Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 58
Real Integration Using Cauchy's Residue Theorem

So, welcome back student to the next class of complex analysis, where we dealing with the complex integration. And in the last class we solve many problems related to this particular form, which we called Cauchy's residue theorem.

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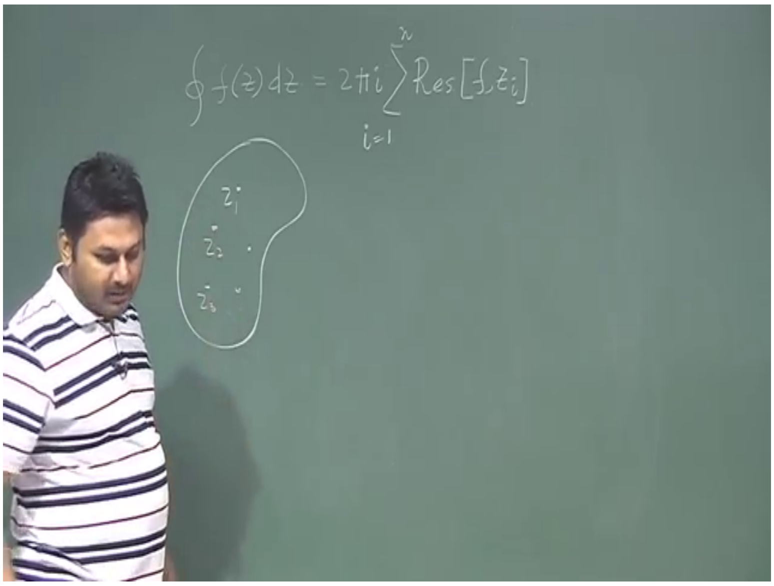

Let me remind you once again, that if complex function is given you can find out the closed integral of this complex function by exploiting this result. Provided the function is defined in this region where some finites singularity are there. In this singularities we calculate the residue, and once we calculate the residue then just add whatever the residue we have in this region.

And then calculate so that was the recipe we use. So, for to calculate the complex integration. So, today we will be going to show something very interesting.

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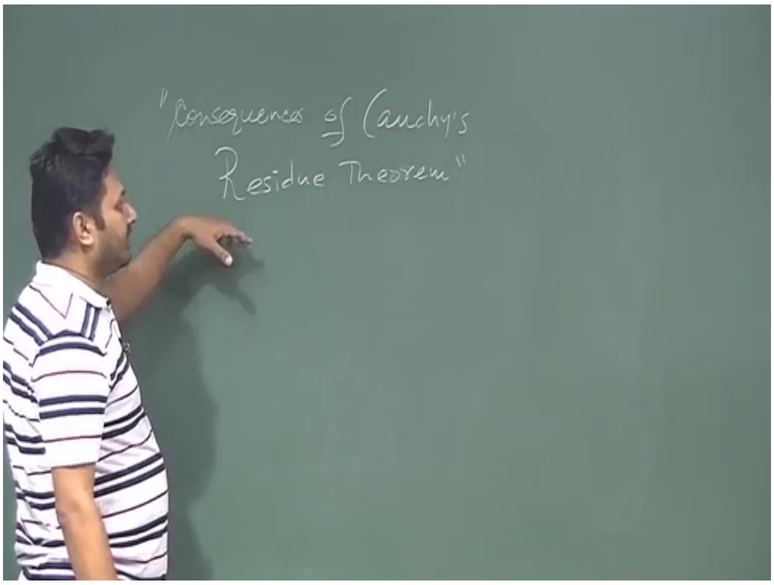

This is a consequence of say it is a consequence of Cauchy’s theorem Cauchy’s residue theorem. So, it is consequences of Cauchy’s residue theorem.

So, we will going to use the Cauchy’s residue theorem once again here, but the consequence means I can evaluate this things for real case.

