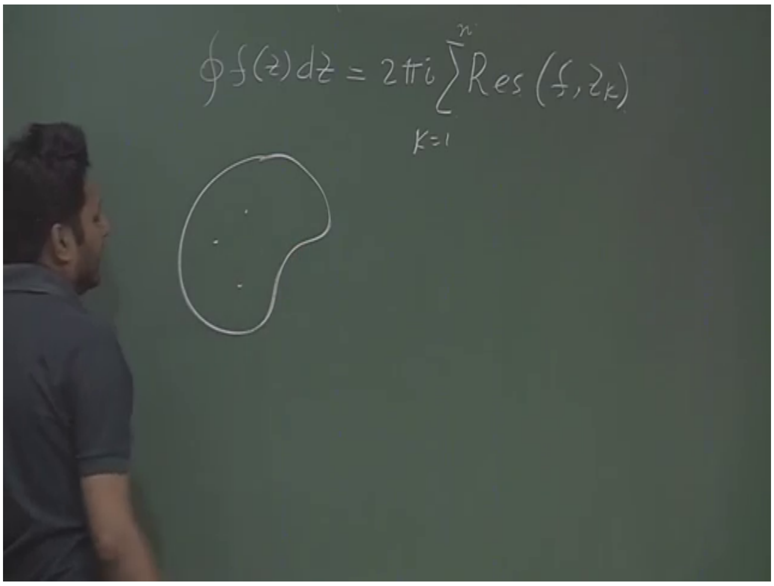
Mathematical Methods in Physics-I

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 59
Real Integration Using Cauchy's Residue Theorem (Contd.)

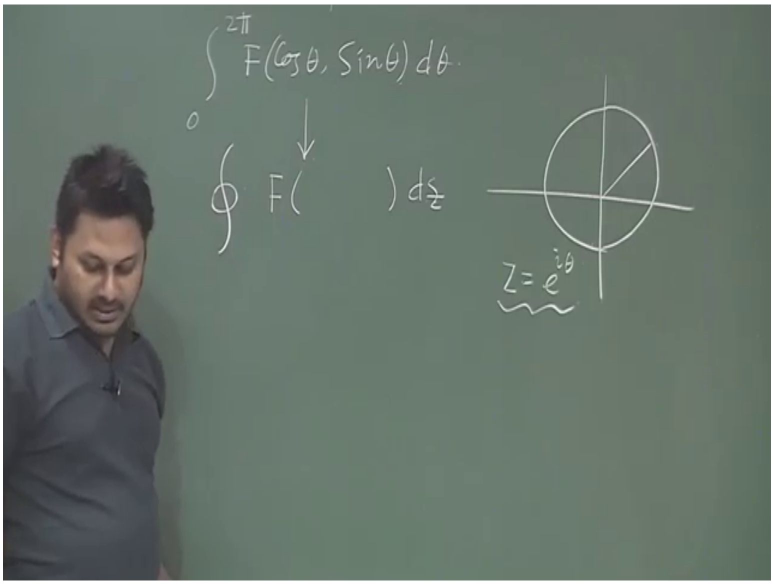
Welcome back student to the next class of complex analysis. We are almost at the end part of our course. So, we studied the Cauchy’s residue theorem which suggests I can do this integration with this recipe.

(Refer Slide Time: 00:30)



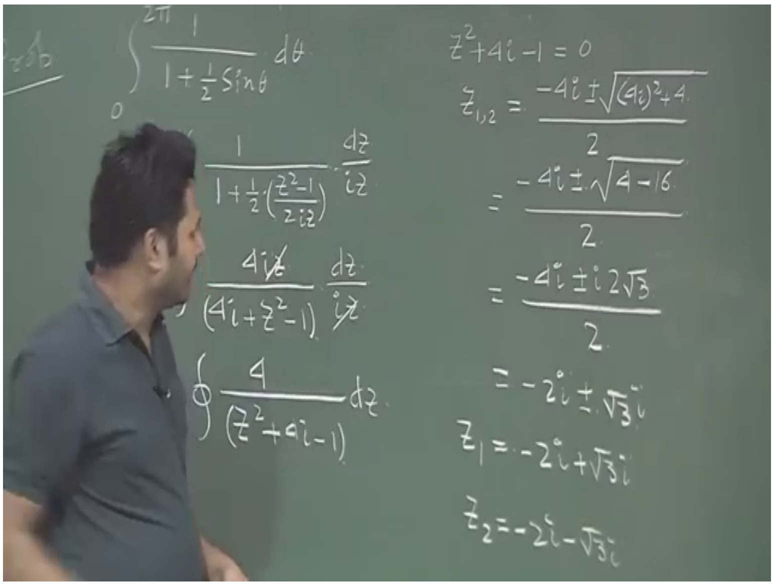
So, if we have some region where we have some finite singularity of this function f z, then I can write this total integration of this f z as 2 pi i multiplied by the residue and how the residue will be calculated and all these things. We have discuses elaborately, but interesting thing is that even we can using this Cauchy’s residue theorem we can even calculate the real integration.

