**Mathematical Methods in Physics-I**
**Prof. Samudra Roy**
**Department of Physics**
**Indian Institute of Technology, Kharagpur**

**Lecture - 60**
**Real Integration Using Cauchy's Residue Theorem (Contd.)**

So, welcome back student to the next class of complex analysis. So, we are almost at the end part. So, probably this is the last class we are having. So, today we will continue with the Cauchy's residue theorem and will now like to solve the problems having the form this that will mentioned in few early classes.

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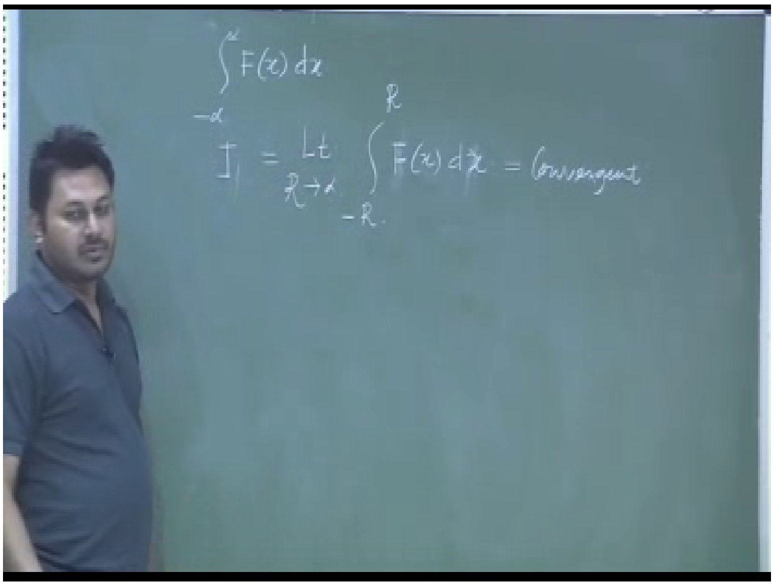

So, this is a real integration where the limit is infinite minus infinity to infinity. So, normally we what we will do that we take the limit of this things. So, limit R tends to infinity then minus R to R and then whatever the function I have F z is F z F x d x because I am doing I am dealing with the real integration.

So, I will have this F x d x. So, limit R tends to infinity minus infinity to minus R to R if this limit exist then I can say the integral whatever the integral value I 1 say or I say is convergence. So, the limit is convergent if the limit is I first calculate this integration with the finite limit minus R to R and then put R tends infinity if this I has the limit at R tends to infinity then we can say this integral is convergence. So, this is one important idea to do that.

Now, I will going to tell how to deal with this kind of integration,

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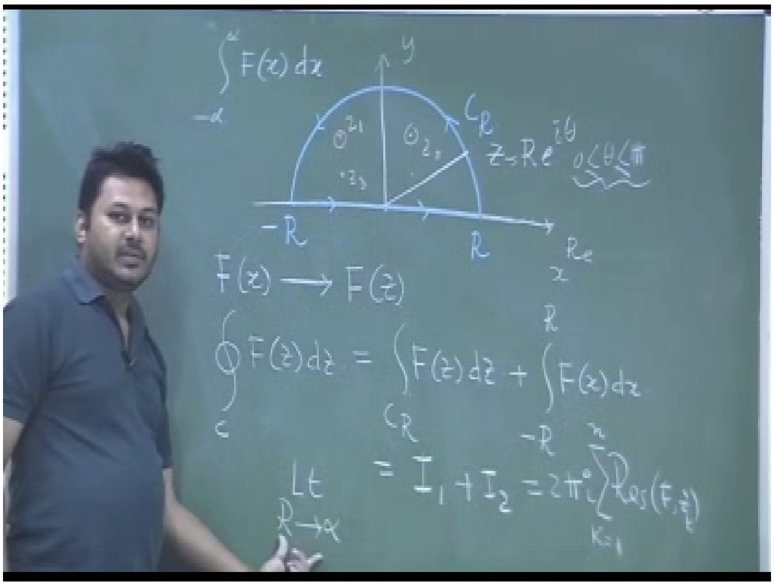


So, if I now draw. So, now, if I draw the real and imaginary axis, this is real or x axis and this is imaginary or y axis. So, the integration is over this real axis right because it is minus infinity to infinity. So, the contour that we will going to choose is something like this, the blue line So, this will be minus of R this will be R and I will take a half circle over that. So, I will go this this return back. So, this entire blue thing is nothing, but a closed look of close circle all my singularities may be here here here here we may have singularities z 1, z 2, z 3 and so on and I write this line as C R right.

