Mathematical Methods in Physics -I
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Lecture - 07
Schwarz Inequality

So, welcome back students. So, let us go to the next class of this linear vector space. In the previous class we learn a very important concept of dual space. And also learn how to define the inner product. So, if alpha is a vector and beta is another vector, then essentially alpha beta is defined as inner product. And this value is essentially alpha i star beta i, i equal to 1 to n, if this is in dimensional vector. Now another important thing we also mentioned in last class, that this structure in n tuple notation. In n tuple notation this structure is alpha 1 alpha 2 alpha n. And when I make the structure like this which is in dual space, it will convert it to something like this.

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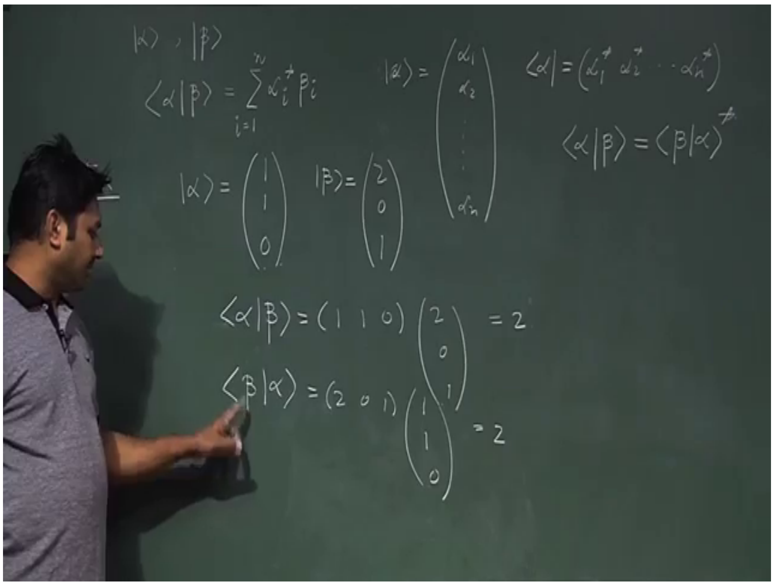

So; that means, essentially what I am doing to go from normal vector space to dual space, that I am changing this structure this is a column matrix. I change this to row matrix; that means, I make a transpose of these things not only that I am making the star of this; that means, complex conjugate. So, now, just give you some example. So, example. So, another important thing, we find is alpha beta is equal to beta alpha star. They are not equal in general, they are related to the star when this 2 are not containing not real coefficient rather they are containing some kind of complex quantity. So, then this star will be playing the role, which is important which is conceptually very important.

So, now alpha just do some exercise. 1 1 0. And beta is 2 0 1. If this is the 2 vectors and if you are asked to make a scalar product or the inner product, then I will have this alpha beta. Alpha beta is just according to my rule, when I make alpha here I need to change the structure like 1 1 0. And this will remain unchanged because I am using these things here, it will be this. And as a result what I am getting 2 into 1 2 1 into 0 0 0 into this things. So, it will be essentially 2. Now if I do the way, I do last time just rotating these 2 vector, alpha beta instead of doing for beta, now I am doing beta alpha. So, then now beta I need to rotate. So, 2 0 1 and then 1 1 0 again. If I multiply this into this 2 0 and 0. I will have 2 you can note that alpha star alpha beta this inner product and beta alpha this inner product both are giving the value 2 the same value, because this these things containing the coefficient which is real in nature.

