and y 2 square if it is not y 1 y yy if it is xy for example, if it is something like this, the result will be x 1 x 2 in general y 1 y 2 in general, then it will be x 1 y 1 x 1 y 2 this multiplied by this, this multiplied by this here, here this multiplied by this multiplied by this. So, this forms a matrix and this is called the outer product it is a name this called the outer product.

So; that means, inner product we are having a from this I am having something a scalar that you know, when I do the outer product of 2 vector by taking this way I am getting something which is matrix 2 by 2 matrix which behave like a operator. So, this is something that is operating over something.

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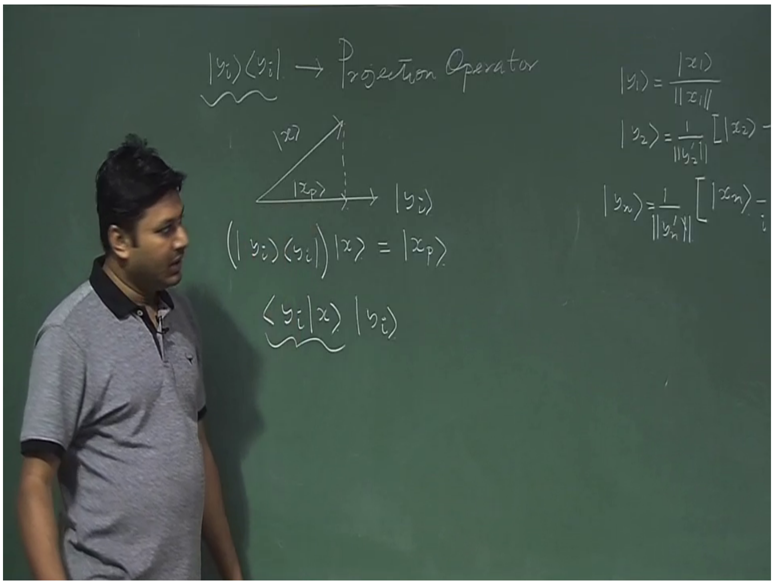

So, what is the meaning in there in this Gram Schmidt Orthonormalization process what is the meaning of that? So, let us try to find out it is nothing but y i this quantity is called the projection operator very important term projection operator. If a vector is given along this direction, and if this is the direction of say y i, if I operate this quantity which is now behave like operator if I operate this quantity over this vector given vector x it essentially means that I am taking the projection. If I say this projection vector xp say I write this as a new vector xp which is the projection of this x vector xp.

So, what will be my xp? Xp will be this quantity and would you which is quite simple if you try to understand that then y i x this quantity is nothing but the dot product or this quantity multiplied by the vector direction this. So, it will give you the vector along this direction so that means, it is projection along this direction. So, in this case try to find out how this projection operator works here specially here in this particular case. So, this equation was y 2 was something x 2 minus y 1 y 1 x 2.

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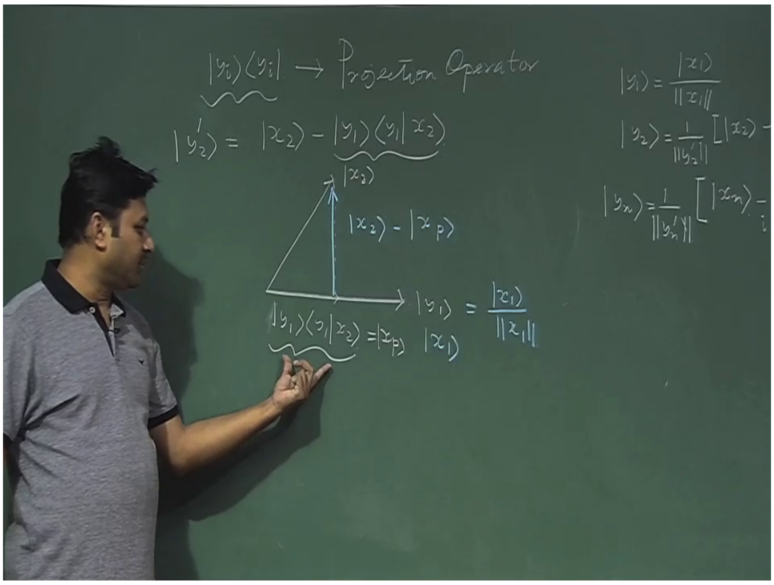

This was my y 2 prime how to calculate the y 2 prime that is a. So, my x 2 was in this direction say this is my the direction of my y 1. So, this vector is the x 2 minus the projection.