So, this closed integration contain 2 part; one is this line which we called C R and another is over the real line which is running from minus R to R and if I add all these things I will have a closed look. So, this integration first I need to write in terms of z. So, this integration minus infinity to infinity, I want to find this integration. So, what should be my consideration? So, first I will change this F z to F x to F z first I change to complex integration, then I will take the complex integration over this circle whatever the circle I have what are not circle whatever the closed reason to I have this is a half circle and then I cut this from here to here.

So, whatever the closed region I have I called it C in total it is C, then this F z I calculate this is nothing, but that combination of 2 integration one is the line integration C R where my function F z will remain same because I am taking the I am doing the calculation for all complex numbers here all complex numbers or imaginary numbers. So, F z will be my function, but when I integrate the another part. So, I go here from here to here and then what happened I go from here to this point to this point.

So, I need to add minus R to R right. So, what function I will write? My function will be simply F x here because the function is lying over the real axis having all y 0. So, since all y 0 there will be no y only the x term is there. So, I will have the integration there will be 2 integration. So, this 2 integration I write say I 1 plus I 2, after having the result. So, what should be result of this integration that is known that is nothing, but 2 pi i again I am using the Cauchy's residue theorem. So, Cauchy's residue theorem suggest that I need to just find out the residue of the function whatever the function is given here F z at z k how many residue are there we need to find out K 1 to z n finite number of residue which are these points ok.

So, that is the recipe to do this problem. So, let us do one example then things may be clear. So, first I need to have a real integration with the limit minus infinity to infinity then what we will do I change is F z, F x to F z my complex term complex function. This then I integrate this complex function over this blue curve closed curve, I called it c it will be divided into 2 part it is divided into 2 part one is C R, which is this where my z over c my z is R e to the power of i theta, theta is ranging from pi to 0 this is the range and then I have another integration which is this line integration minus R to R and the function is now I should not write F z because y is 0. So, I will write F x.

So, my integration is this one, this is the desire integration I have why because the function is giving. So, next things I will calculate the integration at the limit R tends to infinity, this value I will calculate and then I put R tends to infinity and check how these things are diverging or converging and that will be equal to this and I will extract this I 2 from there. So, I 2 will be this minus I 1, I 1 I 2 evolve at R tends to infinity.

We understand this things. So, let us use this concept to find out some real integration whose limit is minus infinity to infinity problems. So, let us take one example or problems. So, first problem I will have minus infinity to infinity the function is given as x square minus 2 x plus 2, d x this is the integration I have in our hand.

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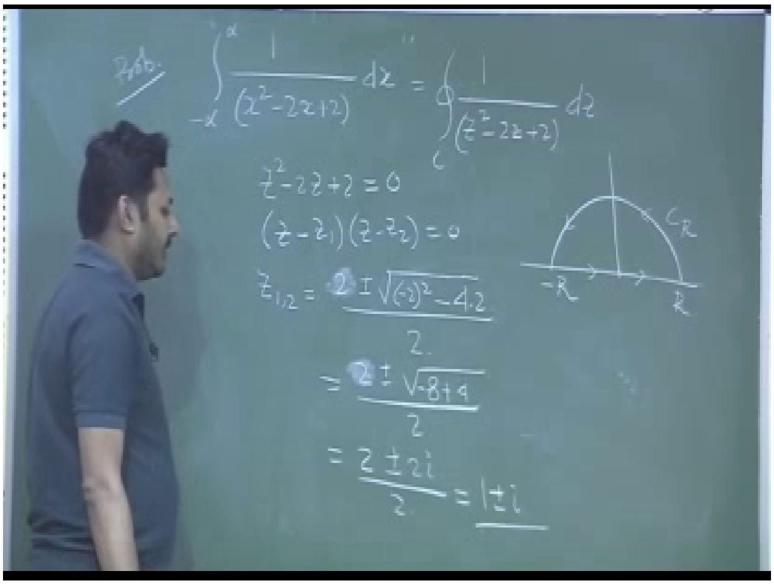

So, now, I need to convert this integration as close integral 1 divided by z square minus 2 z plus 2 d of z. This integration over real axis since I will going to evolve this things over this region that I shown minus R to R and this is C R and this is my closed path.