(Refer Slide Time: 01:27)



So, one and example was given that if we have some form like this, then you can convert this function to as a complex function and then I can evaluate this integration by taking some suitable contour. Here the contour was taken as the unit circle So that my z become e to the power of i theta. That was the transformation we make. And also we need to change this trigonometry function cos theta and sin theta in terms of z and we did it. So, let us continue with that let us continue with that this is the concept we will going to use to solve this problem.

(Refer Slide Time: 02:29)



So, let us take another problem. So, the problem is given as 0 to 2 pi 1 divided by 1 plus half sin theta d theta. This is integration let us check, whether this is this is integration we need to evolve. We need to calculate this integration. And now this is real integration you can see the limited 0 to 2 pi everything is the real here.

So, we need to consider this integrity to convert this integration to the complex integration. In order to do that we need to take our standard contour here, which is the unit circle with z equal to e to the power of i theta, then from here d z was something d z by i z was d theta that we have calculated. Now we need to change this sin theta and d theta in terms of z, then we have an have an function this function will be in terms of z. So, sin theta is nothing but z minus 1 by z divided by 2 i. If I take z equal to e to the power i theta, then it comes as z square minus 1 divided by 2 i.

So, that is the transformation I need to make. So, d theta we will change as d z i z this part, and sin theta I will change this. So, if I do that then this integration will be converted to the contour integration, where the function is 1 plus half sin theta is z square minus 1 divided by 2 i. D theta will be replaced as d z by i z this. So, it will be a calc lengthy the calculation. So, let me erase because I do not need that I will going to use this part of the board for the calculation, this I do not have much space here to calculate. So, then I need to simplify a bit. It will be 4 i, divided by 4 i plus z square minus 1 d of z divided by i of z.

So, here it will be 2 i z. So, sin theta. So, let me write it once again sin theta was z minus 1 by z divided by 2 i. So, it was z square minus 1 divided by 2 i z. So, it should be 2 i z then it will be 4 i z. I should be careful about these things where, otherwise my result will be completely different. Next is I need to now factorise this quantities. So, z i z I seems to be cancel out. So, I have something like 4 my function become simple function which is 4 plus z square plus 4 i minus 1 d z.

So, this is by function. So, this is the function which is the real function in terms of theta. Now I convert this theta in terms of z, I convert sin theta in terms of z as well and as a result I am getting something which is completely over z, and I have the contour integration and this contour this known to me also use of unit circle.

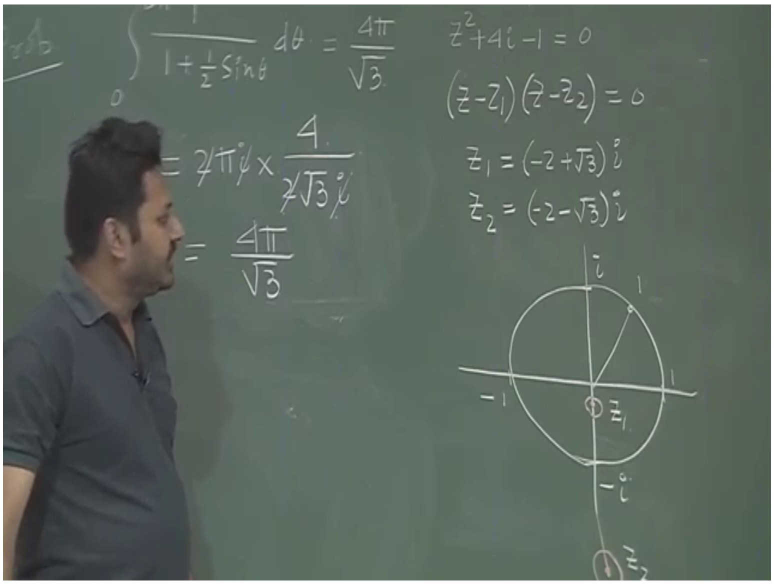
So now on the next thing is straightforward I need to just calculate the residue of this function and used just Cauchy’s residue theorem 2pi i multiplied by residue that is all.

