Mathematical Methods in Physics -I
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Lecture - 06
Inner Product

So let us now start the next class in linear vector space. So, in our from the last class, we can understand partially.

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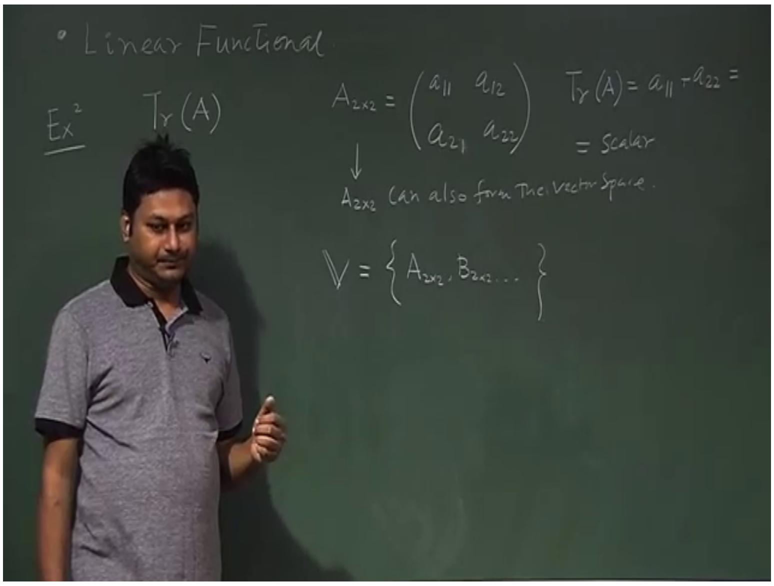

What is we were discussing something about linear functional, and we also give one example of linear functional which was a dot product. And we mentioned that we will give another example which is not necessary a dot product kind of things it will be different kind, but that can also be considered as a linear functional. So, the next example, example 2 is the trace of a matrix my rule is trace of a matrix. So, what is a trace of a matrix? See A 2 by 2 matrix can be represented as a 1 1, a 1 2, a 2 2, a 2 1, a 2 2. This is a 2 by 2 matrix the elements I can write in this way.

So, now the trace if I apply, the trace operation over these things then what essentially we are doing we just add the diagonal elements. Here so; that means, outcome is a 1 1 plus a 2 2. This is just a number this is number and number 2 numbers I will going to add. When

I add 2 numbers a 1 and a 2, I can have a scalar. So, I am adding 2 numbers and as a result I am having a scalar. So, trace of a matrix is a operation that I operate over A and as a result im having a scalar.

So, one thing also you should note that A this things A which is a 2 by 2 matrix can also form the vector space; that means, in the vector space V, I can have the elements A B all 2 by 2 matrices. For example, they are forming the elements they are the elements of a vector space. So, now, what I am doing that I try to operate this trace function, or trace rule over that and when I operate we find that I am getting a scalar. So, my first condition is valid, the first condition of linear functional is that linear function is something which I can operate over the vector element, where the vector element here is A, in such a way that it is giving you the scalar.

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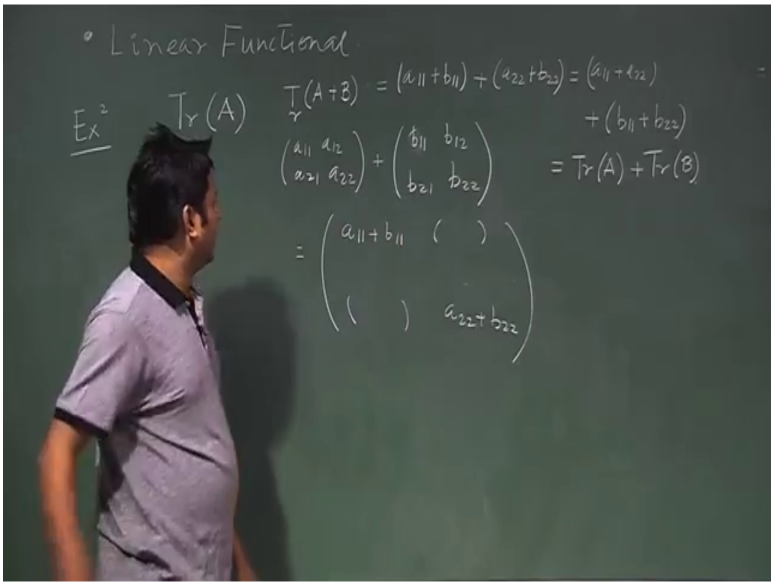

So, now, I will try to find out what is going on when I try to apply that in A plus B.