So, this path is C. So, entire path is C. So, this is the integration C R is only this from here to here it is C R this is not a closed path. So, this is the integration I need to evolve. So, now, again this integration has some values. So, let me first find out what is the value because I know very quickly what should be the value of this integration, the singularity I will find 2 pi R multiplied by residue will give you give the result. So, let us find out what result I am getting. So, z square minus 2 z plus 2 equal 0. So, this is if I write z minus z 1, z minus z 2 equal to 0 then what should be my z 1 and z 2 I need to figure out.

So, my z 1 2 here is minus b. So, it is 4 plus minus root over of b square which is minus of 2 of square minus 4 a c minus of 4 a c 4 into a is 1 c is 2. So, 4 a c divided by 2 right. So, it is how much 4 plus minus of root over of 8 minus of 8 plus 4 by 2.

So, it seems to be something like this will be minus of 4. So, minus of 4 means 2 root i. So, 2 root over minus 1; that means 2 i. So, I will have something like 2 plus minus of 2 plus minus of seems to be something like this, let me check once again. So, here it is not 4 minus b is minus 2 why I am writing is as 4 it is minus 2. So, then it become mine plus 2 this is plus 2, this is plus 2. So, minus of b plus minus root over of b square which is minus 2 square minus 4 is c.

So, minus 4 into a into c, so 1 and 2. So, minus this things it is 4 and this is minus 8 this 2 this will come out to be as 2 plus minus of 2 i divided by 2.

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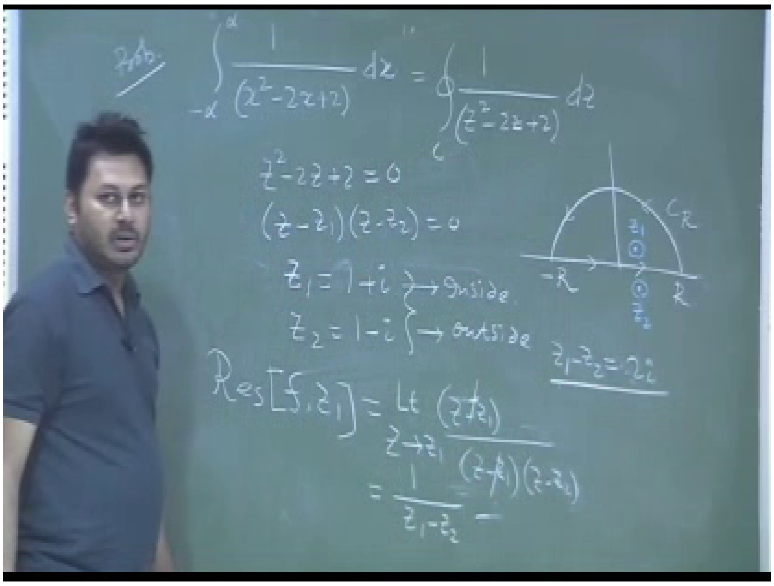


So I will have 2 point 1 is 1plus minus of i that is all, 1 plus minus of i. So, my z 1 here is 1 plus i and my z 2 is 1minus i this 2. So, now, I check where the z 1 and z 2 point are in this particular case. So, I plus 1 is somewhere here 1 plus I somewhere here this. So, this is the point of my z 1 which is inside the region mind it, and another point is replica mirror image of this point which is here z 2 and which is outside the region this is just outside the region not inside the region.

So; that means, this is inside z 1 is inside, this point is inside and this point is outside. So, I know which point is inside and which point is outside. So, next is to find out the residue. So, the residue of this function residue of function at z 1 point is z 1 is inside that is why z 1 point is limit z tends to z 1, z minus z 1 multiplied by the function the function is one divided that.

So, it is z minus z 1, z minus z 2. So, z 1 z one cancel out I will have this. So, when I put the limit it will be z 1 minus z 2, 1 divided by z 1 minus z 2. So, z 1 minus z 2 is 2 of I z 1 minus z 2 to of i. So, the value of this integration. So, value I have already figure out the value of this integration.