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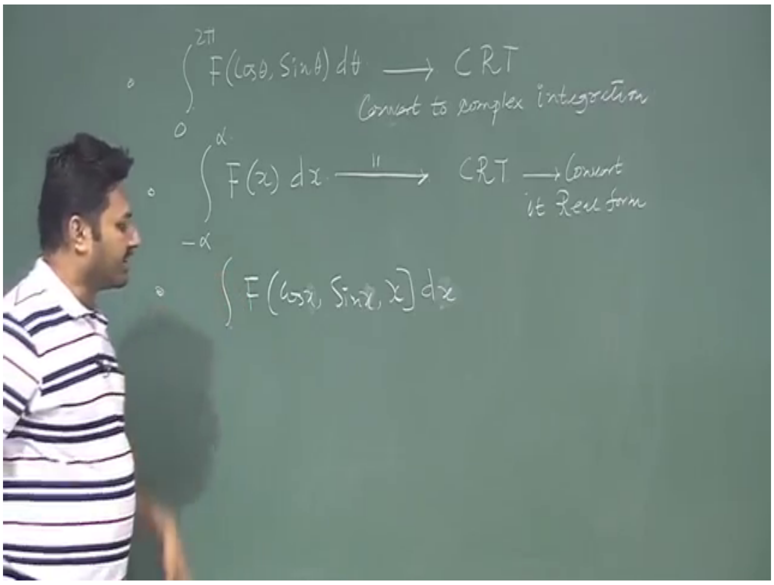

So, let me I am just writing what we try to do let me do that. So, if we have some function in this form, say 0 to 2 pi function of cos theta sin theta; that means, trigonometric function. We can evaluate this function or this integration and use the

Cauchy's residue theorem to find out the value of this. If the functional form is given like this, purely trigonometry. Also we can find this indefinite integrals where I have pure real function.

So, some pure real function is here with minus infinity to infinity. And I can convert this integration in the form of complex integration, and then use CRT to calculate the values. So, the first this is the function that is given this is a real integral, next thing this intermediate step I need to convert I need to make it convert to complex integration.

So, the given integration will be real, I will convert to the complex integration and then apply the Cauchy's residue theorem and get the result. In this case also I will convert this to complex integration apply Cauchy's residue theorem. And then get the result and sometimes we need to convert that to again to the real integration. Again because the result whatever we get maybe in terms of z, then we need to find out if it is in terms of z I need to take all the real value. Or if I take on the real value then I can get only the result corresponding to the real coefficient or real values.

So, sometimes you need to convert it in the real form once again. Apart from these 2 forms also we can have the form like the function can contain that trigonometry function as well as some algebraic function. I should write it as cos x sin x so that my variable remain x something like this. So, again I can convert these to some complex component and then I use the Cauchy's residue theorem to calculate than and return it back if necessary.

So, let us try to do few problems. So, I will start with this first. One type one then type 2 type 3 we will do gradually.

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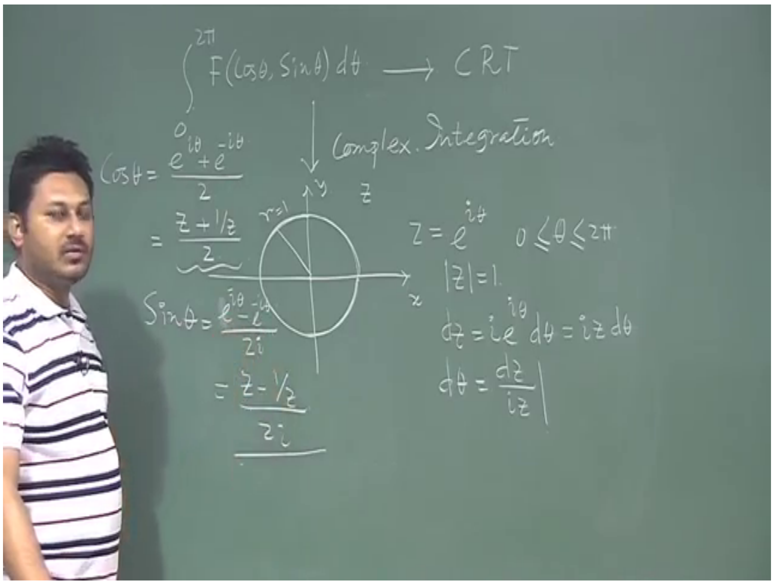

So, let us try to find out how to convert these things to a complex system. I first I need to convert these as a complex integration. The integration that is given to us is real, I need to convert these with a complex integration.