So, in order to do that residue will be at that point. So, it is a quadratic equation we just need to solve the quadratic equation 4 i minus 1 is equal to 0. So, my z 1 2 the solutions 2 solutions 1 and 2 will be minus of d; so minus of 4 i plus minus root over of b square. So, 4 i square, then minus 4 a see. So, this is a minus. So, we have a plus 4 divided by 2 into a is 1 so just two. Then let me sim try to simplify it 4 i plus minus, this quantity seems to be 4 minus 16 divided by 2.

So, this quantity is minus of 4 i plus minus of 4 minus 16 is minus of 12 root over of minus of 12. So, we should have 1 i and 12 root over of 12 is 4 into 3. So, 2 root 3 variable. So, it will be 2 root three, and whole divided by 2 fine. So now, I can simplify more this. So, I will eventually have minus of 2 i plus minus of root over of 3. So, my z 1 is minus of 2 i plus root over of 3 i and my z 2 is minus of 2 i minus of root 3 i.

So, let me write it here exactly what I am getting. So, this function now factorized and if I factorized is as z minus z one, multiplied by z minus z 2 so that I precisely know at which point the singularity arrives is equal to 0 when z 1 is calculate to be this.

(Refer Slide Time: 09:37)



So, it is seems to be minus of 2 plus root 3 i. And z 2 is minus of 2 minus of root 3 i. So, we have 2 points where in the singularity arises. And we need to find out which of these 2 points a inside the circle, because my contour is here. This is the contour I have with unit circle. So, this value is 1, this value is 1, this value is minus 1, this value is i and this value is minus i.

So now if I look to these 2 z 1 and z 2 you can readily find that minus 2 plus root 3 i. This quantity is somewhere here which is root 3 is around 1.7 or something. So that means, minus 2 plus 1.7 something, so minus of something. So, somewhere here it is sitting. On the other hand minus 2 minus root 3 i this quantity is greater than 1, so more of these things greater than 1. Obviously, it will be outside the circle somewhere here. So that means, my z 1 seems to be inside the circle, and z 2 point which where we have the other singularity is outside the region of this circle.

So, whatever the region is there z 2 is outside. If z 2 is outside then you should bother about the z 2 is residue now I can find. So, this quantity this quantity should have the value 2pi i residue of the function at z equal to 1 point, that will be my result. 2pi i residue of function to z i z I means z 1, means this one z 1 which is minus 2 plus root 3 of i.

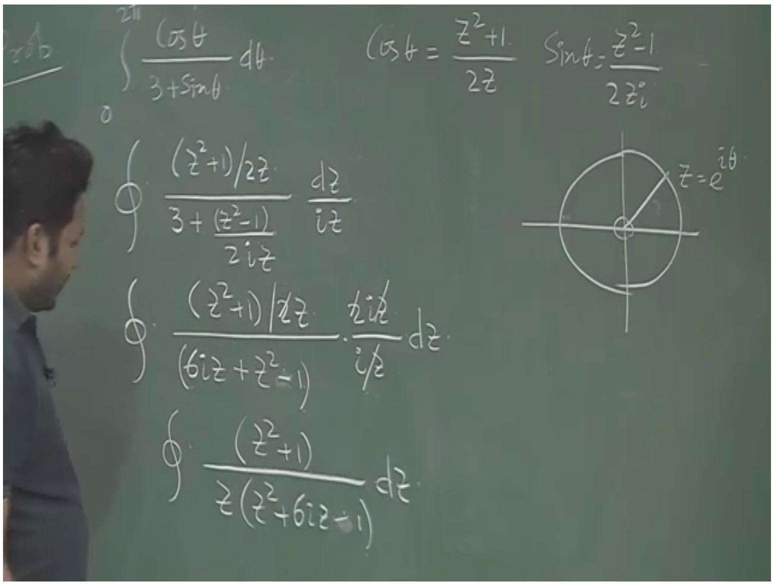
So now I will just put this residue. So, residue will be nothing but residue of f at z 1 is limit z tends to z 1 z minus z 1 multiplied by the function. My entire function is this. This portion I have already written in z 1 and z 2 in this way. So, this function will be 4 divided by z minus z 1 z minus z 2. So, this will be my residue of the function at the point z 1. So, this point z minus z 1 z minus z 1 will cancel out. Now if I go with the limit if I put the limit then it will be 4 divided by z 1 minus z 2, that will be my answer. Z 1 and z 2 we have already calculated here.

