Mathematical Methods in Physics -I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 08
Inner Product Space, Gram- Schmidt Orthonormalization

Welcome back students. So, in the last class, if you remember, we defined few important notation.

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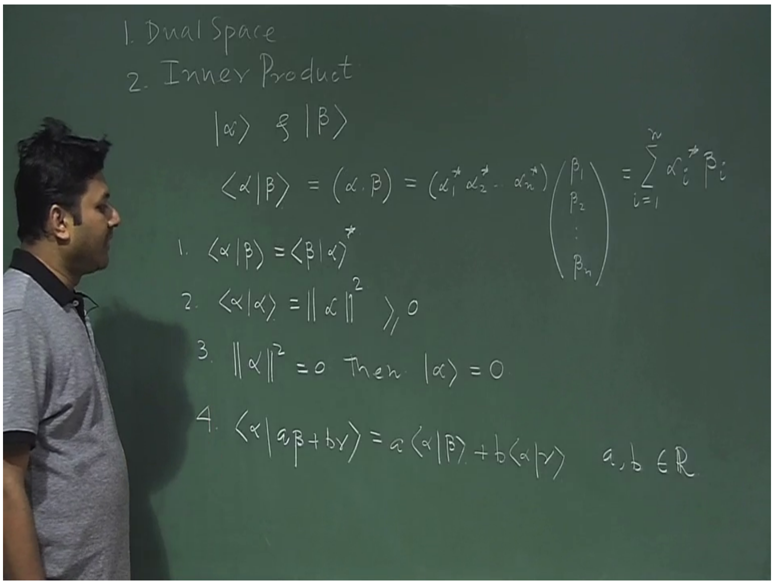

One is dual space and second one is inner product. In linear vector space this dual space and inner product both are very important concept. If you remember if 2 vectors say alpha and beta are given, in a vector space this is a elements of the vector space, then a inner product is defined by say alpha beta. Sometimes the notation is also we can has alpha beta in some places, but essentially these 2 are the same things. This thing can be represented in n tuple notation also. Where I can write alpha 1 alpha 2 alpha n, beta 1 beta 2 beta n, when alpha 1 alpha 2 alpha n are the components of the vector alpha and beta 1 beta 2 beta ns are the components of the vector beta. This is nothing, but this quantity alpha I star beta i, i tends to 1 to n if the vectors has dimension or the space the vector space has a dimension of n, this is known I mean this is not a new thing.

So, now if this inner product follows a few properties, for example, if I write this say alpha beta is equal to beta alpha star, which we have already prove this in in our previous class.

Next thing is that alpha, if I take the inner product of the 2 same vector then we essentially call it as the norm square like this. This quantity is always greater than equal to 0. Third if this quantity is 0, then the vector alpha must be 0. Or it is a null vector and finally, if I write where a and b are 2 scalars that is belongs to the field. So, a and b are belongs to the field or real number here. Then this kind of associated property can be followed by this vectors under this inner product formation. Then if this 4 properties satisfied not necessarily in any vector space this, if I can define a inner product and I can also define this 4 properties uniquely. If I do if I able to do that then we call this space as inner product space.

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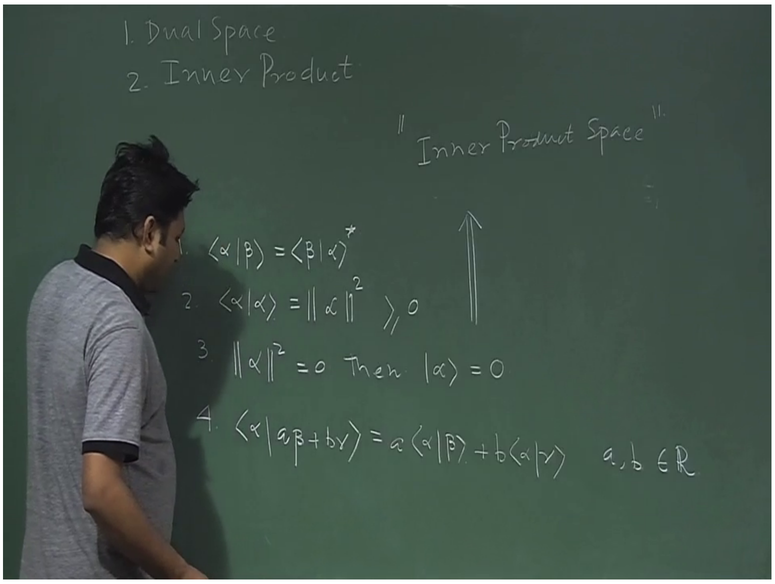