So, in order to do that first I need to calculate the in order to calculate the Cauch use the Cauchy's residue theorem, what I need to do the region. So, here for this particular case invariably we will take the region or the contour where we have the unit circle. So, this is my x this is my y. So, this is in z plane. So, I try to convert whatever the function is given to me in terms of complex integral in terms of complex variable. So, if I try to do that I need to do the integration over some region. So, I take the region invariably as unit circle.

So that means, the radius here is of 1. So, the z value will be essentially e to the power i theta so that mod of z is equal to 1. So, my variable that I am going to change introduce z should be related to theta as e to the power i theta. And since I am moving here to here, theta will vary from this limit. So, exactly I have 0 to 2 pi. So, no problem with this theta limit. Next thing is that how to change cos. And so, when I have z equal to e to the power i theta, then I have d z as i e to the power of i theta d theta, which is i z d theta. Or my d theta can be replaced as d z divided by i z.

So, I can replace this, I can replace the integration to the closed integral where my circle where my contour is given as a unit circle in complex plane up to that is, now I need to convert this cos theta and sin theta which is straight forward and trivial. Cos theta is nothing but e to the power of i theta plus e to the power of minus i theta divided by 2. So, in terms of z because I am changing everything in terms of z I need to change this integration to complex integration. So, my variable should not be theta, rather my variable will change to z. So now, here it is z plus 1 by z divided by 2 this is cos theta.

So, whatever, whenever I have cos theta I can replace this cos theta in terms of z. So, my variable theta is now replaced by z, and sin theta is z e to the power i theta minus e to the power of minus i theta divided by 2, 2 i. So that means, it will be z minus 1 by z divided by 2 i. So, we this transformation I can change my entire function whatever the function is given to me. I can change my entire function to my complex in terms of complex variables. So, this is entire recipe we will going to use to evaluate this things.

So, now if I try to do that let us take some example. So now my so, this things becomes.

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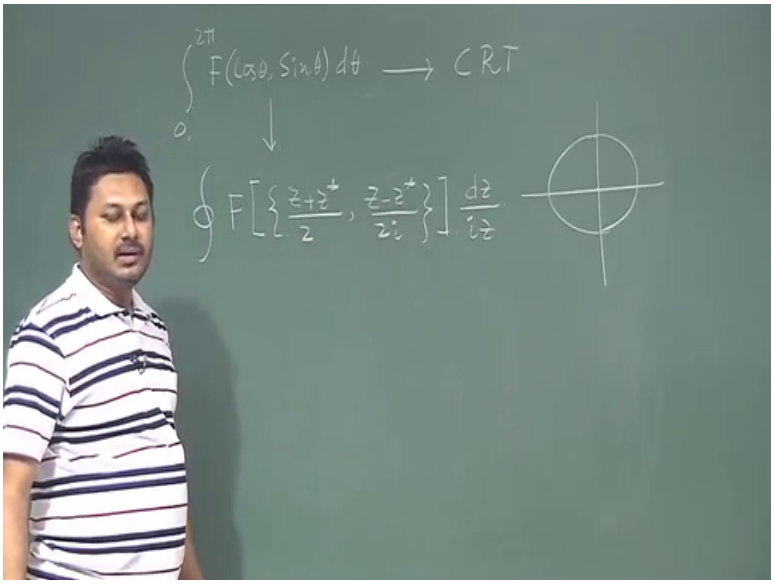

So, now this thing is now totally convert it I know what is the recipe of conversion. So, things become, this integration now become closed integration function of z plus z star by 2. Z minus z star by 2 i d of z by i z. So, the given function was the completely real function I convert this function as the complex function. And now I am capable of applying my Cauchy's residue theorem, and find out the residue of this whatever the function is given over the circle, over the region this. And then I am done, I am calculate, and then whatever the result I can get the result out of that.