So, trace A plus B what will be the result. So, my A is a 1 1, a 1 2, a 2 1 a 2 2, which is a and I am adding this as a another vector b 1 1 b 1 2 b 2 1 b 2 2, a plus B is element of the vector space. And I am adding 2 I am getting a new vector, new vector element I called it c where c is just a edition, if I do the addition you can readily find that the addition is a 1

1 plus b 1 1 plus some terms here. I am not going to write these I am just interested only the diagonal term for the time being.

So, it is a 2 2 plus b 2 2 this structure the structure wise it will be something like this. So, now, if I make a trace over that ill have what value. So, if I write this here. So, I will have a 1 1 plus b 1 1 plus a 2 2 plus b 2 2, which is essentially a 1 1 plus a 2 2 plus b 1 1 plus b 2 2; a 1 1 plus a 2 2 and b 1 1 plus b 2 2 is nothing, but trace of A individually plus trace of B. So, my first rule is satisfied function over A plus B is equal to function A plus function B. This function is a trace function and both the cases it is giving the scalar things.

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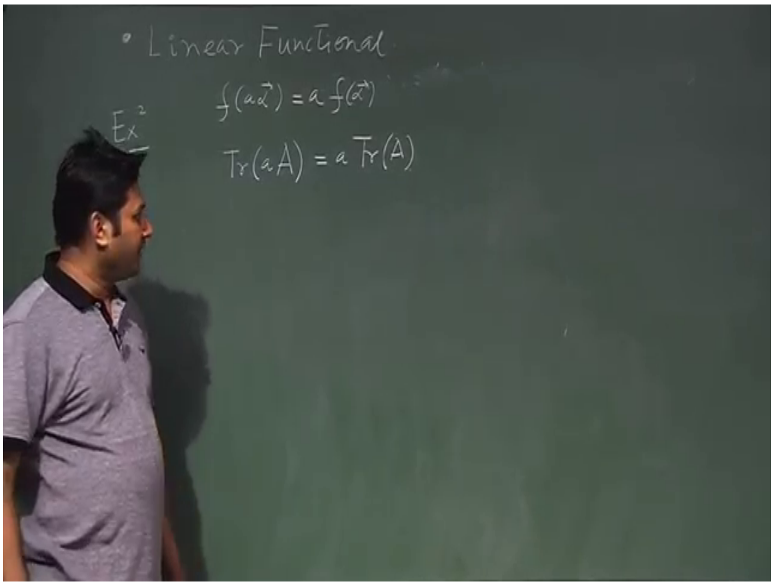
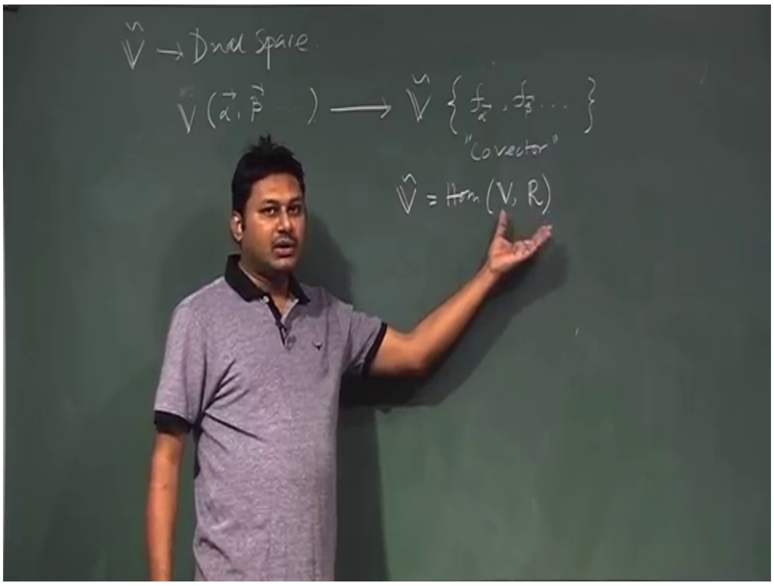


So, rule one is satisfied. Now what about the rule 2? Rule 2 gives me that just write to recall the rule 2 is a, if I over a vector then it will be f these things, if I try to do the same thing with this.