So, the projection I just say that this vector is this one, this is the xp the projection vector that I do and here what I am trying to do I am just trying to find out that what is my x 2 minus this, this is my x 2 vector and if I subtract from this vector then essentially what I am getting is this vector this vector is nothing but x 2 minus xp this vector this perpendicular vector. Now, you can see that x 2 vector is giving to you x 1 was also in this direction because y 2 is nothing but x 1 mod of this quantity. So, it has to be this direction. Now, using these and these vectors I can figure out another vector which is perpendicular to y 1. So, this is exactly perpendicular to y 1 and which is x 2 minus p xp is nothing but this projection operator thing.

So; that means, I calculate this one then subtract from this and this, I will get a new vector which is perpendicular to this. So, this is the concept every time if you able to do that you will start getting a vector which is perpendicular to the previous one and with this you can calculate interstate of vectors which are orthogonal to each other. So, now, let us find out more about this projection operator. So, if say e 1 e 2 en are set of in vectors.

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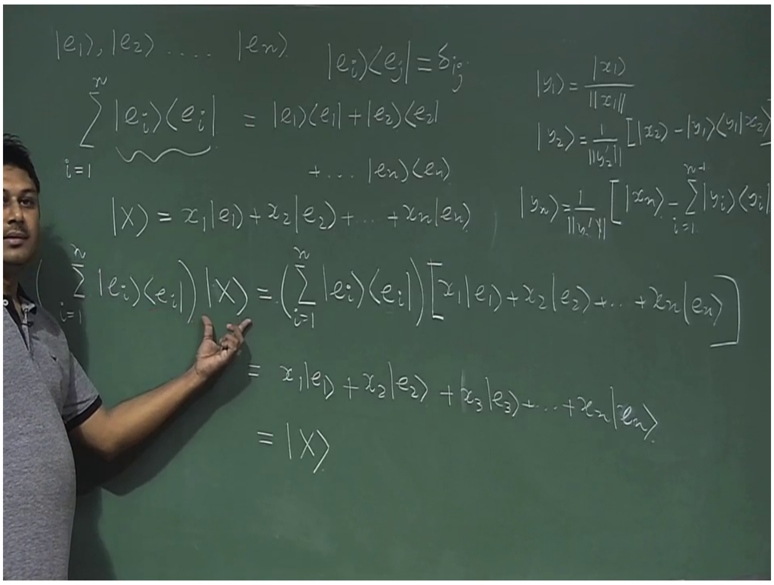

Which are linearly independent of each other as well as orthonormal so; that means they are forming an orthonormal set mathematically means this is satisfied these things is satisfied. If this is satisfied if these things are satisfied then any vector can represented in this form that is one issue, second thing is that I need to find out this quantity. This is a projection operator along any direction I any prefer direction. So, there will be different directions.

So, 1 2 3 is corresponding to different direction. So, i is a some direction and now I try to find out this quantity; so 1 to n. So, sum of all this quantity. So, sum of all this quantity is essentially the meaning of this is that I tried to project all possible direction of a vector. So, what will be the value of these things? This will be an interesting thing. So, this is a projection operator along i-th direction. Now I tried to find out the summation of all the projections if this is a projection in i if I run this i from 1 to n.

So, is essentially mean I try to find out this quantity e, e i e i plus e 1 e 1 plus e 2, e 2 plus e n e n what will be the summation of entire things all the projection operator if I sum if orthonormal to each other then what I am getting. So, let us try to find out. So, I need to find out this. So, what I will do that I will take a vector x, this vector can be decomposed by this basis.

So, e 1 e 2 en; so this is the decomposition of this vector along the bases now I need to use this over that. So, what I will do that I will project this projection operator entire projection operator over x big x and try to find out what I am getting in the right hand side. So, essentially what I am doing I am doing these things what is my x x is x 1 e 1 plus this. So, this is my, this is that thing I want to operate over that and try to find out what I am getting. So, term by term if I look that what will be my first term, it will be I and it operates over the first term. So, i and i it will give 1. So, I will first term will be this, all the second term the second term will start at 2 and a if I say this is one then it will operate over this and this and this. So, if it is one then it operates over this and this 2 and one will cancel out and three and one is cancel out. So, all the other term will going to vanish the second thing is that the next term which is e 2 e 2.

So, e 2 e 2 when I operate over this first term it will going to vanish because e 2 and e 2 will going to merge and gives a 0 value e 2 over that we give one because this thing over this operation give you some non zero values. So, my next term will be something like this in the similar way my third term which will going to survive is something like this up to n. So, what essentially after operating this over that what essentially I am getting my old vector which is x. So, operative projection operator over x as a result I am getting the my old vector x.

