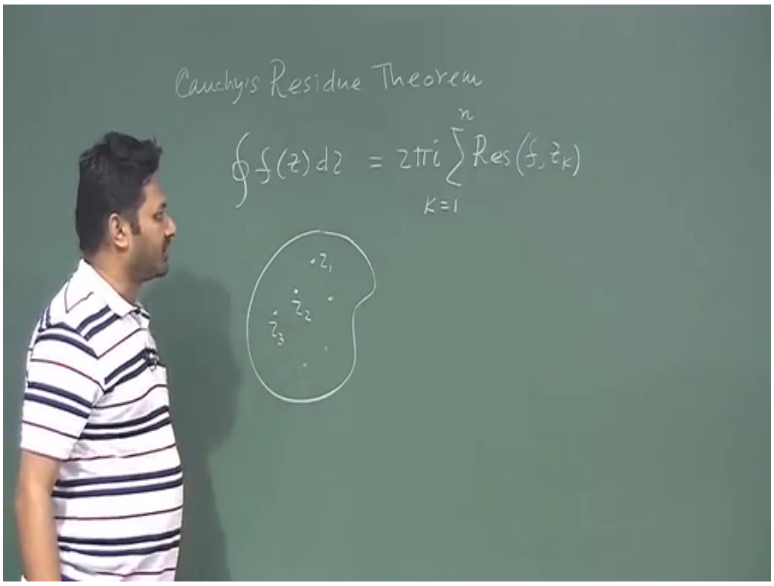
Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of physics
Indian Institute of Technology, Kharagpur

Lecture-57
Cauchy's Residue Theorem (Contd.)

So, welcome back student to the next class of complex analysis.

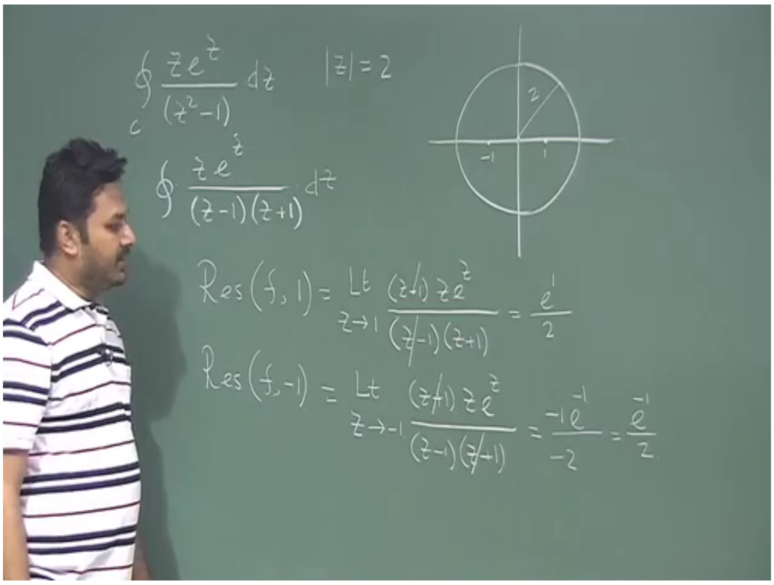
(Refer Slide Time: 00:27)



In the previous class, so, we started at Cauchy's residue theorem; Cauchy's residue theorem while we find that we can evaluate disintegration with the form this.

So, I can eventually find out the integration of the function f z which is analytic in a region except few points which we called this z k say z 1 z 2 z 3 are few points finite number of points were the function is not analytic, but still I can evaluate this function if I able to find out the residue at that point of this function multiplied by 2 pi i. So, we have already applied this formula to different problem. So, our aim here is to do more problems. So, today we will like to do more problems in board.

(Refer Slide Time: 01:58)



So, let us take another example or another problem the function is given as z e to the power of z divided by z square minus 1; this is the functional form that I have and also we need to define the domain over which I am calculating that is; so, the domain is defined as mod of z is equal to 2. So, as usual let us try to find out where the domain lies and where this simulative lies for this domain. So, mod of z equal to 2 means; I have a circle of radius 2 like this centre at z equal to 0.