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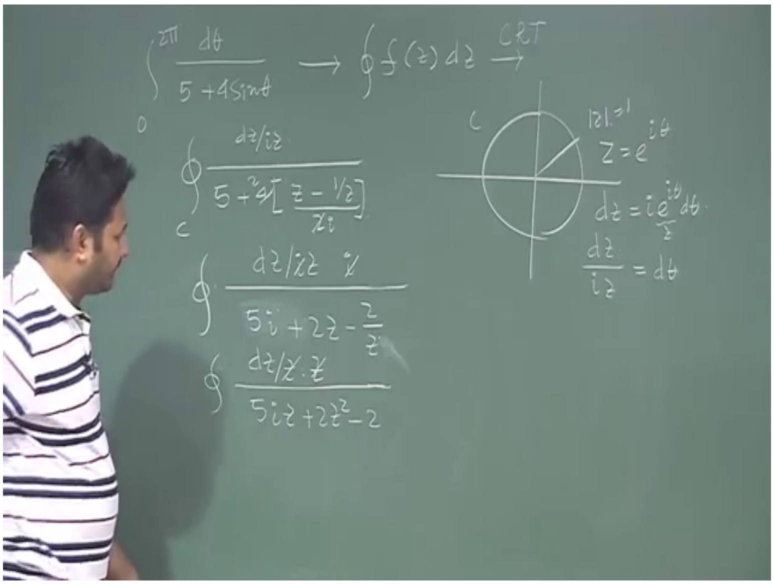

So, with this concept, let us try to find out some problem and check how it is working. The problem that is in our hand right now is something like this. 0 to 2 pi d theta 5 plus 4 sin theta I need to evaluate this integration. 0 to 2 pi 1 by 5 plus 4 sin theta.

So, may recipe is that I need to convert to as a function of z d z, and then apply the Cauchy's residue theorem. So, let us do that. So, I will change that. So, my region in very invariably I can take this region. So, I want to calculate this integration with unit radius with z equal to e to the power of i theta. So now, I if I put it here d theta d theta in place of d theta I should put d z divided by i z because from here d z is equal to i e to the power of i theta d theta. So, d z divided by i z this is z is equal to d theta. So, first I put this next I need to change this. So, 5 plus 4 sin theta I should write in this way z minus 1 by z divided by 2 i.

So, first I need to evaluate this function and then with the close integral, where the c is given as my unit circle. So, let us evaluate this. So, my function. So, it is 5 plus this. So, I will have this as 10 i plus 4 z minus 4 divided by z this portion with divided by 2 i. So, it is better directly to do it here. There is no point that separately I can calculate.

So, it is d z divided by i z divided by 10 i plus 10 i divided by anyway this 2 will be cut to the 4. So, I can I should not write. So, this 2 and this 4 will be cancel out. So, things will be easy. So, I have shift 5 i plus 2 z minus 2 divided by z square. This is in my hand and one I will be multiplied here.

So, 5 i plus 2 z minus 2 z square and the i is divided entirely divided. So, I will take I upsatir. So that this I and this I will now cancel out. So, next over d z d z is already here. So, we have d z divided by z. Then 5 i this is 2 by z not z square just 2 by z. So, it is 5 i z plus 2 z square minus 2 and this z will go upstairs. So, I will haves z here. So, d z divided by z multiplied by z. So, this z and this z will cancel out again. So, I will have something like this, let me erase this part.

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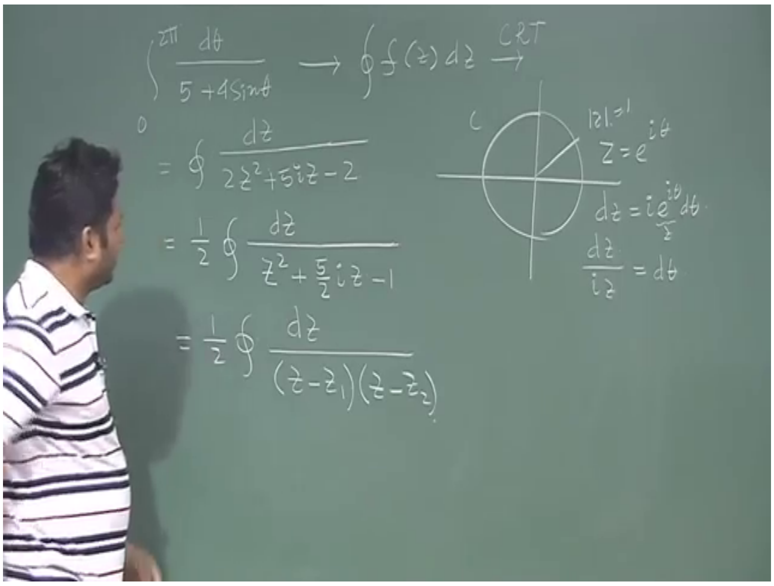

So, finally, I am getting d z divided by 2 of z square plus 5 i z minus 2. This is the form I have. So, this is my f z right now. Still I can take 2 common. So, I can have half integration of d z divided by z square plus 5 by 2 i z minus 1. And I can write this as multiplication of I can factor factorise this and write in this form. This is convenient minus z 1 and z minus z 2. I just write this particular thing in this form. Because then I know at where my singularities obviously, my singularities as z 1 point and z 2 point.