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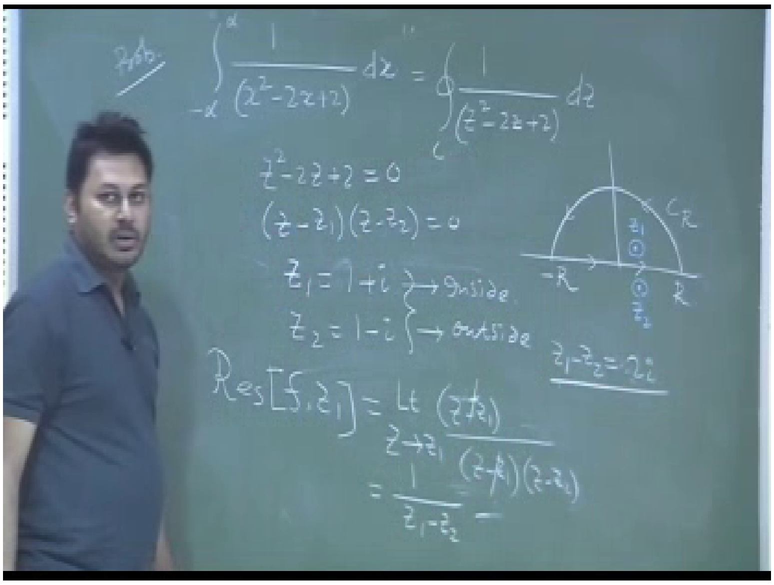


So, this residue is 1 divided by 2 i, the residue is 1 divided by 2 i. So, the value of integration let me write it here 2 pi i multiplied by 1 by 2 i. So, the integration seems to be only pi the integration seems to be only pi.

So, now this integration let me erase this now the next part we need to take care.

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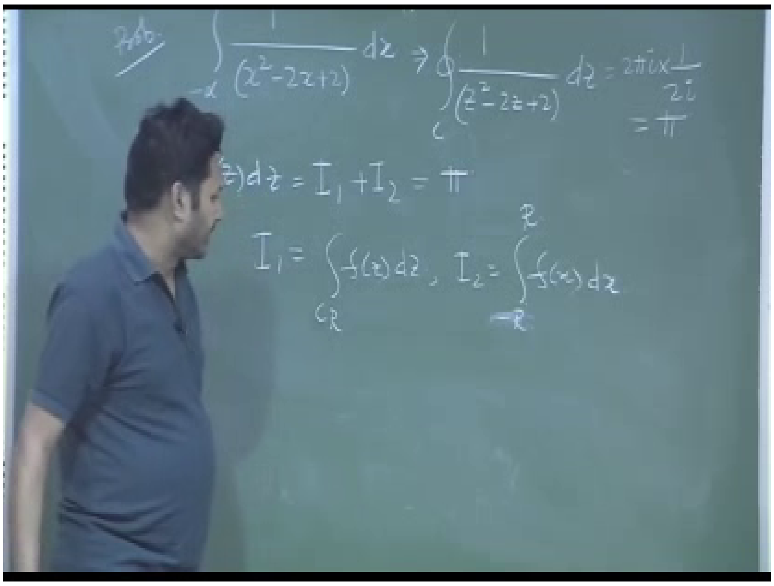


So, this integration of containing 2 part. So, this I convert it this is not equal to this mind it I convert this to close integral. So, this integration I write it function. So, this integration c if I write this is the function of z d z is equal to I one plus I 2 which is equal to pi right.

So, where is what is I 1? I 1 is integration of C R this function of z, d z and what is I 2? I 2 is integration minus infinity minus R to R function of z function of x d x. So, I need I 2 here because I 2 is my integration and then I put R tends to infinity that is the concept. So, let us find out what is this C R.

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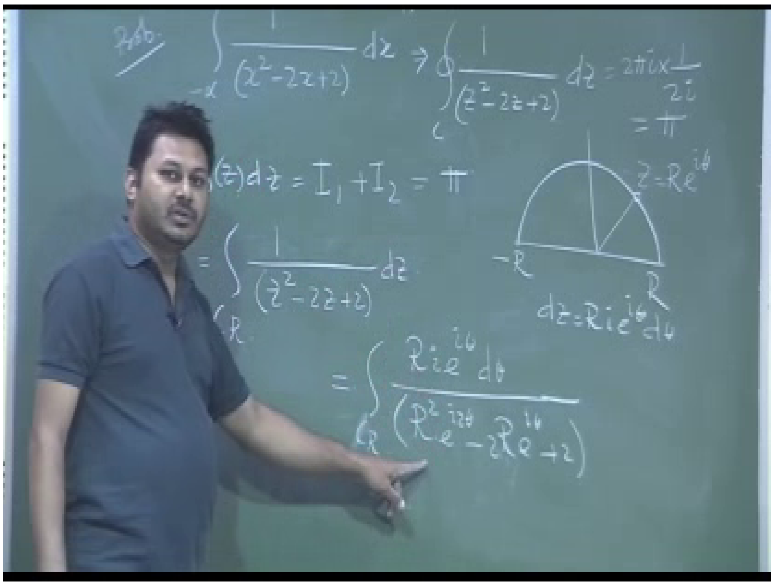


So, integration C R if I put this value here it will be z square minus 2 of z, plus 2 d z the contour once again this something like this is minus R this is R. So, this point, my z point here. So, z is R e to the power of i theta.