So, trace of some multiplication a with A is nothing, but a trace of A. I am not doing meticulously I suggest you to do and check whether this is valid or not. I believe it will going to valid and if it is there then we will have a another example of linear function. So, linear functional is I give you the 2 examples where the essential rule is that if you apply that over a vector, the vector will transform to a scalar quantity. And following these 2 independent rules that if we operate a plus B and ill get individual things and a scalar multiplication can come outside. So, now, I can understand what is linear functional mean. (Refer Slide Time: 07:05)



Now after understanding the linear functional I let me go back to our original problem, which is dual space. And by definition in the last class, we mentioned that dual space is a collection of all linear functionals.

So, if I if a vector space is defined with a vector say alpha beta and so on. Then vector space is defined like this. So, the corresponding dual space will be containing some other vectors like this. Now we call this vectors and this case they are also forming a vector space, one can prove that I am not going to prove this here they are also forming a vectors this set of linear functionals also forming a vector space, but they have few a different name. If I say this is a vectors sometimes they call this is a co vector, also some technical term associated with this dual space. That dual space is a homomorphism of the space vector space with R. So; that means, dual space can operate over a vector space as a result I am getting a scalar quantity in terms of R. I am not going to that integracy mathematical integracy just here my goal is to understand exactly what is going on.

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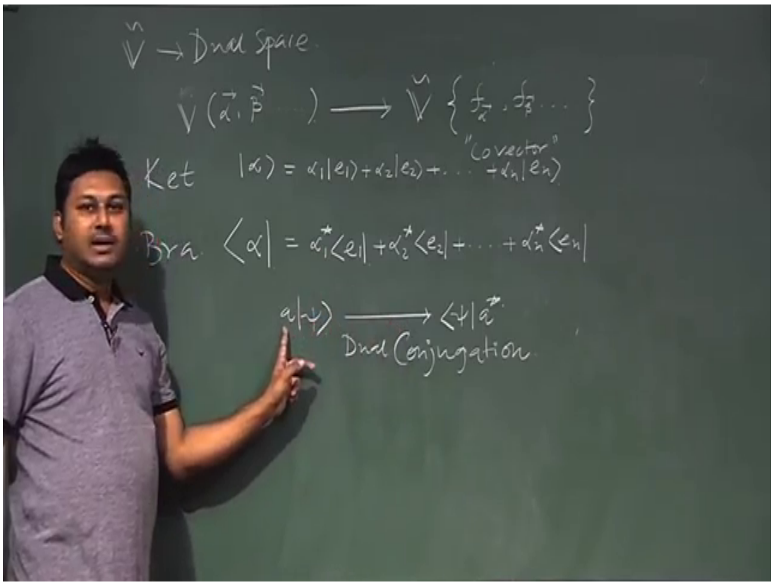

So, now the question is very important question, how to represent a co vector in dual space. For example, say alpha is a vector, that I defined in my vector space and which is alpha 1 e 1 plus alpha 2 e 2 say alpha n en. This is the representation I want to make. So, this representation is with some basis vector and this representation in vector space V.

So, now the question is what will be the counter part of these things here. So, which is important the counterpart look something like this. Alpha now I change this notation this notation is called ket notation. And this notation is called bra notation. So, in bra notation what we are doing essentially I am rotating these things. So, this is now in this direction I am rotating these things, not only that in the right hand side there will be a change what kind of change will be there alpha 1.