Now, I need to check z 1 and z 2 which one is inside and which one is outside that is all, but before that I need to calculate. So, the sim in order to find out the singularities. So, let me erase this one in order to find the singularity, I need to evaluate these quantities.

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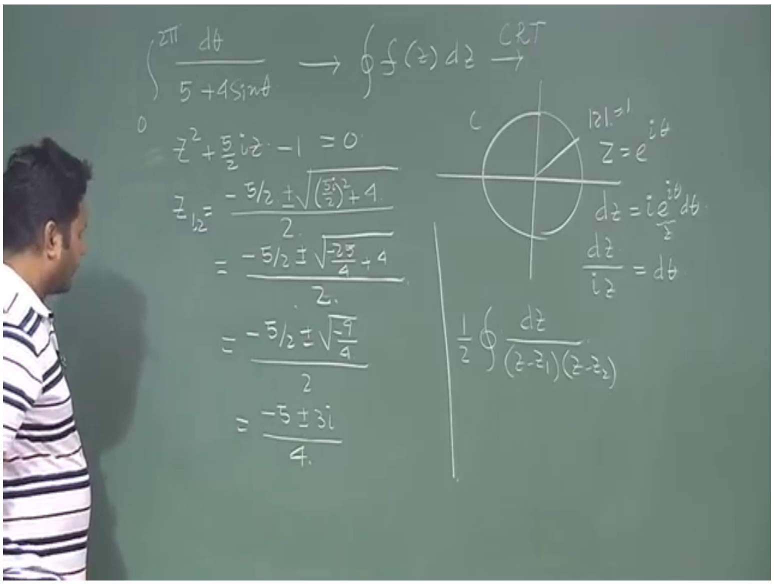

So, z square plus 5 by 2 i z minus 1 is equal to 0. So that means, my z. So, let me calculate it here. So, here let me erase this. So, this quantity z 1 is minus b which is 5 by 2 plus root over of plus minus. So, z 1 and z 2 together I am taking z 1 and z 2 so that I can put the plus minus sign here. Minus b plus minus of root over of b square, b square is 5 by 2 whole square minus 4 a c. 4 is minus 1 4 c is minus 1. So, minus 4 is means plus 4 divided by 2, 2 a a is 1 here, so divided by 2.

So that means, if I simplify that. So, it is minus 5 by 2 plus minus root over of 25. Here 1 i is here. So, I mistake this i. So, when I make a square of that. So, it is 5 by 2 i square. So, i is here. So now, if I square that it will be minus of 25 by 4 plus 4 divided by 2. Let us erase this one half time is there we should remember one half time is there. So, integration let me try to before I forget. So, it is something like this z minus z 1 z minus z 2. It is something like this.

So now this quantity I will simplify further it is 5 by 2 plus minus of this is 4 into 4 16. 25 minus minus 25 plus 16, it seems to be 9. So, 9 root over of minus 9 by 4 divided by 2. So, it is minus of 5 plus minus of 3 i divided by 4. It seems to be something like minus. So, my z 1 and z 2 i evaluate let me write it here.

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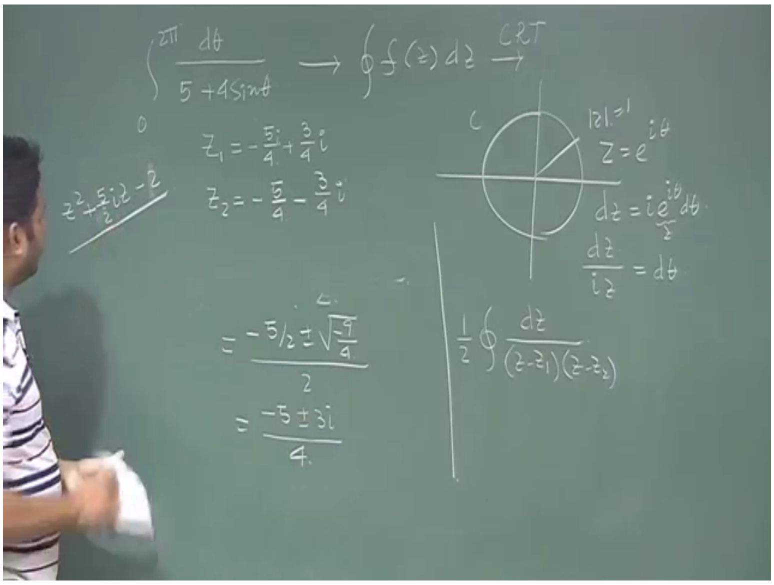