So, now if I convert that then I said this is my I 2 and now if I convert in terms of R over C R I do that over C R. So, it will be d z, d z will be R I e to the power of i theta it. So, it will b R i e to the power of i theta d theta divided by this quantity. So, this is R square z square which is R square e to the power of i 2 theta square minus of 2 of R e i theta and plus 2.

So, now what I am not going to do the entire integration over that, but I am just looking what will be the form of the integration how the integration will be look like when R tends to infinity. In order to do that I can simplify one thing that I take the mod side of both sides.

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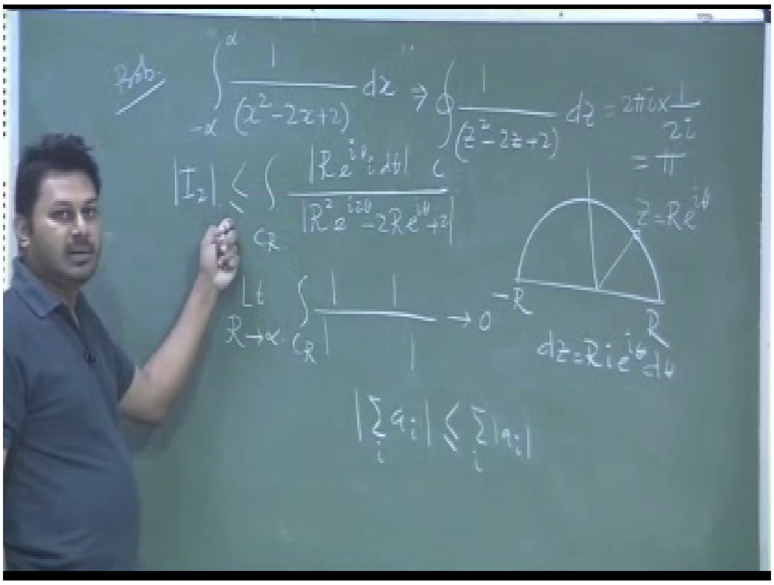

If I take the mod of both side, then it should be less than equal to the integration C R mod of this quantity divided by mod of that quantity R square e to the power of I 2 theta minus 2 of R of e to the power of i theta plus 2.

Now, as I mention my aim here is to find out the integration when R tends to infinity now if I try to find out what is happening here R tends to infinity. So, limit at limit R tends to infinity this equation, this function we have R in the denominator. So, 1 by R is a constant. So, it is not depends on theta. So, integration cons in the integration R is constant. So, if it is a constant and if I put this limit R tends to infinity will find that the denominator is changing with the order of R square.

So; that means, the right hand side R tends to infinity this quantity integration C R whatever the function I have this, this it will going to be 0. So, right hand side which are mod. So, when I take mod then this mod will be outside integration, when I put the mod inside then this quantity; obviously, greater than this point because summation of something this is always greater than equal to this quantity.

So, this is always greater. So, I am doing this things here. So, this when I put the mod of this things I should put mod of this things here and here outside integration, now I put this mod inside integration then this quantity should be greater than this I want. So, now, at R tends to infinity if this quantity goes to 0 then; that means, I have I 2 is 0 also, I 2 will be 0 with a condition limit R tends to infinity that is my outcome.

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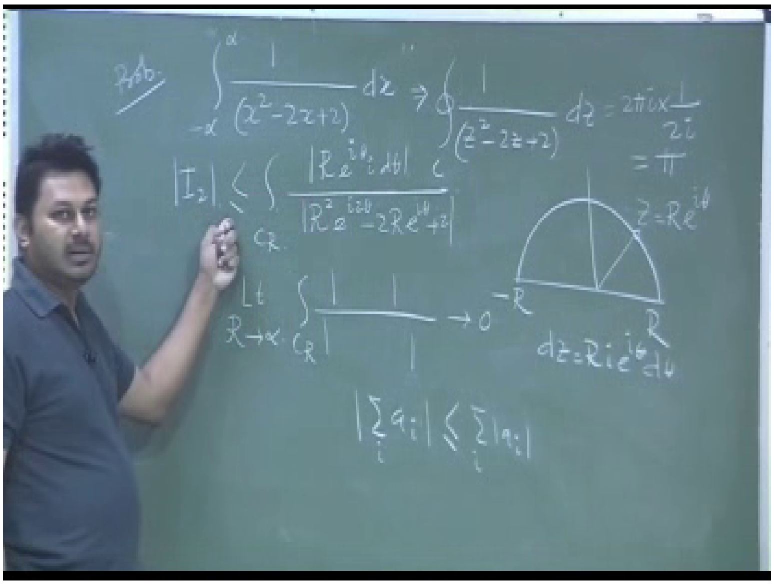