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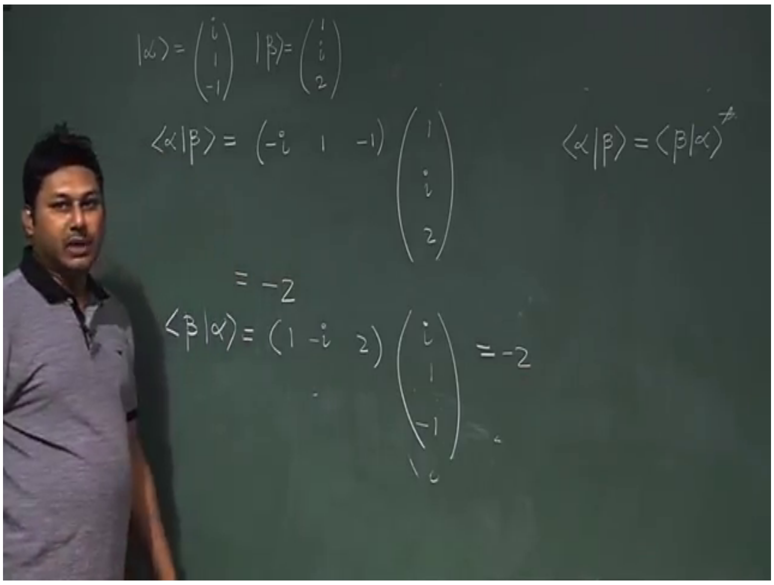

Now, change this coefficient to complex and just check whether this is following this things or not this is I need to check. So, say alpha is i 1 say minus 1. Beta is vector beta is say 1 i 2. Whatever that value you want you can put. Then alpha beta, how do I write this? Alpha I need to rotate it. So, it will be minus i because when I rotate this I need to put the complex conjugate of these terms. So, I should be equal to minus i 1 minus 1, b will remain unchanged because I am not doing anything over beta. Beta is not going to change.

So, now, what will be there it is, now I am taking the example in such a way that this become something real actually. So, go with that then I will change that. So, I multiplied by 1 is minus i minus i. And then there is a plus I which is absorbing the thing. So, now, I will have minus 2, I have real thing. So, expected when I do beta alpha it will not going to change, i 1 and minus 1. If I do then I sorry when I make a change has should be a minus 1, because I am making the beta vector into making a dual conjugation of the beta vector.

So, the coefficient again I multiplied by i minus i, and I will gain come it is boil down to a value minus 2. So, there will be no change because these and there are both are real quantities. So, it will I mean that is why alpha beta alpha are became same. Now if I change I need to change because. So, make it as 2 i. Let us check whether by putting it 2 i am getting something extra or not.

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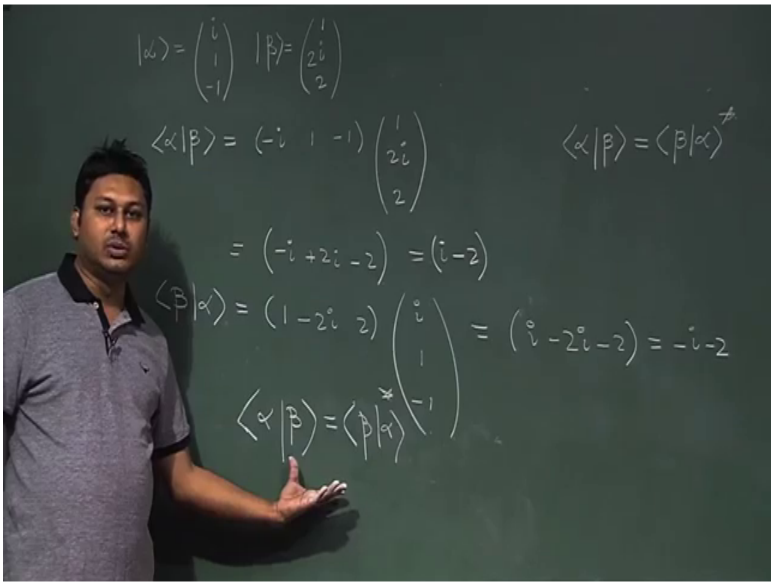

So, what will be my alpha beta. Alpha beta will be minus i 1 minus 1. And it is 1 2, i 2. What will be the result here, this is let me calculate meticulously this is minus i this is plus

2 i. And this is minus 2. So, what will be my result my result is i minus 2, I am getting result i minus 2. This is the scalar quantity even though this is a complex, but this is a scalar. So, this this a scalar thing that is valid, but now I am going to do that other way I will change that.