So, if this things are followed then we call it as inner product space; that means, vector spaces is there. In vector space we have 2 vector alpha and beta I also define for alpha beta I also defined inner product which I did in my previous class. And now what I am trying to do is just try to find out this properties and if this 4 properties are satisfied then we call the there is they the vector is forming a space called inner product space. So, inner product space in general a normal vectors. They are forming inner product space I j k when I put the normal vectors in 2 d and 3 d then they are forming the inner product space, if you find that all the 4 properties they are valid there. So, this is important to know that there is another kind of things called inner product space which will based on the inner product definition.

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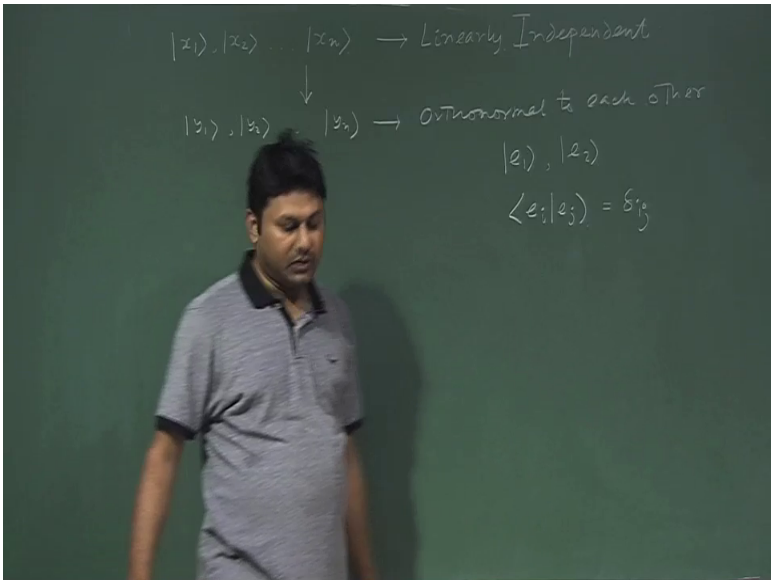

And it follows that this 4 properties are obeyed by these things. In spinners inner product space the next very important thing, I like to mention is suppose a vector is given set of vector is given like x 1 x 2 xn. These are the set of vectors that is given and also I mention that they are linearly independent. So, the problem is a set of vectors given to me and also it is given that these vectors are linearly independent to each other; that means, if I have an expression, let me remind once again if I have an expression like this the only solution for that is cis are 0. It essentially means that this vectors are linearly independent, but the question is different, the question is different I want to form from that I want to form another set of vectors. Say y 1 y 2 yn which are orthonormal to each other.

In last this class also, I define something orthonormality means if a vector say e 1 and e 2 when I am saying these 2 vectors are orthonormal. That essentially means that ei ej I can put any value as i and j one or 2 is equal to delta ij. So, when I put same I; that means, e 1 e 2 or e 1 e 1 then it will give one. If I put e 2 e 2 it will give one, but if I put e 1 e 2 or e 2 e 1 it will give 0; that means, e 1 and e 2 are normal to each other. Not only that their norm is one. So, that is that is called orthonormal or orthinormality. Now orthonormal basis these are forming a basis because we know that a set of linearly independent vector always can form a basis is very important. Orthonormal basis gives you very important ideas or say any vector if I want to expand in orthonormal basis like this.