So, once I have this then this integration will become minus R to R limit R tends to infinity disintegration is become pi.

So, now this is converging because right hand side I have pi. So, I can write this is called Cauchy's principal value. So, I can now write safely minus infinity to infinity 1 divided by x square minus 2 of x plus 2, d x is equal to pi. So, this is the result of integration pi, but you should also take care of the issue that what will be the value of integration in C R you need check every time whether it is going to vanish or not if it is going to vanish then ok otherwise you have a problem.

But normally the problem is given in such a way that in all the cases you will find this things will going to vanish. Quickly we will take another example where 2 functions are associated with that. So, let me check what example is in my hand right now.

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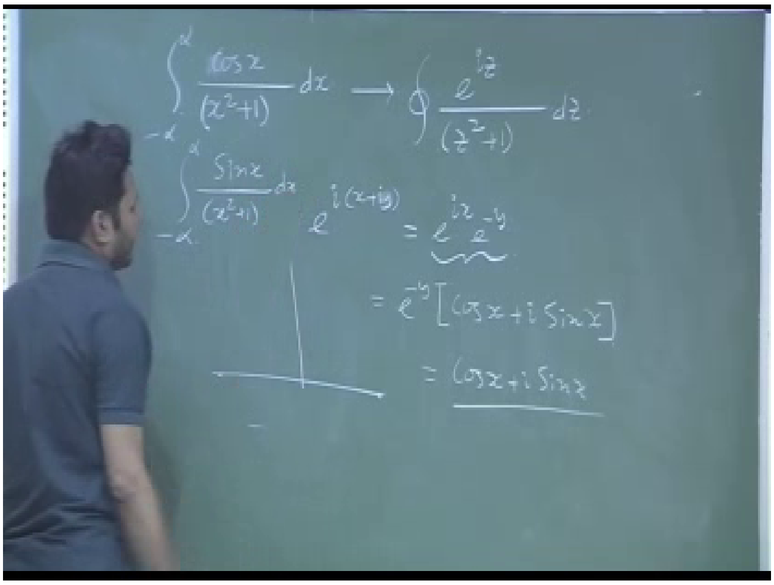

So, I have this integration. So, minus infinity to infinity e to the power not e to the power it is cos of x now x square plus 1, this is the integration I need to evolve. So, here the integration is slightly different from the previous one because in the previous integration the everything was the function of x, but here find it is combination of a trigonometric function and the algebraic functions. So, these 2 are there, but will should we should not bother about that because we need to convert this as a complex forms. So, when I in convert to a complex form cos x we will write at e to the power of i z divided by this quantity, I will convert this and this quantity I will convert z square plus 1 d z.

So, why I write this e to the power of i z cos z. So, if I do e to the power of i, it will be x plus x plus i y something like this. So, it will be e to the power of i x multiplied by e to the power of minus y right this quantity is e to the power of minus y multiplied by cos x plus i sin x.

So, this is this integration if I do when I do this over real axis, this is the real axis over real axis then y is 0. When y is 0 over real axis we have 2 component one is cos x and another is i sin x. So, these if I write in this way I should eventually have this expression when I am evaluating it over the real axis, now when the result this is the real part if I have a result which has a real and complex part, if I deal with only the real part then the result will come for cos x, if we deal with the complex part the result will come for this case. Once we have this and these together then I can evaluate this quantity as well as this quantity.