So, then I need to write it 1 minus 2 i, 2 multiplied by alpha which is i 1 minus 1. Again meticulously if, I calculate then first term is i, second term minus 2 i into 1 which is minus of 2 i. And third term 2 this. So, minus 2 which is minus i minus 2. Now you can check the difference, you can see the difference. That initially it is I minus 2, now it is minus i minus 2. The meaning is that alpha beta is nothing, but beta alpha star, which I have already shown in general way. So, just this is a just merely some example to show how this inner product you should calculate. And also you should you should note that when the elements are complex. So, there is a rule or there is a property, that this 2 things are equal to each other with this this relationship. So, now, with that also we can prove or we can show few things. So, let me do that straight way that.

Now, I know how to calculate the inner product, which is not a very new concept to you, but still the way the structure is presented here maybe something new to you that is dual concept of dual space and all these things.

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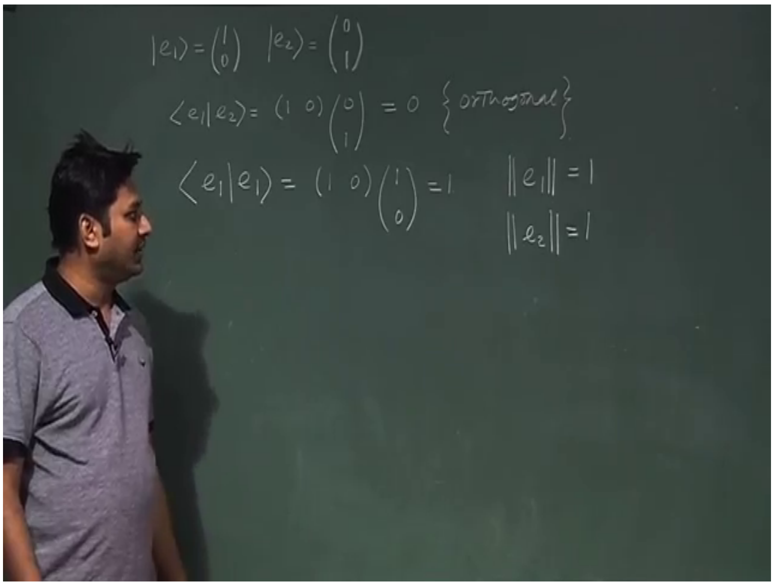

So, let me just check few things that I am saying that e 1 which is 1 0 and e 2 which is 0 1 is a natural basis and orthogonal to each other. So, they are orthonormal. So, the; that means, this quantity readily we will check this is a quite straight forward thing, but it is important that what is going on here one value I am getting this, this it is 0; that means, when I (Refer Time: 10:33) I change one and 2 and this coefficient is changing it is not the same coefficient I am having 0 in my hand. So; that means, they are orthogonal to each other. The first thing they are orthogonal. The next thing is that I will do the same thing, but I am changing e 2 to e 1, I am making a inner product of the same quantity.

So, it will be one 0 and it will be also 1 0. So, what I am getting is 1. So; that means, when 2 vectors are multiplied to each other in a inner product notation, I will have a scalar quantity if the scalar quantity is 1 I can say the norm of e 1 is equal to 1. Norm is the root over of these things this is 1. So, if I make a root over of these things. So, this is one in a similar way, you can show that e 2 will be 1. So, the conclusion is or not it shouldn’t be the conclusion I should say that whenever I am taking the natural basis it is following the orthonormal property straightway. Straight way it will be following orthonormal property.

Now if I change this basis to some different basis, because you should to remember that you can construct in principle infinite number of basis in 2 d or 3 d or nd depending on the value of the vectors, only necessary criteria is that in order to form the basis they should be linearly independent. Since they are linearly independent they are forming a basis, but not only the basis is important also important their relationship between the vectors that is forming the basis. So, here we find the relationship is ortho they are normal to each other; that means, they are orthonormal not only that their amplitude is 1. So, that is why we say this is orthogonal to each other orthonormal to each other; that means, normal amplitude and orthogonal.

So, 2 property together. So, they are orthonormal. So, now, I will going to change this things to some other values say 1 1 minus 1 1. This is also a linearly set of a linearly independent vector in r 2.