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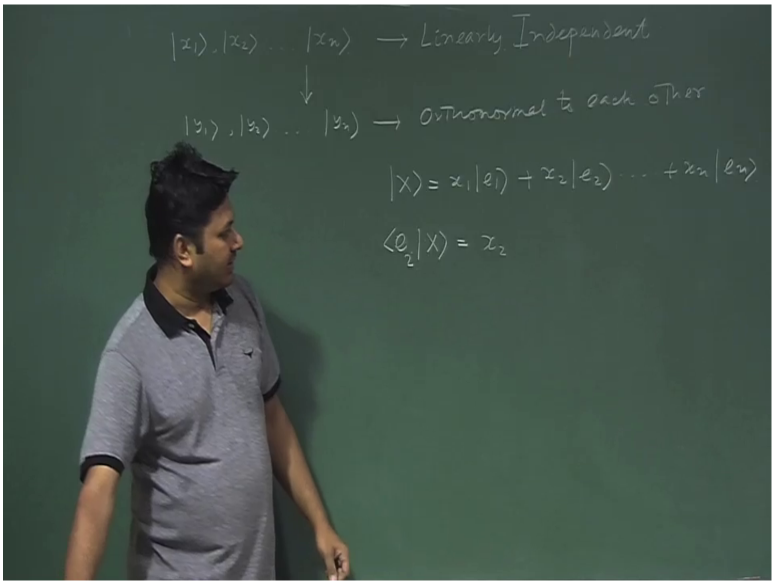

If e 1 e 2 e n are orthonormal, we can have very important information from this for example, if I want to find out any of the element say x 2.

I just if I make this quantity say e 2 this quantity, if I make a projection over e 2 from this side then all the term will going to vanish except this one. So, I will have x 2 straight away. So, the component of the vector can readily be figure out if I use orthonormal basis. So, orthonormal basis is very important I will come to this point again. So, the question is, if a set of linearly independent vector is given it is possible to find out the orthonormal set of vector from that. So, that is the problem. So, let us do the problem whether it is really possible or not so; that means, x 1 x 2 xn is given. And I need to find out from that linearly independent using these vectors which are linearly independent, I going to find out a set of vector which are orthonormal to each other. This process by the way is called gram Schmidt process, n gram Schmidt orthonormalization.

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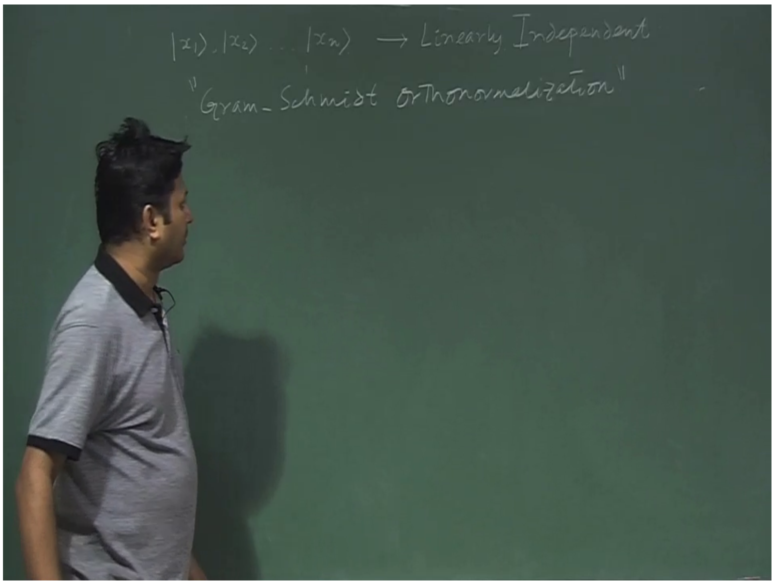

This is called this specific process coth called gram Schmidt orthonormality orthonormalization. So, let us start with that.

So, how do I choose my vectors y 1 y 2 y 3? So, that they are orthonormal to each other from this linearly set of independent vectors.