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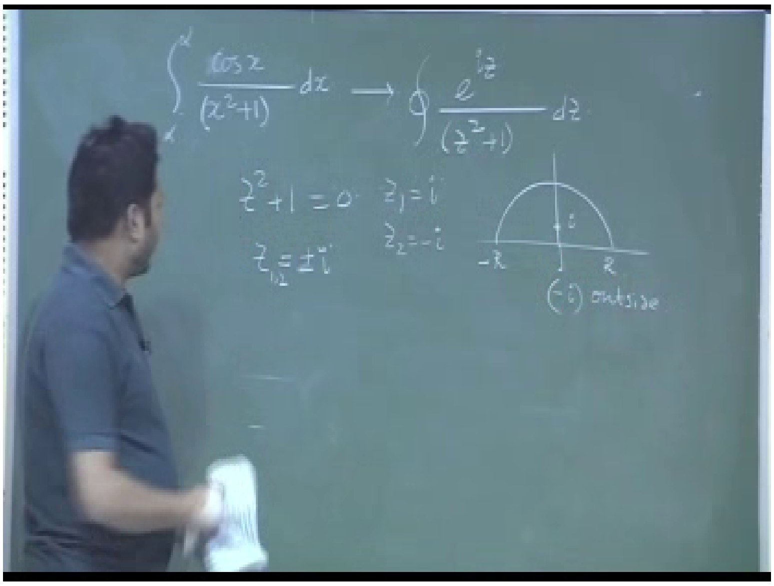

So, this is the general way to take when whenever you find cos x or sin x and then take do the integration and find out whether this is a real value or the complex value and then you will convert this accordingly. Let me do the problem then things will be clear to you. So, I know this is my. So, this same contour I am going to use minus R to R, because my limit is minus infinity to infinity minus R to R is there. So, now, if I try to find out what is the residue it is z square plus 1 is equal to 0.

So; that means, z is z 1 2, 2 is plus minus i. So, one residue is at z 1 is equal to i another residue is z 2 is equal to minus of i right so; that means, one point it here I is here and minus i is here. So, minus i is outside. So, the residue what will be the value of this integration?

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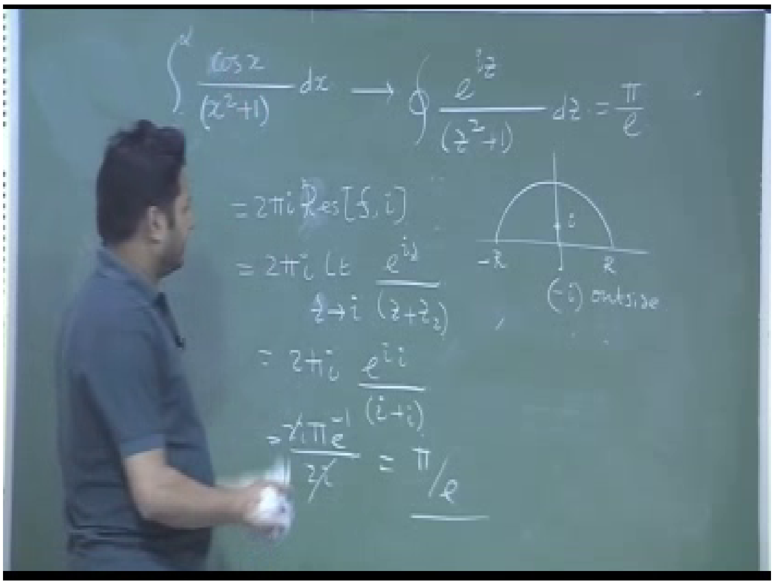

Value of this integration is equal to 2 pi i then summation then residue of the function at i point so; that means, 2 pi I what is the residue what is the function.

So, limit z tends to i e to the power of i z divided by z minus z 2, z plus z 2 where z because z I have z 1 plus z 2 to this is i z, z 1 z minus plus z 2 equal to 0. So, that was the condition my z 1 is I and z 1 is 1 is i another is minus i. So, if I put this here then we have 2 pi I and z 2 is i. So, it is e to the power of i into i divided by i plus i. So, it is 2 i.

So, I will have pi e to the power of minus 1, 2 i divided by 2 i. So, 2i 2 i will cancel out. So, I will pi by e. So, pi by e is result. So, now, you can see the integration of these things are pi by e; that means, this is the real entirely real value. So, when it is a real value; obviously, it should be related to this cos x. If I find the sin x then the sin x integration is zero. So, the rest part is straight forward now this integration has 2 part as usual one is I 1 and another is I 2 which is pi by e, this I 1 part is integration of C R e to the power of I z is R cos theta plus I sin theta divided by.