Now became alpha star if alpha is a real quantity then alpha 1 and alpha 1 star will be the same thing, but since this is a component of the vector which is a scalar thing. So, scalar maybe this is coming from the scalar field. So, scalar field may be a real quantity or a complex quantity. So, for the time being we consider the general case that these are complex, if these are complex I can write it as this. So, it will invert it will also invert and so on.

So, this is the notation of a vector in dual space. So, this is the vector which is in normal space I am changing notation and I am saying this is a new notation. And this is not a vector are the co vectors I mentioned, but it is a vector element in dual space. So, now, if this is the case now ill going to operate this over, that when I operate these over that what happened that I will get something which is not a vector quantity rather scalar quantity, but you should remember that from a say psi. This is a vector in vector space if I go to it is dual space.

It will be converted as; psi a star I am going to this vector I am going to the dual space for corresponding vector and it will change like this with this things is called dual conjugation. In quantum mechanics we heavily use these things you should remember, that whenever you are making a dual conjugation; that means, you are not in a vector space. You are going to the corresponding dual space a exactly the same thing I am doing here I am just rotating this. So, I am rotating this to in dual space I am making this as this which is the thing, but this, but a is a scalar quantity. Now I define these things and now I operate this over that. When I operate this over that my things will look like something.

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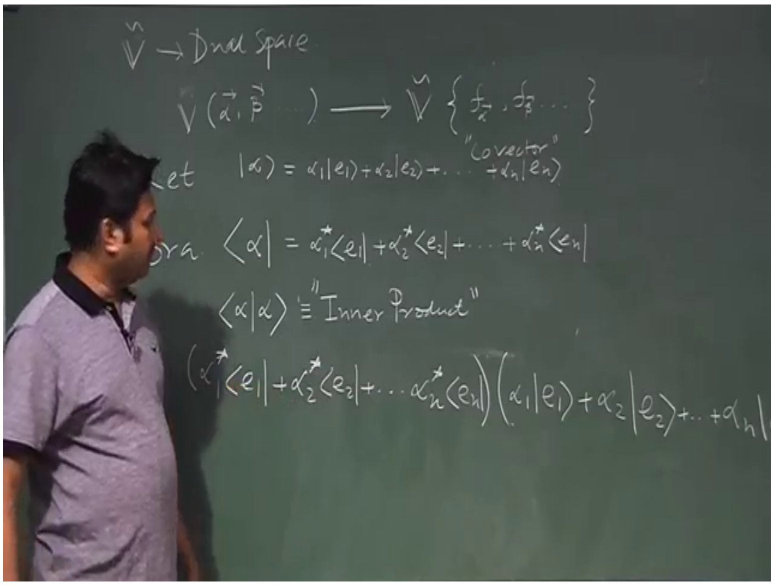

So, when I operate that over these and these. So, it will be looking like alpha, this is a notation of inner product. So, I am doing the inner product here it is inner product. I am making an inner product and I am operating this which is a like this is operating over that.

So, if this is the case then I will operate this intact stuff to here. If I do then what expression I will expect I am writing these and these in terms of this. So, in my left hand side it is alpha 1 star e 1 plus alpha 2 star e 2 and so on. Alpha n star en this which is this intacting, I will operate over my original vector original vector was decomposed with some basis vector like this. So, alpha 1 e 1 plus alpha 2 e 2 plus alpha n ens. So, these multiplied by these will be my inner product. And now term by term by term I will calculate.

So, let us do that here what will be in the my first term alpha 1 e 1 alpha star one e 1 alpha 1 e 1.

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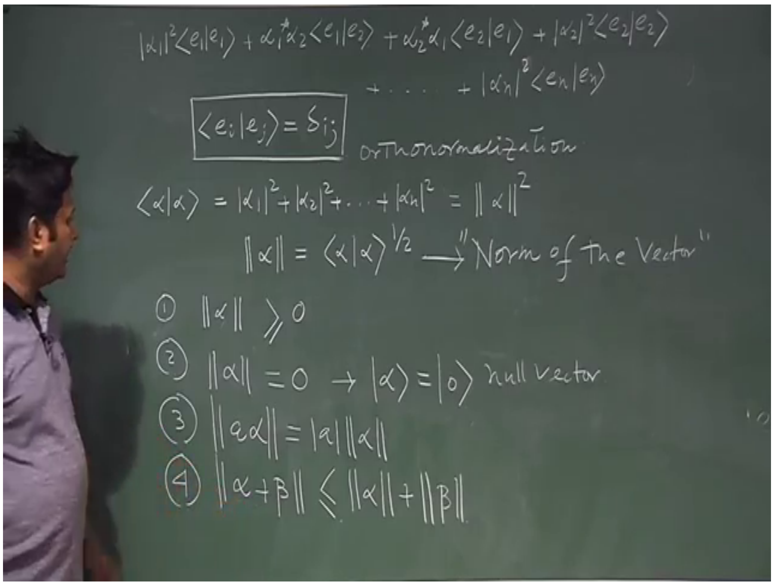