So, my z 1 I evaluate as minus 5 by 4 plus 3 by 4 i. And my z 2 is minus 5 by 4 minus 3 by 4 i minus. So, here let me check once again this 5 by 4, I just erase that here should one i term, because my total integration was something like z square minus plus 5 by 5 5 by 2 i z minus minus 2 it was something like this, this remember. So, one i term was there. So, I miss this i term. So, here this i term will be there.

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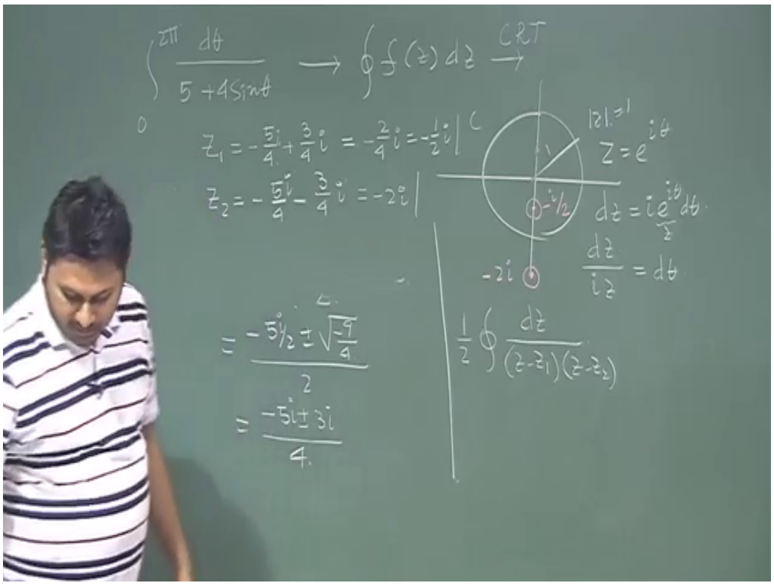

So, I am just miss this i term, now I remembered. So now, this is something like this. So, I can slightly So, this I gives me 5 by 4 i plus 3 by 4 i. So, 5 by 4 i plus 3 by 4 i, it will be just minus 2 by 4 i or minus half i. This is my z 1, and z 2 is 5 plus 3. So, it is 5 plus 3 8 and 4, 4. So, it is just minus of 2 i. So, one residue after doing all this detail calculation what I find that one residue is sitting as minus 1 by minus 1 by 2 i. And another residue is sitting as minus 2 i.

So now I need to check that which point is sitting where in terms of this figure this circle. So, half i this is a radius of 1. So, half i is somewhere here minus half i, sorry minus half i is somewhere here. This point is minus i by 2. What about the other point which is 2 i? 2 i if somewhere 2 i is minus 2 i is somewhere here. This is 2 i minus 2 i which is; obviously, outside the circle whatever the circle. So, I should not take care of this residue I only take this residue, if I take this residue then the results is coming like this.