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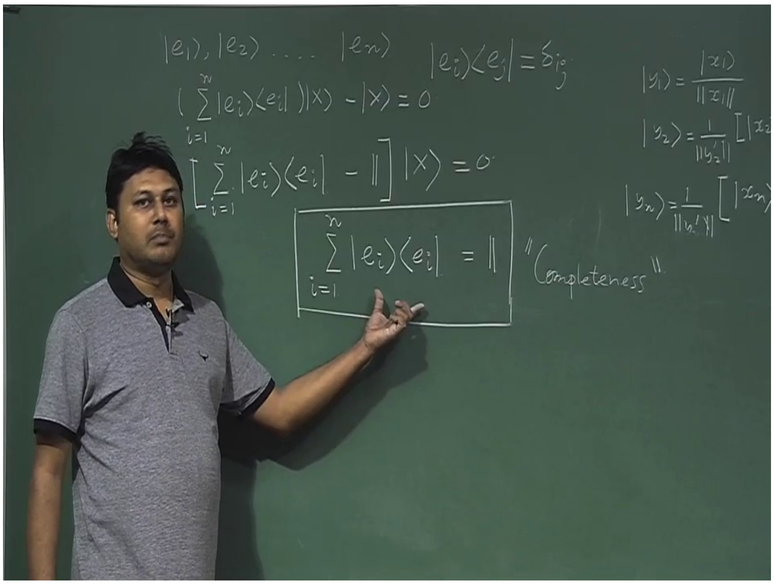

So, if I write this here, then I should write it 1 to n e i e i that operates over x minus vector x is equal to 0, I take this x this side and my equation is something like this from here you can write a very important expression which is essential in quantum mechanics also. This one is basically a matrix, because just before I show that these things this quantity gives you something which is matrix is a matrix form. So, I should write it one because I am taking x common out of that.

So, if I take x common out of that as if this is one normally, but since this is a matrix operation I need to write one in this way you should understand that this is a unit matrix. So, now, from here I will have a very important expression, this is a special name of that this expression that project operation over entire all the direction if I take it essentially gives you the unit vector a the something unit matrix. So, this is called this special property called completeness a special thing called completeness. So, now, I can check that in principle that weather this is giving you some kind of completeness things or not with my usual basis so that quickly to understand how this is working.

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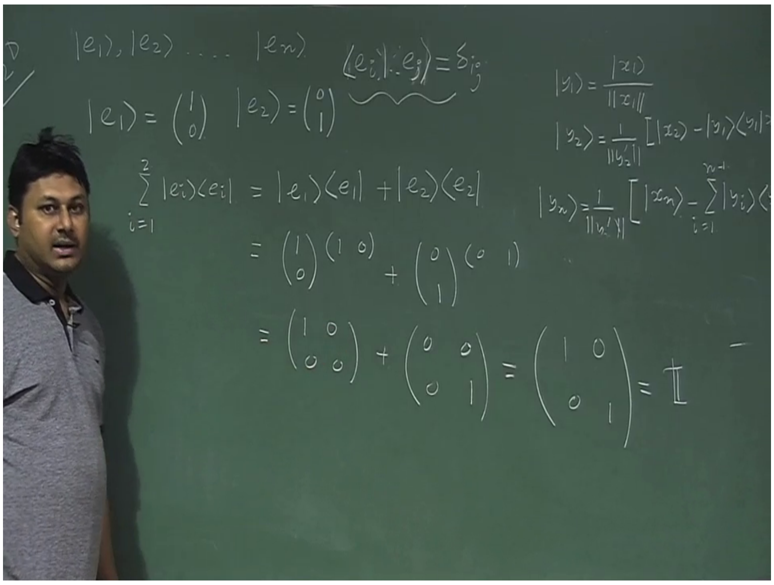

So, let me do that for 2 D which is easier e 1 e 2 I know that this 2 are orthonormal basis.

If they are farming orthonormal basis readily they will follow this rule sorry here I am writing wrong expression somewhere it should be something like this orthonormal this I by mistake I write projection operator notation please note that, if you note that please correct it. So, this is the orthonormal the concept of orthonormal basis.