So, if I write this alpha 1 star and this quantity gives me mod of alpha 1 square with the term e 1 e 1. Then if I multiply this to the second term, I will have something alpha 1 star alpha 2 e 1 e 2 first one. It the first term is e 1 e 1 second term is e 1 e 2 in the same way I can find another term which is alpha 2 star alpha 1, which is e 2 e 1 and so on and so on. The point is, I have one term which is e 1 e 1 I have a cross term like this also different kind of cross term, I find and not only that also I find something like alpha 2 star e 2 e 2 this term and so on. Finally, I will have alpha n star alpha n square mod of alpha n square en now after that I will put some condition over that.

So, e 1 e 2 is chosen in such a way that they are orthonormal to each other; that means, e 1 ei in general ej is data I j, this is the condition I want to impose, I took the basis in such a way there is a relationship between the basis. And when I apply the relationship to the basis then I am saying that for this basis this is delta I j so; that means, they are forming a orthogonality not orthogonality it is orthonormal, when a 1 I is I j is I then I have one. So, this property is called orthonormal a orthonormalization for the timing just note this term and will again explain this. So, if I apply this to here all the cross term will going vanish. So, essentially what we are getting essentially, what we are getting left hand side is alpha right hand side is what alpha 1 square plus alpha 2 square this quantity.

Now, I know this quantity this is nothing, but the square of the component of the alpha square of the component of the alpha. So, this quantity I can write it as alpha square. This alpha is 2 mod alpha. Which is the root over of this quantity is simply called norm the vector. Norm of the vector, which you know it is not a very new thing again a vector, if you just realize a vector in just I j k form what is the norm of that.

So, a vector is even in I j k form the norm is the coefficient square coefficient square plus coefficient square and root over of that. So, here I am doing the same thing coefficient square plus coefficient square plus coefficient square whatever the value I am getting I am making a root over of that I am getting what is called norm this is a very important. So, next it is the conjugation of the same vector with the same vector, and this has some this has some properties this norm has some properties.

So, what kind of properties. So, let me discuss right here this nor has some property. So, first I am defining a vector. Then I go to the dual space then I operate with the inner product notation then I find a vector from the vector I am finding some scalar quantity now I am operating the same vector with the same vector then as a result I am getting something which I called norm up to this is fine. Now what I am doing next is to find out the properties of this. So, you should remember this quantity is always greater than 0 first of all this is a scalar no need to ex explain why it is scalar because already I have shown that this is the scalar quantity. So, this has to be scalar not only that this quantity had to be greater equal to 0 first property. So, norm will always be greater equal to 0 norm of a vector is always a greater than 0 second property if it is 0.

Say I am saying greater equal to 0. So, there is a possibility that it will be equal to 0 if this things is equal to 0, then I must say that these vector is a null vector this vector is a null vector. So, this is a second property. And third property a if I multiply some a with the vector and then try to find out the norm of this things.

If I multiply some scalar and with the vector and then try to find out the norm of the vector, then it is essentially the norm of the vector multiplied by the mode of a which also you can prove very easily. This is the fourth third property we have fourth property which is interesting that I have alpha plus beta 2 vector, I add and then try to find out what is the norm of this 2 vector, which is which is always following this rule it is less than equal to individual norm of this 2 vectors. Now this is also not a very new thing to you this is also not a very new thing to you because this follows simply the triangular inequality.