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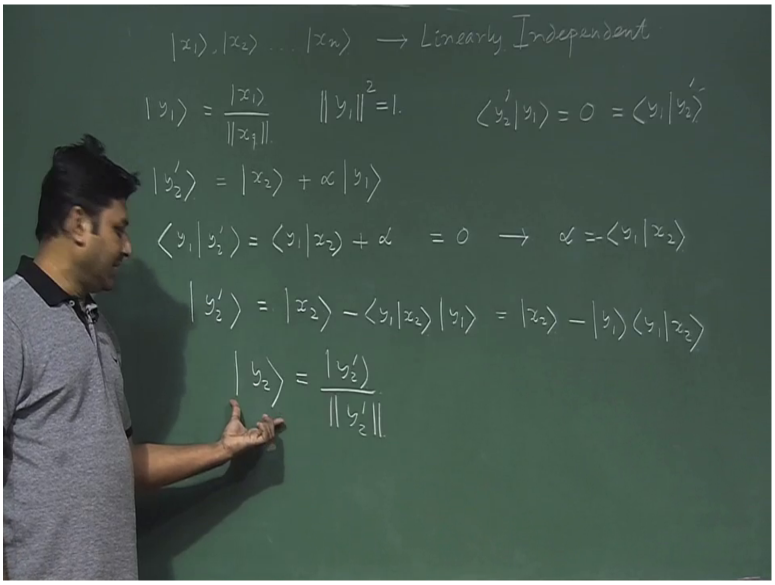

So, let me do that in do that in that in this way. So, first I choose y 1 in the direction of x 1 and divided that by the norm of this vector. So, if I do that then readily I find that y 1 is a unit vector along the distance along the direction x 1 no problem with that. So, right hand side is known. So, I can automatically generate my y 1 how to generate y 2, I write y 2 prime why I am writing y 2 prime ill explain shortly. I choose the vector like this. Please note that when I am forming y 2 I am in the right hand side, I know what is my x 2 because x 2 is given here I know what is my y 1 because just before I formed y 1 what I do not know is alpha. Now this alpha is important because by putting this alpha in suitable way I can find out my desired y 2.

How let me explain. So, my idea is y 2 and y 1 should be orthogonal to each other. So, if that is the case this has to be 0 this quantity has to be 0. Or in other way y 1 y 2 is 0 both the cases the meaning is same we are dealing with real vector for the time being. So, now, what I do I will just say here I am using prime. So, let me put the prime here, then I will explain why this prime is taken you can understand readily. So, if I now operate y 1 from both the side it will be y 1 x 2 plus alpha, why it is alpha because now, x 1 y 1 y 1 term will be coming here which is the mod of y 1 square and here I find that mod of y 1 square is 1. So, from here I know that this quantity is 1. So, if I do that I will get this 1. So, ill not going to now my demand is that alpha 2 will be such that that this quantity is equal to 0.

Because I want y 1 and y 2 to be perpendicular to each other from here readily I find my alpha which is y 1 x 2. So, what will be my final vector what will be my final vector my y 2 prime vector is nothing, but x 2 minus the value of alpha. Which is there one should be minus sign because alpha plus this is equal to so there will be a minus sign here. This now this expression I write in a slightly different way. So, let me write that then, I will going to explain why I am writing these things let me write this slightly different way. So, this is the vector. So, if you if you if you look in the right hand side you will find all the values are now known x 2 is given y 1 I have already generated this quantity I can generate. So, overall it is given a vector which is known and this is my y 2 y. Now y 2 prime is written here let me explain quickly. So, this vector is not normalized this vector is not; obviously, it is not normalized.

So, in order to make it normalize. So, I will write my y 2 the original vector is y 2 prime divided by norm of y 2 prime. So, if I divide at the norm of this vector, then ill get another vector which is in the same direction, but these vector will be now with unit. So, I will form y 2 and this y 2 is y 1 pra y 2 prime divided by the norm of that so; that means, y 2 is my unit vector. So, I can form y 1, I can form y 2 and my y 2 is this form. So, let me write somewhere what the value I am getting. So, my y 1 somewhere here I am getting like this my y 2 is 1 divided by y 2 prime multiplied by this quantity which is nothing, but the y 2 prime in the similar way let me find out find out y 3. Because I need to find out y 1 y 2 y 3 up to yn. So, ill not going to stop here.