So, if I now say z 1 minus z z 1 minus z 2 it will be just 2 this things minus this. So, it seems to be 4 divided by this minus this it will be 2 of root over of 3 multiplied by i. I have this quantity. So, this is the residue. So, actually I have calculate. So, what should be the final value of the integration? Then I need to multiply that by 2 pi i. So, I have calculate the residue this is the residue of the function at z 1 point, that I calculate. And the result will be 2pi i multiplied by the residue that mean this quantity which is 4 divided by 2 root over of 3 i. So, this I will going to cancel out, 2 2 going to be cancel out, we will have 4 pi divided by root over 3.

So, this is the result of this integration. So, by the value of this integration what I evaluate is use 4 pi root over 3. This is a real integration and the result that is coming is also real. So, that is consistent with the fact. And we can say that this integration we just transferred to into the complex form and then calculate the residue and use Cauchy's residue theorem and get the result.

So, let us go to another example before going to the next just to another of similar kind. So, the next example that is in your hand is something like this. 0 to 2 pi cos theta divided by 3 plus sine theta d theta.

(Refer Slide Time: 15:42)



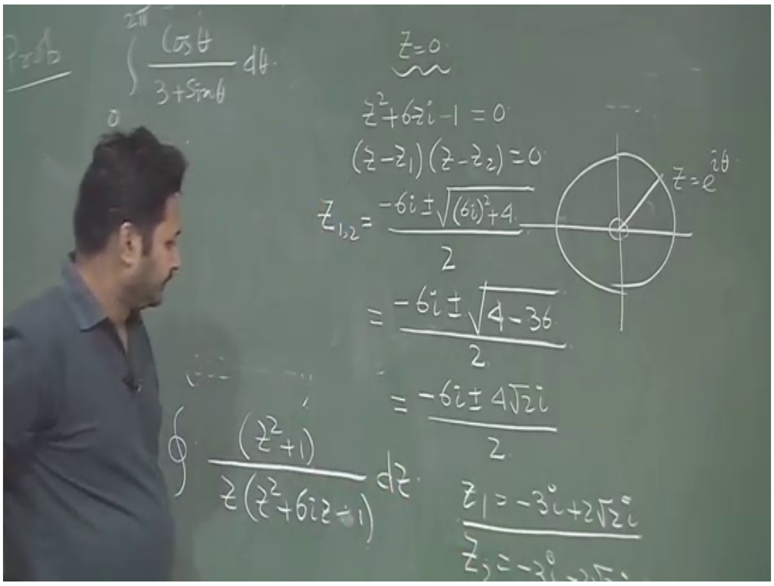
So, again I need to convert this in terms of z. So, cos theta is z square plus 1 divided by 2 z and sin theta z square minus 1 divided by 2 z 1. So, I just convert this to and my contour is the standard contour I always used for this problem as I mentioned this is unit circle. This is equal to 0 I have the unit circle. So, z is e to the power i theta is changing to 0 to 2 pi, and this is the contour i I going to use.

So, this is z square plus 1 divided by 2 of z, 3 plus sin is z square minus 1 divided by 2 i z plus sin theta I will going to replace which is d z divided by i z 2 z, I not 2 i 2 z, I and 2 2 z. So, it will be z square plus 1 divided by 2 z divided by 6 i z and then plus z square plus 1. And this 2 i z will be sitting here divided by i z and d z.

So, few terms will going to cancel. So, for example, this 2 i z and here i z is there. So, i z I z will cancel out. This 2 and this 2 also cancel out. So, what I finally, have is something like this. The function is z square plus 1 divided by z multiplied by z square plus 6 i z plus 1 d z. This is the function I finally, have in my hand

So, let us check once again everything one sine is there, it should be minus i can see this negative sine is sitting so this minus. So, 6 i z plus z square minus 1 and other things are there. So, again this function should have some kind of singularity. So, let us check where these singularities are. So obviously, the singularity is at z equal to 0 first thing is z equal to. So, let me erase this part. So, the singularity at z equal to 0 we have a singularity that is for sure.