So, how it is following the triangular inequality, let me quickly show then you will realize exactly what is going on it is nothing very tough thing to understand at that moment. Say this is my vector alpha and this is say my vector beta with my vector sign.

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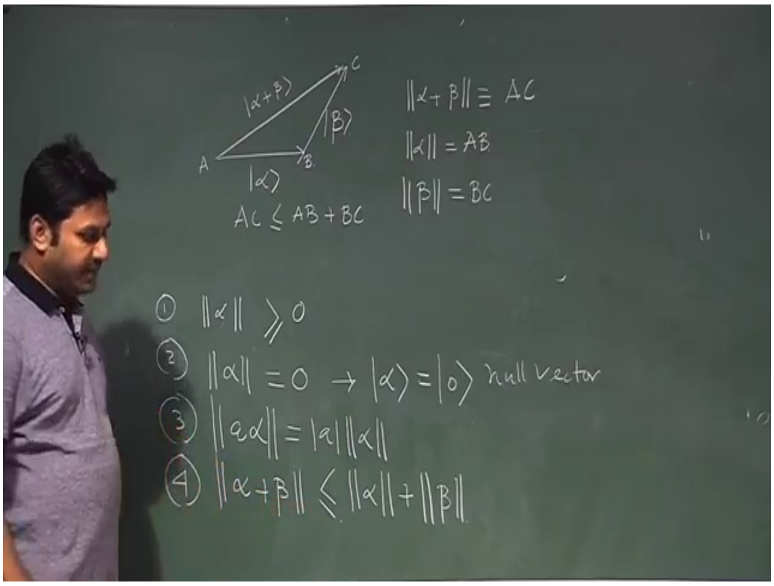

So, if I join that it will be the vector alpha plus beta. So, what is the norm means norm means, this length of this of this side. So, norm means is basically if I say A B C the norm of alpha plus beta is nothing, but AC in a similar way norm of alpha is nothing, but AB and norm of beta is nothing, but the length BC. So, from a triangle we know that this always follows AC is always less equal to AB plus BC which is essentially the fourth rule which is essentially the fourth rule. So; that means, the norms are following certain rules sometime there is a name on that since the norms is following this few rules if a norm js following this 2.

If a vector space have vector element this is a vector elements, I can calculate the norm of that vector elements with this formalism and as a result what I am find that some kind of scalar property and this scalar property the scalar thing has following this rules, whatever the rules I just show if they are following that 4 rules then they are forming a space also individual space.

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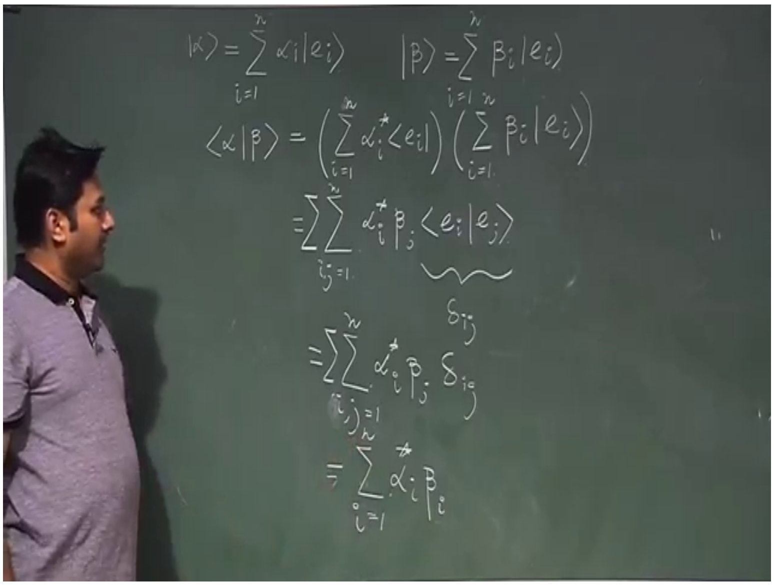

Which is called norm space norm space. So, norm space is nothing, but a space where the norm is following that 4 role for the timing, you just note that if you note that is sufficient. So, after the calculating the norm, another thing is important which is I represent alpha as a now I am using the summation sign because now we should be familiar with these things I am not going to write interest of. So, this summation is nothing, but the expansion of these things, I believe all of you are aware of this summation sign. So, alpha I am writing this with the summation sign, I am also writing another beta another vector beta which will be defined in way that beta i ei, i equal to 1 by n. This is another vector whose components are different, but I am expanding the same vector with the same basis ei and ei.