Now, from the orth concept of orthonormal basis I have e 1 and e 2 which are orthonormal to each other now I try to find out this quantity summation ei ei, I runs from since this is a 2 dimension n is two. So, it will be 2. So, what value I will getting e 1 e 1 this will be my first term if i expand this summation sign and my second term will be e 2 e 2 no cross term will be there is no ij please note that there is no i j. So, this term now you know how to calculate this outer product. So, it is 1 0 1 0 this is 1 0 1 0 plus 0 1 0 1.

So, my first term is 1 1 1 0 0 0 0 0. So, 1 0 0 0 my next thing is 0 0 0 0 0 0 0 this one and this multiply one. So, if I add I am getting just 1 0 0 1 which is nothing but the unit matrix of order 2 because I am taking just 2 second order. So, I will get if you do that for 3D you will also get something 1 1 1 and so on. So, I can show that they are farming these things. So, let me write down the properties of the basis e have. So, so far we have 2 properties one is ei ej is delta ij which is called orthonormal property, I write o n second property that I have which is called completeness.

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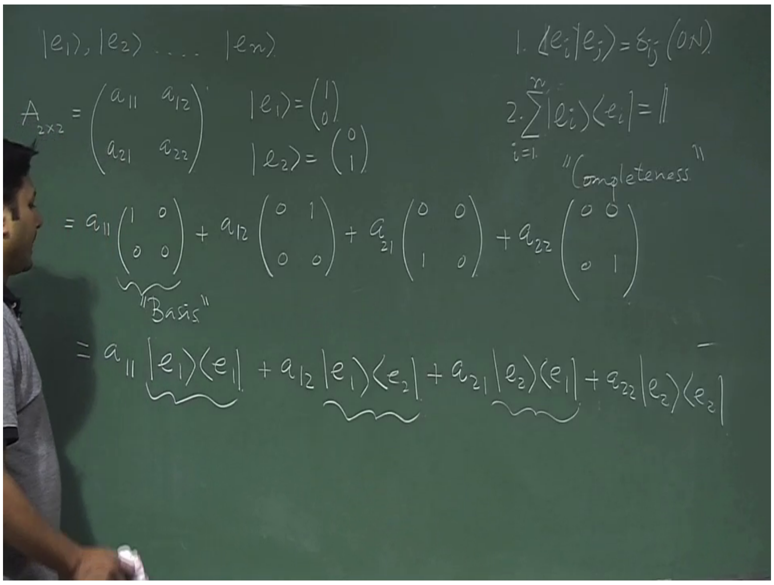

Which is called completeness apart from this 2 another important property that also this basis holds. So, let me write it say a matrix in 2 D a 2 by 2 matrix, I write this matrix in a component form where these are the components.

Please note that if I write this matrix in this form I can decompose this matrix in this way exactly like a vector I can decompose my matrix in this way, this is total decomposition of the matrix. So, what I do that I will just multiply with the some matrix which is this form this is called the basis like the vector this is also called the basis, but this basis is for the matrix decomposition. So, this basis is coming from this matrix is decomposition. So, this is one basis this is another basis, this is another basis the mot most natural basis one can have. In 2 D by the way e 1 is one 0 natural basis e 2 is 0 1 natural basis I can write this expression this quantity in terms of e 1 and e 2 also just before I have just did it.

So, 1 1 will be how much? It will be e 1 e 1 1 2 will be e 1 e 2 if I take the outer product of e 1 and e 2 please do that then you will get this this matrix 2 1 is e 2 e 1 and finally, 2 2 is e 2 e 2. So, I can divide my entire matrix in terms of these kind of operator which I define as a projection operator here this is a projection operator, but this is not a projection operator because one and 2 are here, this is not the same direction. So now, what I am trying to find out that if this is given how to find out the matrix element out of that.

So, matrix is given where these are forming a basis. So, how do I get the matrix element?

So, this is my entire matrix A. So, in order to find out ei what I will do? I will first operate ei over A in this side and ei over A in this side sandwich I make a sandwich A with operating from this side and this side in right hand side or.