(Refer Slide Time: 19:19)

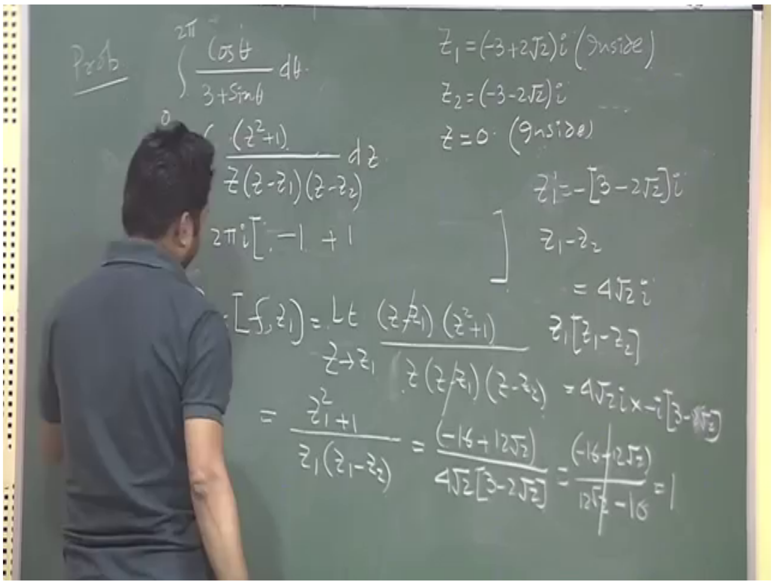


Another singularity I need to write it as equal to. So, z square plus 6 z i minus 1 is 0. So, the solution if I write is at z minus z 1 and z minus z 2 equal to 0, then z 1 2 at before minus of b, so minus of 6 i plus minus root over of b square. That mean 6 i square minus 4 a c. So, this is minus or plus 4 a is one see is one. So, it is just 4 divided by 2.

So, this quantity is minus 6 i plus minus of root over of 4 minus 36 divided by 2. So, it is minus of 6 i plus minus of 4 minus 36 seems to 32. So, 32 is 16 into 2. So, we should have 4 root to i divided by 2. So, z 1 and z 2 is something z 1 seems to be minus of 3 i plus 2 root 2 i and z 2 is minus of 3 i minus 2 root 2 i. These 2 points I have as my z 1 and z 2 one is root 3 plus 2 root 2 i and another is the complex conjugate of this quantity with a just minus 3 i plus 2 root this things.

So now I need to put it here to find out what is I mean which point is outside. So, one points.

(Refer Slide Time: 21:46)



So, let us me write it once again here (Refer Time: 21:47). So, z 1 is simplify it minus 3 plus 2 root 2 i, and z 2 is minus of 3 minus of 2 root 2 i. So, if you see these 2 points. You can really find at minus 3 plus 2 root 2 i, this quantity is inside. Because 2 root 2 is 2 into 7 1.7 4. So, minus 3 is should be around here somewhere very close to 0. And this quantity is minus both the things a minus. So, obvious it is way out to the so some where here negative of this quantity is below. So, only this is insides. So, z 1 is inside this things is outside.

So, the residue is calculated only into 2 points. So, this are inside z equal to 0 is another residue which is also inside. So, this is inside z is 0 is also inside. So, this function I will be write as z square plus 1 divided by z into z minus z 1 z minus z 2 d z, z square. So, I write this in this particular form. Now what I will do I will try to find out the solution of these things, which is 2pi i multiplied of residue calculate of this function at 0 point plus residue calculate of this function at z 1 point, because z 1 is inside.

I need to calculate these to residue at this and then multiply 2pi i will get a result. So, let erase this parts. So, first I need to calculate residue at 0. So, if I do residue of this function at 0 point this limit z equal to 0. Z multiplied by the entire function which is z square plus 1 divided by z into z minus z 1 z minus z 2. This z z it will cancel out and then I can put the limit if I put the limit it will be 1 divided by this will be 0 it is 1, this will be 0 this will be 0 z 1 into z 2 z 1 into z 2 if I calculate.