So, now try to find out what is the value of alpha beta. I have already done the calculation for alpha. I am doing the same thing, but just changing another thing in just changing alpha to beta. So; that means, essentially I am doing this. So, what I will do alpha 2 n sorry this is the notations changing. This is in Kate notation now I am operating this over b. So, now, it is in bra notation. So, bra and Kate ill operate and this is my the inner product this is my the inner product. So, when I am changing that I need to change that also. So, if I change according to the rule, if you remember it will be alpha star ei now ei will also change because I am taking the covector multiplied by summation of I equal to 1 to n. And then beta I ei it will not going to change because I am operating this you should also note that when this mult this kind of multiplication is there I can say that this for this kind of multiplication, when I conjugate these things always you should remember that this index you need to change.

So, in totality I have i j which is running from 1 to n if I write this as a index I. So, it will be alpha I star then index j, I tried bj and then this with operate with this. So, ei ej this is the total thing I have after operating that you should remember that when I multiply I have already taken care of this issue by changing the index that I have already mentioned. So, now, what is this quantity this conseq quantity, I have already wrote just before that this is delta ij if these things are orthonormal to each other these things are orthonormal to each other. Then I can write this is delta i, j if this is delta i, j then my equation is i j alpha I star bj delta ij. If I run the now there are 2 summations, it should be one to n one is over j and 1 is over I. So, if I have the 2 summations simultaneously I can also put another summation sign, but ij.

Let me put it another summation sign to remove the confusion that one summation whenever I do the summation for one I. So, it will be equal to 1, when i is equal to z and other case it will not going to because if I do the same thing for j when j is equal to i, then this thing will be one and other case it will not. So, what value I will eventually get ill eventually get this value, alpha i star beta this will be my final result. Because I am doing the summation when I am doing the summation delta i j will be converted to 1 when i is equal to j. So, only these components will be remaining other components will not be remaining and ill be ending up with this expression, but the question is. So, I have the expression this equal to this this multiplied by this I show that the value.

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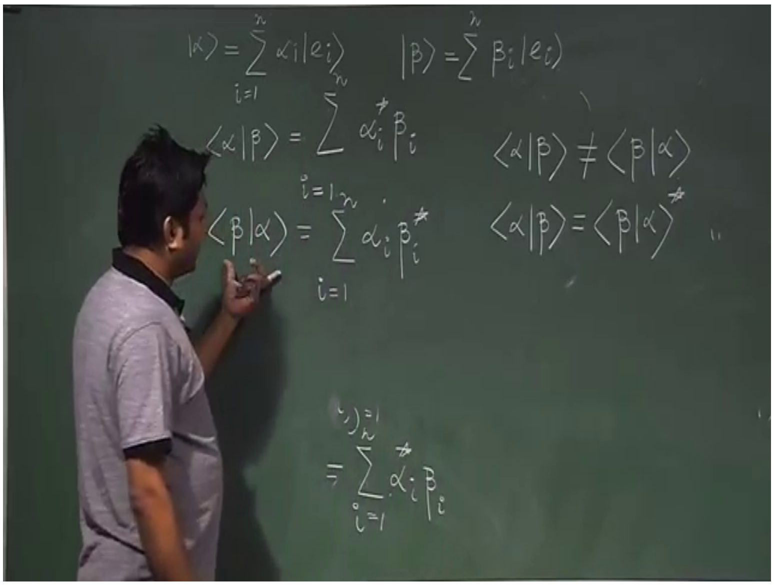

I am getting is summation of i equal to 1 to n alpha i star beta i. I am having this now the question is the question is please take it as a regular basis home work a homework exercise, beta alpha, I will just change this ordering for the first case I operate alpha over b and I am having some quantity like this. Now I am rotating these things I am now taking beta as my co vector and as I am upholding this alpha also I am having some kind of scalar.