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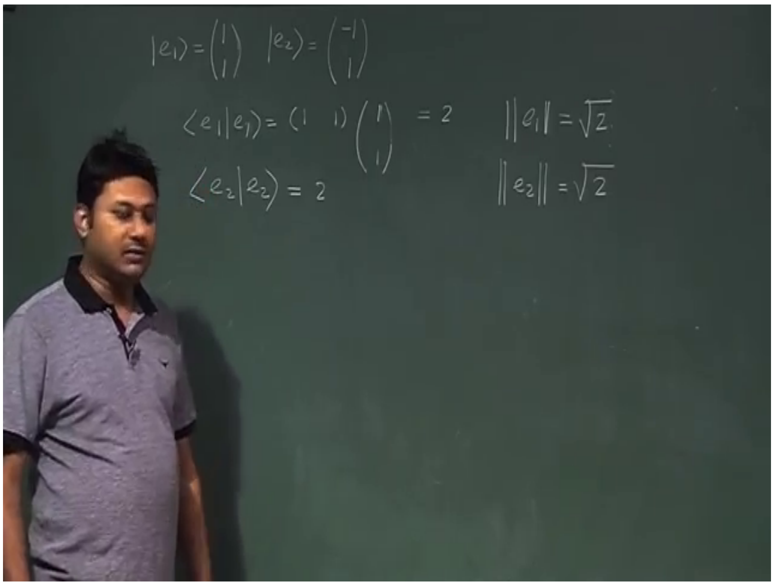

Now, if there are linearly independent in r 2, the question is their what is the relationship between e and e 2. So, let us just find out. So, if I find e 1 e 2. It will be 1 1, minus 1 1 1 multiplied by 1 minus 1 1 multiplied by 1 1. So, minus 1 plus 1 it is 0. I need to check the way around e 2 e 1. So, e 2 e 1 is whatever the value, I have the complex conjugate of that here I have 0. So, I do not need to calculate, I can say this is say please 0 2 vectors e 1 and e 2, when I make a inner product I will getting the results 0; that means, they are orthogonal to each other no problem with that.

Now, the question is if they are orthogonal. So, what is the difference between these and previous vector. The natural basis we mentioned what is the difference between these 2? The difference is the norm. So, let me calculate the norm. So, I have changed that e 1. So, I will make these things 1. And then then what is the value, I have I will have value 2 readily. So; that means, these 2 has a norm e 1 the norm of e 1 is root over of 2. In the similar way if I calculate e 2, e 2 i also find that this value is 2. So, norm of e 2 is also root over of 2. So, even though they are orthogonal to each other, but they are not orthonormal. They are orthogonal to each other; that means, the angle between these 2 vector is 90 degree or whatever because the inner product gives me 0. So, in general will we say this as the orthogonal property, but the norm value of this individual vectors are not equal to 1.

So; that means, they are not normalized they are not normalized. So, in general they are not orthonormal, even though they as orthogonal they are not orthonormal. So, now, let us give a very simple example of another vector say e 1 is 1 0 and e 2 is say 1 1. This 2 vector again this these 2 vectors are different. And again this 2 vectors are linearly independent to each other you can you can check that they are linearly independent. If I cannot write e 2 in terms of e 1 which I do not I cannot have. So, then these 2 vectors is a linearly independent vectors so; that means, they can in principle formed basis, but what is the difference between these things and previous 2.