Then let me erase this. So, let me erase this point. So, it is not quite. So, z 1 and z 2 is so that means, I need to calculate this points if I take. So, it is 3 minus 2 root 2 minus i this is z 1 multiplied by 3, minus of 3 plus 2 root 2 of i. This minus and this minus will be plus i, i will be minus. So, it will be 3 square a plus b into a minus b it is 3 square minus 3 square minus the square 2 root 2 square. So, which is 9 minus 8. So, it is seems to be minus 1.

So, the residue that I am getting here for z equal to 0 seems to be minus 1. So, this residue I calculate and this residue this residue is coming out to be minus 1. Now next about this residue, so let me erase this. So, residue of this function at z equal to 1. So, residue of function z equal to 1 is limit z tends to z 1 z minus z 1 into z square plus 1 divided by z into z minus z 1 into z minus z 2. So, z minus z 1 cancel out. So now, if I would the limit I need to calculate only z square plus 1. So, if I put z 1 square plus 1 this is 1. And down stairs it will be z 1 multiplied by z 1 minus z 2 i need to calculate this quantity and to calculate this quantity.

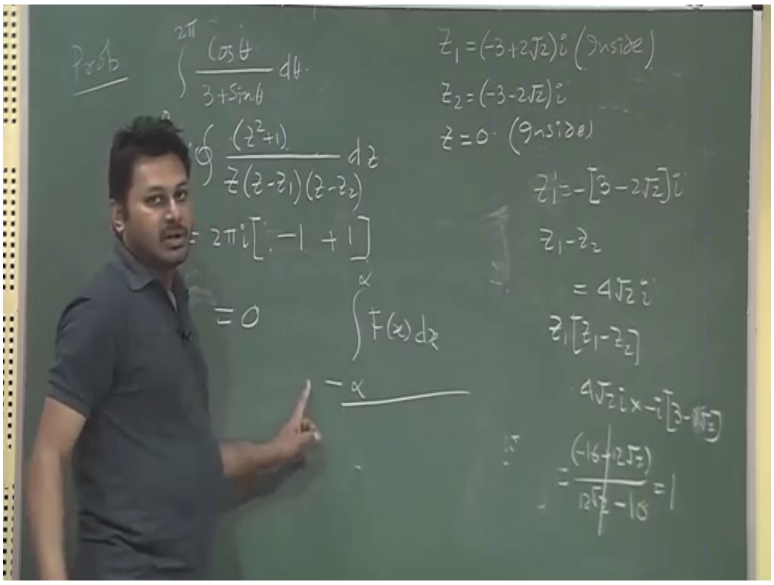
So, let us calculate here. So, z 1 square is how much? Z 1 because minus 3 minus of 3 minus 2 root 2 i. So, z 1 square is minus 3 square minus 3 minus 3 minus 2 root 2 square of that, so minus of 9 minus 3 into 2 12 root 2 and then plus 8. And I will have then minus of 17 minus 12 root 2.

So, z square plus 1, if I put z square plus 1 then this quantity z square plus 1 I need to make a plus 1 here. It seems to be minus of 16 plus 12 root 2 i am getting this point. Now z 1 minus z 2 multiplied by z 1. So, z 1 is z 1 is that. So, minus z 1 is minus of 3 minus plus minus 2 root 2 i. And z 1 minus z 2 is this minus this one make this minus this, I will have I will have 4 root to i.

So, once I have this minus this 3 will cancel out only this and 4 root i. Now if I put it here z 1 is this quantity. So, it is. So, z 1 multiplied by z 1 minus z 2 is nothing but 4 root 2 i multiplied by minus of I multiplied by 3 minus 2 root 2 2 root 2. And if I put it here then I will have 4 root 2 multiplied by 3 minus 2 root 2 ok.

Now, if I do this calculation it upstairs it is minus 16 plus 12 root 2 and down stairs if I multiply. Then it will be 4 into 3 it is 12 root 2 minus into 2 multiplied 4 into 8 multiplied by 2 it is 16. So, again it will cancel out and I am getting plus 1. So, the residue here whatever I calculate is plus 1. So, after doing all this calculation we find that one residue is minus 1 an another residue plus 1. And if I add these 2 things I will get the results 0.

(Refer Slide Time: 30:22)



So, integration of this thing after doing all these things I find it is 0. So, here where like to stop. In the next class I will go with this Cauchy's; this Cauchy's integral this residue theorem to find out more different kind of integration. So, the next day we will go with this integration minus infinity to infinity f z d z how do calculate this type of integration.

So, with this note let me conclude the class. So, see you in the next class.

Thank you for your attention.