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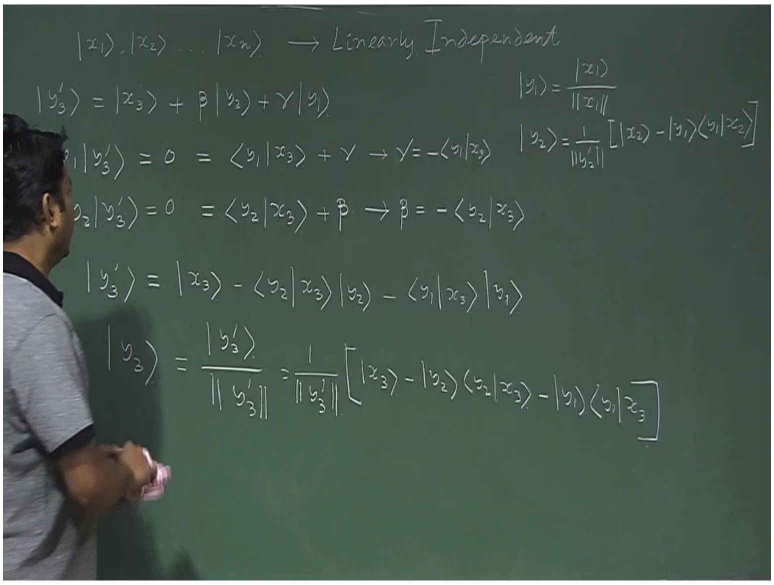

So, let me find out y 3. So, again in the same logic ill put y 3. I will take x 3 here plus beta is some constant gamma is some real value or scalar constant this and my demand is please note that when I am forming y 2 y 3 prime all the values, here is known to me except the beta and gamma which I will try to find out by using the orthonormality. So, the first thing is that if I say this quantity, I want that this has to be 0 in the right hand side what im getting this this term will not be there because y 1 and y 2 are perpendicular. So, beta term will not be there. So, only I will have gamma y 1 y 1 again y 1 y, 1 is nothing, but the unit vectors. So, I will have this as one. So, ill get this which is 0. So, what will be my gamma from here? I will find that gamma is equal to minus of y 1 x 3. Also if I do the same thing when operating y 2 from the left hand side to y 3 prime because y 2 and y 3 are perpendicular to each other.

I want that that they should be perpendicular then it is equal to 0. And it gives me if I operate this term will going to vanish first term will be this and the second term will be simply beta because again y 2 y 2 term will be coming here which is one, and this term will going to vanish. So, from here also I can find out my beta which is this. So, I can construct my alpha construct my gamma construct my beta alpha. I use in this case here I put 2 a new scalars which as which is this and I can also figure it out. So, now, what ill do I will find out what is my y 3 finally, y 3 prime. So, first y 3 prime y 3 prime is x 3, this one then beta I figure out which is that and gamma is this also I calculate this form. So, right hand side everything is known this is a vector this is a scalar this is a inner product. So, it will give you a scalar quantity this is a vector multiplied by a scalar quantity no problem with that. So, overall they are giving a inter vector which are perpendicular.

So, final the final step in the previous way y 3, which is which has to be with unit norm then I need to just divide whatever the y 3. I am getting with the norm of that, if I do that then whatever the vector I am getting is basically the unit vector. So, in the similar, way I can write that this in the form I need to divide y prime note that y 3 prime. So, y 3 prime and then the entire thing I write again I write a slightly different way. So, this is my total vector. So, if I do that for other say y 4 y 5 and y n. So, the ma my final expression of y n should be something like this we do not need to do all these things meticulously, we can we can understand how this things are working then we can form our yn.