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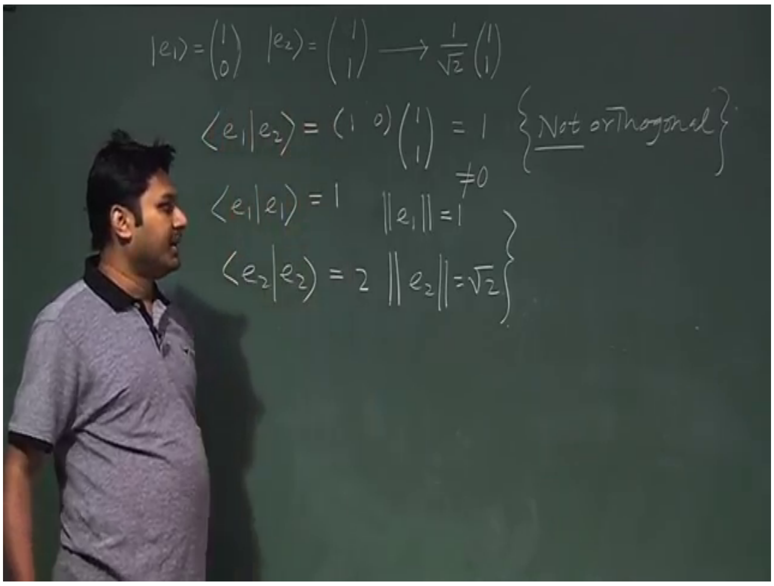

Let us check, e 1 e 2 1 0 1 1. It is one. First time we are getting that e 1 e 2, which is 2 different vector when I make a inner product it is not equal to 0 it is 1; that means, they are not orthogonal also. Not orthogonal. Because this is not equal to 0, if the inner product between these 2 vector is not equal to 0. Which is in this case it is 1, they are not orthogonal. What about the individual norm of this? So, e 1, e 1 this we have already done. So, this is one. So, the norm of e 1, here is 1 and also if I do these things which we have done in the second example I will find this is 2.

So, norm of e 2 is equal to root over of 2. So, this 2 are farming basis that is the first case second thing is that even though they are forming a basis they are not orthogonal. When they are farming a basis you need to understand that a vector in principle can be divided into can be represented in terms of these basis there is no such issue with that. So, there will be no problem that a vector cannot be represented in this case. So, it is represent it can be represented. The thing is that they are not orthogonal, neither they are normalized these unit vector this vector is normalized we call it unit vector, but this vector is not normalized. See in order to normalize normally what we do we just multiply this with his one by norm.

So, then this become normalize. So, if I want to make this vector normalized. So, I will change this vector by multiplying, this now if I take a new vector which is the multiplication of these vector with it is in norm value one by 2 the normal value, then this vector essentially be a normalize director that is the standard way to make a vector no normalized, but important point is that even though they are not orthogonal, even though they are not normalized, one of them is normalized, but in general you can expand a vector in this to a linearly independent vectors, but there is some problems because you cannot readily it is very difficult to find out what is the coefficient that is the different issue will come to that later.

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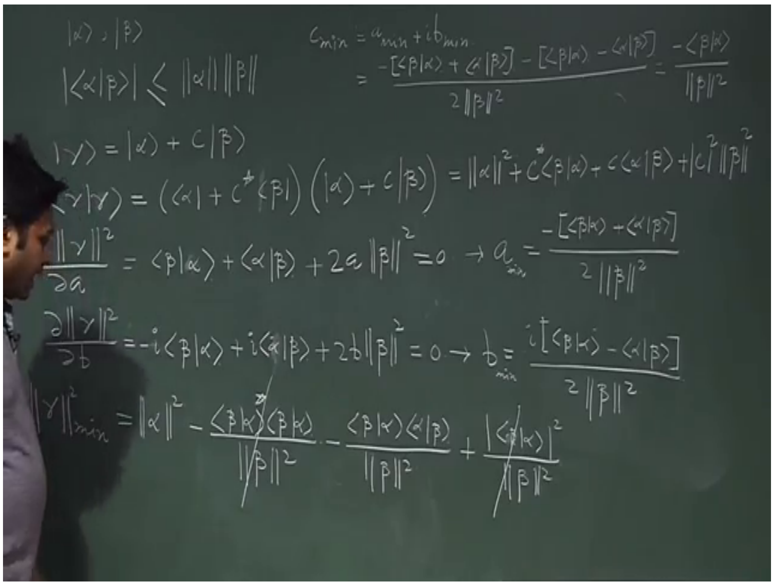

So, after having this. So, let me now we know what is the inner product and all these things. So, let me write down one thing. So, alpha and beta are 2 vectors. So, what is the inner product between alpha and beta? Alpha beta is a inner product is the scalar value.