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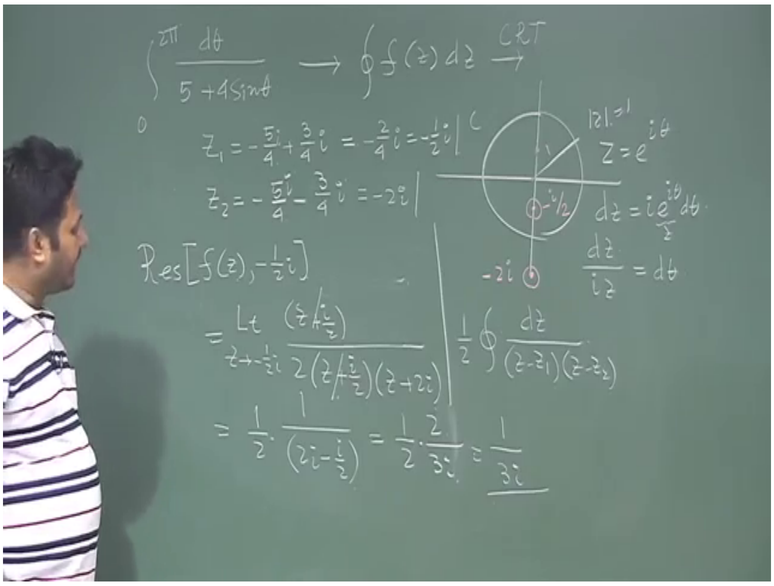

So, I need to find out. So, this is my function this is my function I have z 1 and z 2. So, residue of this function f z at minus i by 2 is limit. Limit z tends to minus of z minus z plus i by 2 the function is 2 divided by z minus i by 2 and another is z plus 2 i.

So, at this point this is. So, if I calculate it will cut. So, it will you now if I put in the function. It will be just half 1 divided by in place of z I just put, 2 i minus of i by 2. So, 2 i minus i by 2 is here. So, it is half 2 2 4. So, 3 so, it is 2 divided by 2 will go upstairs. So, it is 4. So, it will be just 3 i. So, it is 1 divided by 3 i at the end of the day I find the residue at the point minus half i is 1 by 3 i.

So, once I have the residue at 1 by 3 i the next thing is just to evaluate that and then if I evaluate then readily I can have the result. So, just put the result here, residue I find.

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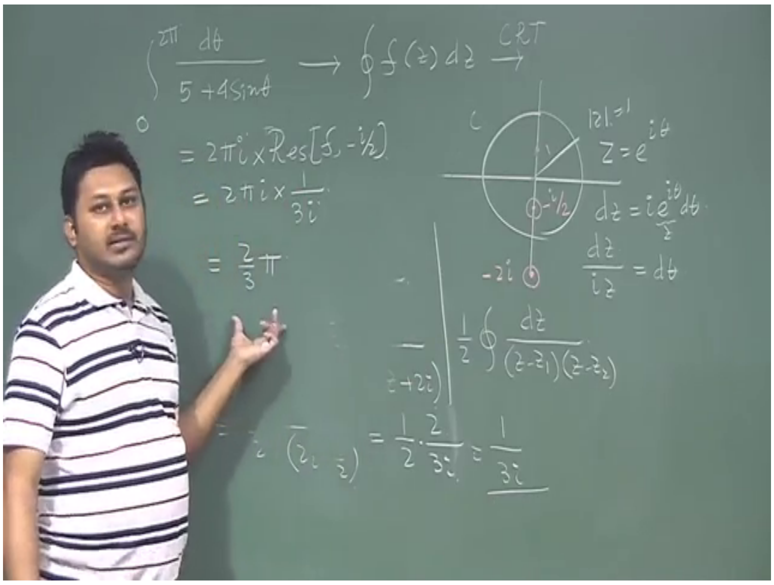

So, this value this integration should have the value 2 pi i integration multiplied by residue of the function f at the point which is inside here is i by 2. So, it is 2 pi i multiplied by 1 by 3 i and it seems to be 2 third pi. The results is coming out to be 2 third pi. So, this is the integration that is purely real. So, let us now try to find out how try to appreciate how these things are happening.

So, this is the integration that is real. You need to convert these 2 complex integration by changing theta as d theta as this d z by i d z. And then sin you can change in terms of z as z minus 1 by z divided by 2 i, then you will have a expression here. And this expression you can factorise and try to find out where is the residue. You find the residue and this is your contour for this contour you find the residue is sitting one residue is sitting inside and other is outside. And you just take the residue an inside in calculate and you find the result that is all.

So, I did all the problems all the steps meticulously so that you can understand how to do this kind of problem, in the next class. So, we do more problem related to that and also we target to do some other problems, which is the real integration from minus infinity to infinity how do tackle with their how do reduce this problem to a complex integration and all this things.

So, with that note let me conclude the class here in the next class we start from here. And then we calculate more and more integration and do all the problems in the board.

So thank you very much for your attention. See you in the next class. Good bye.