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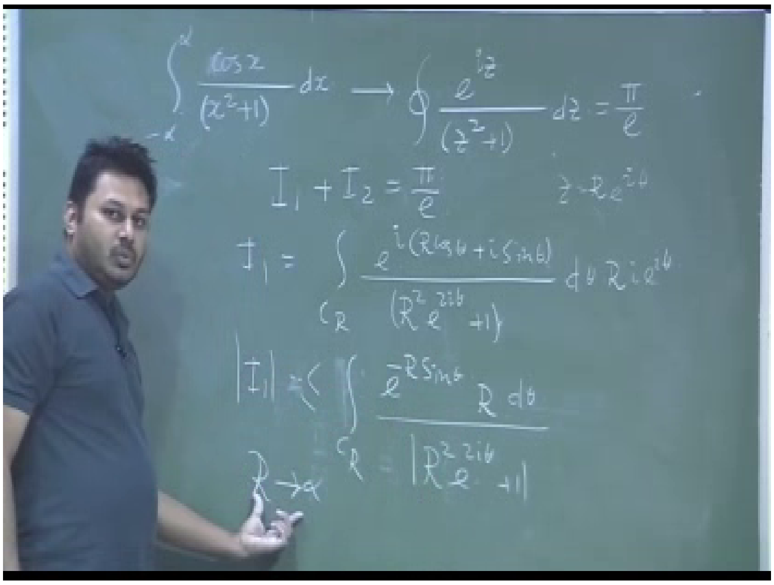


Now, I am putting the value of z as R e to the power i theta. So, R e to the power i theta is R sin theta cos theta. So, I am writing this only. So, this is equal to z and in the downstairs I just z square. So, I write R square e to the power of 2 i theta plus 1, d theta and when I put d theta R will be here with i and e to the power i theta. Now if I put the limit here I 1 again mod of this things will be less than equal to C (Refer Time: 29:49), limit of this things mod of this things the mod of this things is e to the power one term will come out as R sin theta and R will be here and other term will going to vanish d theta also here when I took the mod and the downstairs I have mod R square e to the power 2 i theta plus 1.

Now, again you can check what happened of this integration when R tends to infinity. When R tends to infinity this quantity will going to vanish because e to the power of minus R will decay very rapidly on top of that we have R square here also. So, R square also tend the integration to put the integration towards 0 with the limit R tends to infinity

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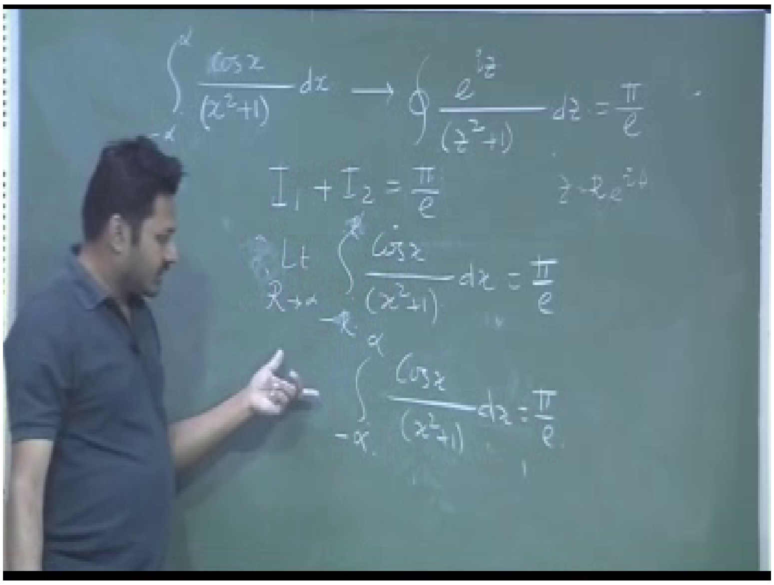

So, then I can write that under this condition R tends to infinity, limit R tends to infinity minus infinity to infinity cos of x, x square plus 1 d x is equal to pi by e this is converging.

So, now again I write this is minus R to R. So, minus infinity to infinity cos of x, x square plus 1 d x is pi by e where I 1 is tends to 0 that I showed. So, student I have shown almost all the different types of integration using the rule of Cauchy's residue theorem. So, that is the syllabus so far we have in our hand. So, almost all the problem I have covered. So, I believe you have enjoyed this course thoroughly and I would like to give you a best luck of this course. Please carefully to read all these things to all these things and that is the things should probably today is the last day of this class. So, I will give you all the best luck thank you for your kind attention and taking this course.

Thank you and best of luck.