Now this scalar value mod of the scalar value rather is always less than equal to the individual norm of this. So, this is the rule, this is the relationship between these 2 vectors always follow.

That if I making inner product of 2 vector and they then try to find out what is the norm of that norm means what is the magnitude of that value, that value will be always less than or equal to if I take the norm of that individual vector and then multiply it to them. This inequality is called is a famous inequality is called Schwarz inequality, Schwarz inequality. So, quickly let us try to prove this inequality. So, try to prove this inequality. So, let me do that in this way. I need to prove this. So, let me construct a vector gamma, which is a linear combination of these 2.

So, alpha plus some constant c beta. This constant c is a scalar quantity and I put c as a plus i b, a complex term I can put that because this is a scalar. So, scalar can be real and complex for the time being I am putting is a complex. So, now, if I do, then if I calculate this thing. So, what equation I should write quickly this plus c star b multiplied by alpha plus c b. Mind it, c star will be a minus ib. Which is if I write expand which is norm of alpha square because I conju make this multiplied by this. Then this multiplied by this is plus c star b alpha plus c alpha b beta c multiplied by aloha beta. And finally, this into this there will be 4 term. So, finally, I will have mod of c square into the norm square of beta. This will be my expression. In the left hand side, I will have this. So, these is essentially the norm square of this quantity which depends on the value of c 1, c 2. Because c 1 c 2 is something which containing a b, and this ab is according to my choice.

So, I can change the ab in such a way that I can manipulate this quantity. When I manipulate this entire quantity, I can manipulate the value of this. So, I want to make this value as small as possible. So, the minimum condition is this. So, in order to find the minimum condition, what is the value of this minima. So, I will try to make a derivative and this derivative e equal to 0 gives me the minimum condition. That is a usual when I make a derivative with respect to a this is 0 because there is no a. So, this term will be a simply beta alpha. The second term will be plus alpha, beta because when I make a derivative with respect to a c is a plus ib beta ib. And I am a when I make a derivative with respect to a the a term will be there the other term will not be there.

So, partial derivative is there. So, I will be get this plus mod of c star. So, mod of cs the mod of c square means this multiplied by this. So, it is essentially a square plus b square. So, when I make a derivative with respect to this, I will have 2 a of b square. Now I am saying that this is 0, for this term has to be a minima. For the minima of this quantity if I make a derivative over a then this term has to be 0. So, this gives me a an expression of a which is a, is equal to minus of beta alpha plus alpha beta divided by 2 of beta square.

This I can do the same trick same treatment for b, also because b is another ah variable that I put in my equation in through c. So, if I do that then the first term will 0, but the second term will have a negative sign. Because it is a c star c star is a minus ib so; that means, this I will be there. So, minus of i b beta and then alpha plus, i then alpha the ordering is important here beta whatever we have plus 2 b, a similar kind of expression with 1 iwe having here.

So, this gives me the value of b, I should say this is minimum because I try to find out the value of a and b in such a way that this quantity becomes a minima. So; that means, is a specific value because I this is a specific value I put a term minima here. So, now, if I do then, I will find that this is i multiplied by beta alpha and then minus of alphabet because I am taking this that side divided by 2 of mode of beta square, norm square. So, now, I have a and b separately now I have a and b separately. So, these a and b.

Now I am going to put here to I am going to put here to make it minimum. So, now, I trying to find out what is the minimum value of this norm this is essentially this quantity. So, if I do then it will be this quantity minimum because I am putting the a and b value that I extracted which is the condition of minimum. So, this minimum value is essentially alpha square plus c star c star is containing this and this. So, what is the c star c star is a minima of this, and this if I construct with a plus ib.

So, let me construct it here it will be easier. So, let me what my c minima c minima is a minima plus this a minima plus ib minima which is if I multiply i here. So, it will be minus of beta alpha plus alpha, beta then I multiply I with this quantities. So, I there will be a negative sign here beta alpha and this minus alpha beta whole divided by this quantity 2 this. So, now, you can see that one value will going to cut this alpha beta with a negative sign and this is a minus sign and the plus this minus will be a plus sign. So, this alpha beta will not be there.