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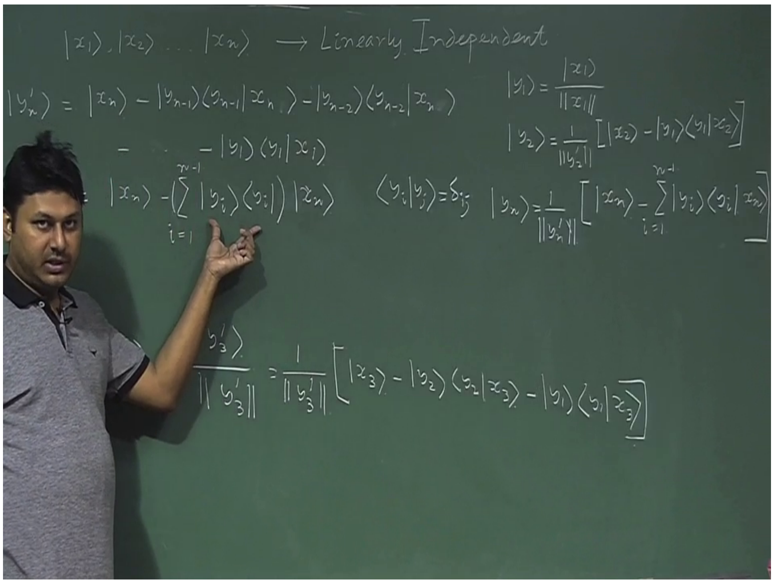

So, yn prime will be something like xn, if I use in this way then n minus 1 yn minus 1 x n minus 1 minus yn minus 2 yn minus 2 the order is reducing here. Here 2 one initially it was just one for y 3 it is 2 1 for 4 there will be another term in this way it will change. And finally, I will have this is my entire yn.

So, now I can write this in a more convenient way in the summation form. So, if I do then it will be just xn minus say summation of yi, yi I take it xi, i tends to 1 to n minus 1. This is a compact form if you look this is a compact form. So, I this things which changing. So, im put this change in terms of i, and I will get something like this. So, this is my one thing I am doing some mistake here. So, both the cases it should be not n minus 1. It should be just n minus 1 should be this n minus 1 should be this because n minus 1 is this quantity. And this is x 3 this is x 3 like this. So, here I should have like this. So, this is my complete expression this is my complete expression. So, now, once I have this complete expression in my hand. So, let me write it yn once again, which is one divided by yn mod of that which is this quantity mod of whatever I have and then xn I have and then sum over this quantity and this this is my complete picture. So, please appreciate that y 1 y 2 yn I have generated by using x 1 x 2 x 3 which are given to me.

So, all the value x 1 x 2 x 3 these are given to me. So, I know all the value and then I use the value in such a way that I can generate y 1 y 2 yn in the form such that anything they are forming this. So, they are forming a orthonormal set y 1 y 2 y 3 are the orthonormal set. So, ill come back to this point with in a moment. Why I am writing in this form before that let us use this formula to find out quickly how this things is working. So, let us do that quickly let me put this here. For example, one example say 2 vectors say e 1 say e 1 prime better to write it in normal notation xy because it will be easier.

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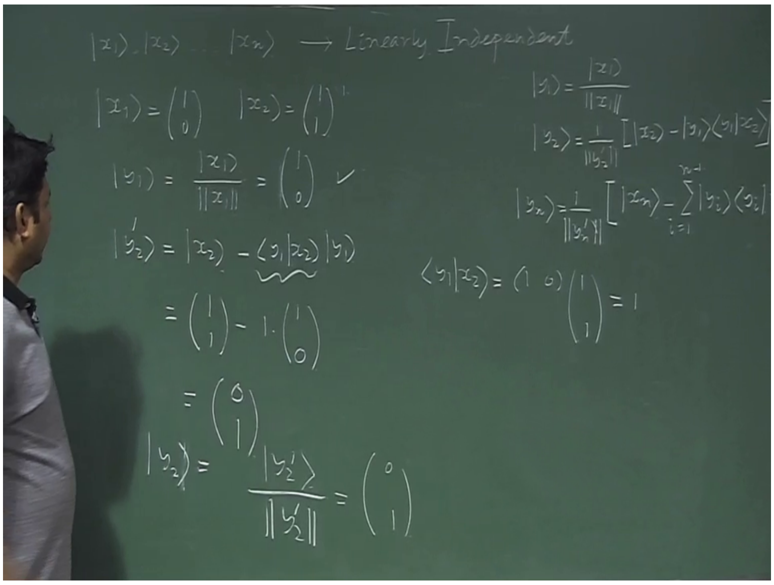

So, 2 vector say x 1 which is say 1 0 and x 2 which is 1 1 is given these 2 vectors are given. So, now, you can note that these 2 vectors are not orthonormal to each other these 2 vectors are not orthonormal to each other. So, I need to find out 2 vectors y 1 and y 2 which are orthonormal to each other using these 2. So, what will be my y 1 y 1 will be straight forward according to my rule it will be something like this which is nothing, but this vector straight away I can find out because this is a vector which has unit norm.