If you do the calculation you will find these things. So; that means, essentially these 2 things are not same; that means, alpha beta is not equal to beta alpha what is the relationship between alpha beta. This is true only when they are real if beta and alpha are real quantity then both the things are same you should know that normally, that is the case a dot b is equal to b dot a only when the components of the vector of a and components of vector of b are real. If that is not the case, then the actual thing is alpha b is equal to beta alpha with a star. If I make a whole star over there this star will be gone another star will be there.

So, this is equal to this. So, this is the relationship between alpha and beta when they are not the real component rather they are different components, I mean they are complex components. So, now, we understand with the bracket notation what is going on. So, quickly I need to show what is going on in another notation n tuple notation.

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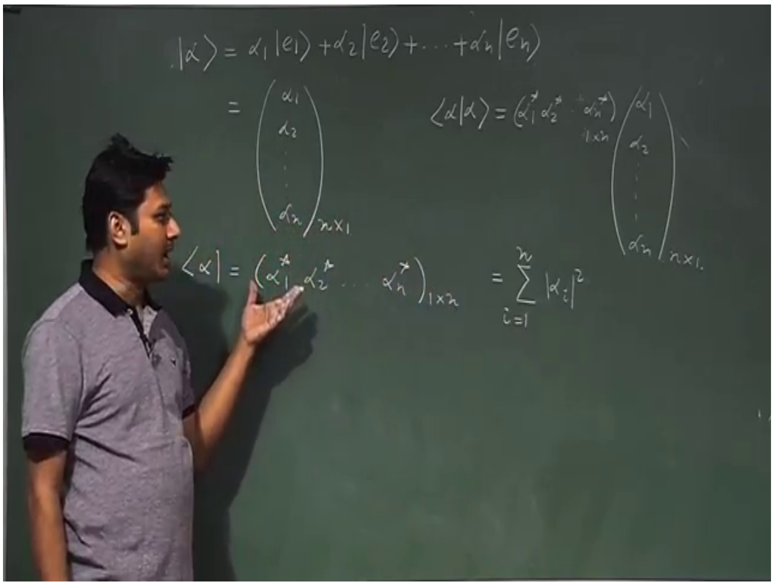

So, let me do that quickly. So, also I can write it as alpha 1 e 1 plus alpha 2 e 2 plus alpha in en. So, in n tuple notation this is alpha 1 alpha 2 alpha n. This is n by 1 vector. So, what happened if I rotate these things what will be the feature of this notation. So, the feature of this notation will be something like this alpha star alpha 2 star alpha n star it will be 1 by n. So, in this is the notation n tuple notation when I am making the alpha when taking the alpha as a vector quantity from a vector space, but if when I go to the dual space something need to be different and that is whatever the structure I have which is a column vector. Now became a row vector not only that my coefficient has to be a there will be a complex conjugate. So, the coefficient has to be complete conjugate because of this dual conjugation.

You may remember the dual conjugation is as I mentioned that when I make a dual conjugation it will be rotating like this with a star; that means, when I go to my dual space these are the elements fitting here it has to be star. So, this is the case. Now if we exploiting this if you calculate alpha, alpha, which will be the multiplication of this with this then eventually this is a in one cross n vector. This is n cross one vector. So, if I multiply these 2 things these matrix we will have something which is a scalar. And I can write this as mod of the same result that we derive in earlier bracket notation, if you do the same thing for alpha beta you will getting the similar kind of result. So, with that ill like to conclude. So, today we are now learning in a very important process.

We are now in a process of learning a something very important which is dual space and in dual space. We find that the vector can be rotate and what the formulation will be and how the inner product is constructed you should know these kind of things in detail. So, that is why todays class, is important in the next class; however, we will start from here only. And will show some example and some elementary example and so how the things is working. So, with that, let me conclude.

So, see you in the next class. In the next class we start with this inner product definition, and then we do some kind of problems and show some kind of inequality important inequality which is called the Schwarz inequality how this inequality is working and all these things we will discuss in the next class. So, with that let me conclude today. So, see you in the next class and thank you for your kind attention.