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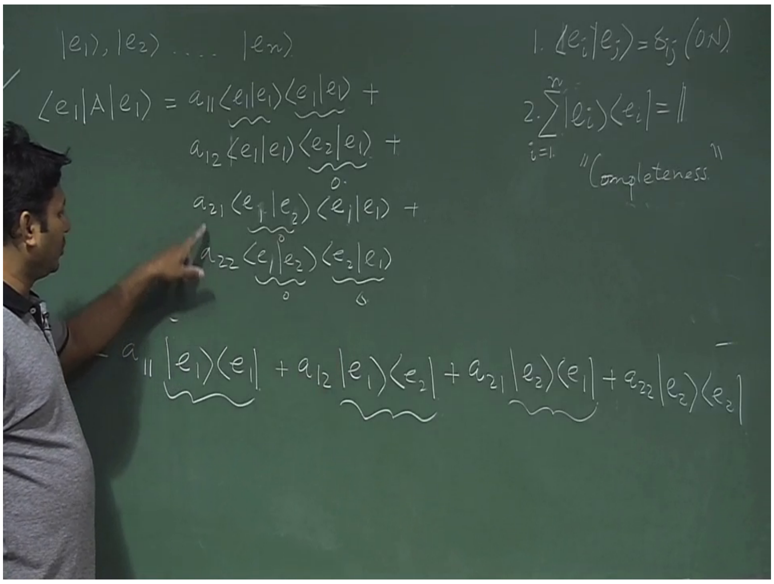

I will get when I operate this from the left side I will have a quantity like e 1 e 1 when I operate from the right hand side I will have another quantity e 1 e 1 this is one this is one.

So, I will have a one what about the other terms if you look that when I make a sandwich of this it will give one, but when I make a sandwich of e 1 to e 2. So, let me write it then it will be may be clear. So, a 1 2 is simply e 1 e 1 it is, but then e 2 e 1 this is not this is 0 because they are perpendicular to each other. So, dot product or inner product between e 1 and e 2 is 0, in the similar way I can also find 2 1 which is e 2 e 1 and then e 1 e 2 let me. So, it is e 1 e 2 and then e 1 e 1 using this again I will find this is a 0 and finally, the last value 2 2 which is both e 2 sin and I am sandwiching then e 1. So, I will get e 1 e 2 and e 2 e 1 e 2 and e 2 e 1 this is also 0; so this 0 this 0 this 0 this 0. So, 2 a 2 2 2 1 1 2 will not going to come into the picture only thing that is there is a 11.

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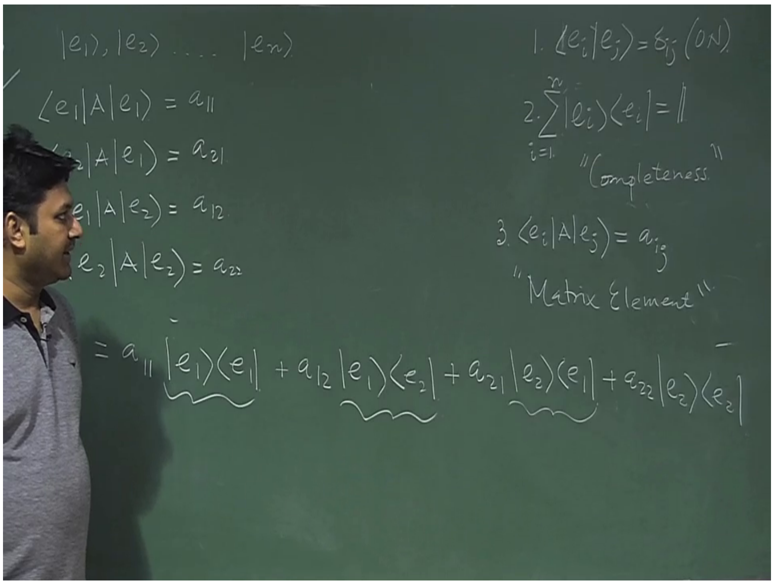

So, I will have my a 1 1 component readily when I sandwich this. So, if I go with that then I will find with the same treatment I can find a 2 1 with the same treatment I can find a 1 2, with the same treatment I can find a 2 2 this. So, now, another important property the e 1 e 2 e 3 has which is there if I write the third property, which is if I operate this ei over a matrix ej I will get back the matrix element a ij because this is called the matrix element. So, this is called matrix element.

So, what in conclusion what I find we find that e 1 e 2 e 3 en are the n basis n number of basis where dimension n it follows three important property which is orthonormal property. Second thing is that they are complete it is a completeness property and third property that this is a matrix element. So, matrix element can be find out by these things. So, when this three properties are followed by this basis we call it natural basis; obviously, because naturally it is coming and it is very important to know that this three properties are followed are coming from the concept of inner product only. So, inner product if you know correctly then you can find out all these things all the concepts are coming from inner product.

With that today I like to conclude here in the next class we will go with more important thing which is called basis transformation where we may require to have the concept of this few things. So, let us find out how the in the next class, how the basis transformation is working, how a vector is changing in another basis if I a vector changing to another basis, how the vector formation can be put into that. That are the few topics that will like to cover in the next class with that.

Thank you, and see you in the next class.