So, I will find out I will figure out a vector, which is this second thing y 2, y 2 first I need to find out y 2 prime y 2 prime is x 2 minus y 1 x 2 and then y 1, y 1. I already find this quantity I need to find this quantity I need to find. So, this this quantity I know. So, if I write it will be 1 1 this quantity isa inner product of y 1 and y 2 y 1 I already figured what will be this value. So, let me write do that here y 2 is 1 1. So, this give me simply one, and this this will be a scalar quantity. So, it will just give me one value one the inner product of these 2. So, now, it will be just one multiplied by y 2 which is 1 0 it is. So, it is comes out to be simply 0 1 minus

1 it is 0 and 1 minus 0 it is 1.

So, 0 1. So, my y 1 is this and y 2 is this y 2 prime is this and y 2 is simply y 2 prime divided by the mod of y 2 according to my rule, but here mod of y 2 is one automatically. So, I will have simply 0 1.

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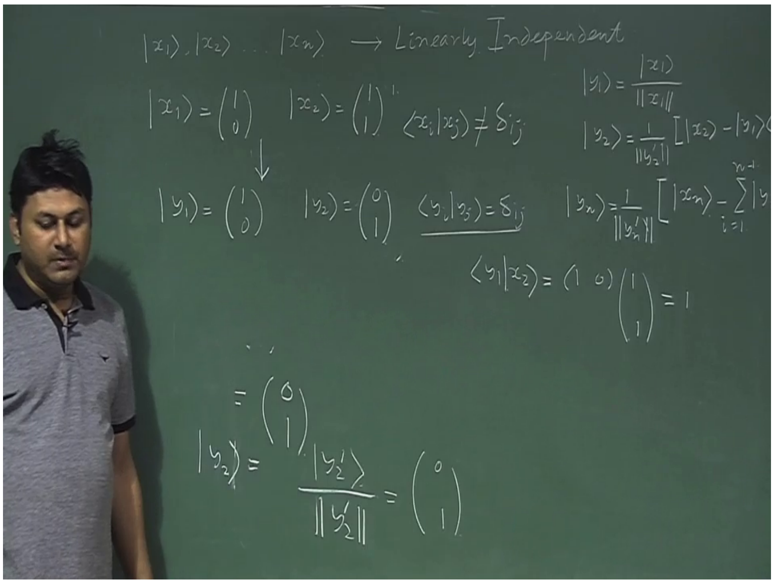

So, now this 2 vectors, I find this 2 vectors is given which are linearly independent, but they are not forming the orthonormality property it is not equal to delta i j using the gram smith orthonormalization process. I figure out 2 vectors y 1 which is 1 0 and y 2 which is 0 1 which is the natural basis in 2 d. Now yi yj are forming delta ij. So, using these 2 vector by using gram Schmidt can. So, this is the 2 d example you can do that for 3 d 4 d 4 d will be more complicated 5 5 d will be more complicated. So, you can deal with the 3 quite nicely 2 d it will be more easier, but in general this is the formulation. So, next class. So, let me conclude in this class what I am doing here.

So, 2 important thing we learned here, first how to find out a set of orthonormal basis when a set of linearly independent vector is given. So, this is given from that, I always can form a basis which is orthogonal to each other not only orthogonal, but orthonormal to each other and. Secondly, I use this inner product concept. So, the basis which is forming delta ij is orthonormal and in order to do that I use one definition here. One different notation here this in the next class, I will going to explain what is the meaning of that what is the physical meaning of that. So, with that let me conclude today's class. So, see you in the next class, where I will discuss this things once again thank you.