So, finally, what I will get is minus beta alpha divided by this quantity. So, this is my c minimum, when I make a c minimum here in this equation then I need to multiply that. So, this will be minus of beta alpha multiplied by beta alpha, because already one beta alpha term is there by the way. I need to make a star here. So, when I make a star here this should be star it should be star divided by beta square then I will have another term which is c.

So, c is simply minus beta alpha multiplied by alpha beta. And finally, I have mode of c square and these things. So, mod of c square is plus that mod. So, mod beta alpha square divided by this things because one it will be to the power 4, but one beta square is sitting here. So, it will cancel out. So, finally, I will have for this quantity and this quantity will cancel out. Because this is mode of this square this is also mode of this square. So, I will cancel out this quantity. So, this is cancel out.

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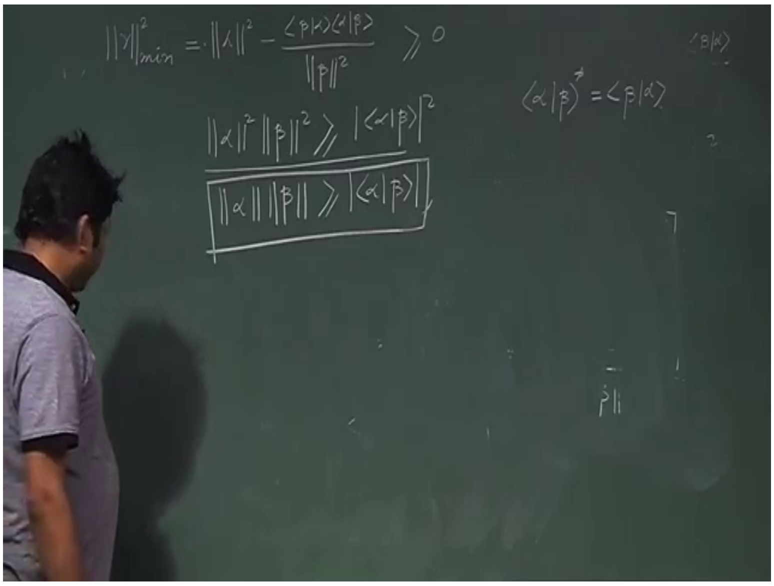

So, finally, what I am getting essentially is a simple expression. So, quickly let me show that what I am getting is gamma square minimum is equal to alpha square minus this quantity, which is beta alpha multiplied by alpha beta divided by mod of norm of this b square. So, now, this minimum value we already explain the fact we have already explained the fact that, this value has to be greater equal to 0, this value has to be greater equal to 0.

Now, if I apply this 2 things then readily, I will find that this is greater equal to this quantity this quantity is nothing, but mod of alpha beta square because alpha beta star is equal to beta alpha. So, if I change this 2 star it will be beta alpha and mod of square. So, it is alpha beta mod of square and beta alpha mode of square will be the same thing. So, essentially I am having this. So, from this expression I readily can say that alpha beta is greater than equal to this quantity. So, we prove one very important identity which we called the Schwarz identity, which is this that when a we are calculating the inner product of 2 vectors. This is the inner product the mod of inner product of 2 vectors is always less than the individual norm of the 2 vectors when they are multiplied. So, with that I would like to conclude to this class. So, so far we learned few important concept regarding the inner product and how it is formed.

If how it is formed and all these things in the next class. We will try to start to finding out how to form these basises, because will now going to play with the basises, if a say it of linear independent vectors is given to you can I make the set of a linear independent vector to a basis which are orthogonal to each other. This is a very important aspects to form a basis and we always use this kind of because, we always want basis to be orthogonal to each other and normalize. So, we in general we always try the basis to be orthonormal to each other through into find out what is the procedure that, I can make a orthonormal basis. So, next class we will start from here and try to find out what is the procedure to make this orthonormal basises so with that.

So, thank you very much for your attention. So, see you in the next class.