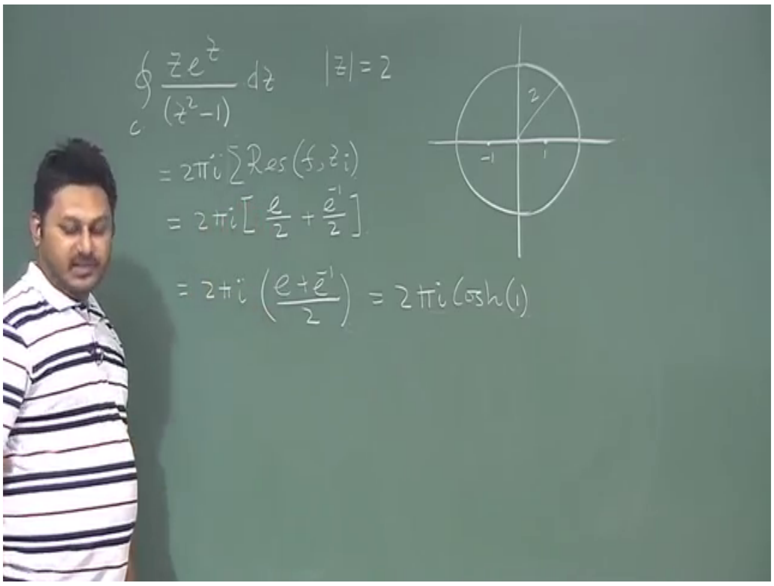
Now, this point the singularity at; so, where z square minus 1 is 0; we have the singularity so; obviously, one singular point is 1 and another singular point is minus 1; 1 and minus 1 are the singular point singularity live in this particular point. So, 1 minus 1 if this radius is 2, then 1 and minus 1 is eventually this point and this point; so, in this example, we find that both the points 1 and minus 1 are inside the region. So, I need to calculate the residue for this both the both this points.

So, let us try to do the thing. So, this function I can write this function as this and the result is something like. So, first we calculate the residue of this function at point 1. So, is nothing, but limit z tends to 1 z minus 1 z e to the power of z divided by z minus 1 z plus 1; I am doing all this thing meticulously. So, that you can understand, but when you are familiar with the system event do readily and do not need to do this steps you can directly find your result.

So, here when I this point will going down I mean this z minus 1 z minus 1; this term will going I mean, then cut and then z tends to 1, I just put z tends to 1; it will be e to the power of 1 divided by 2; this is my first residue at 1 point. So, I need to calculate the residue of f at minus 1 point because 1 minus 1 point is also inside the circle. So, I need to take care of that residue as well. So, z tends to minus 1, I will need to multiplied by z multiplied by z plus 1, z e to the power of z divided by z minus 1 z plus 1 z plus 1 z plus 1 will cancel out. Now, I put these values. So, it will be for z I have minus 1 e to the power of minus 1 divided by this I could is it will be minus 2.

So, I will eventually have e to the power of minus 1 divided by 2; this is my first residue; this is my second residue 1 is e to the e divided by 2 and other is e to the power minus 1 divided by 2. So, now, just is the next is just to put this value that is all.

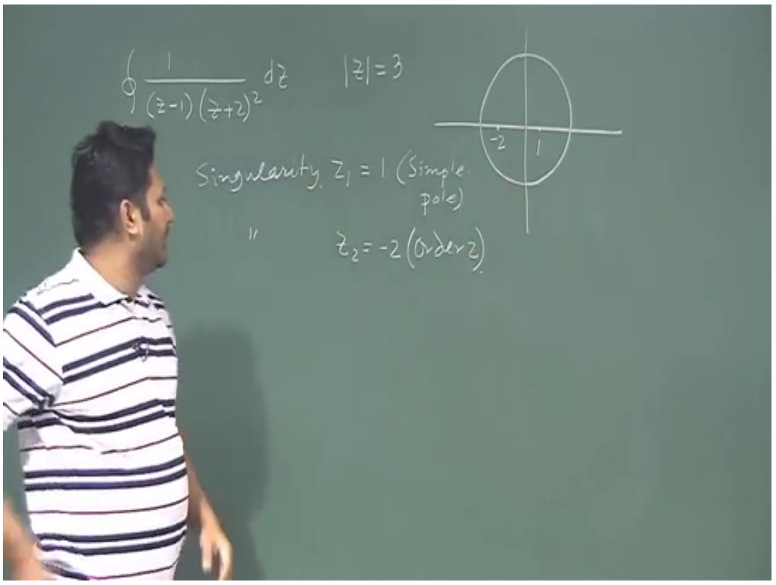
(Refer Slide Time: 06:22)



So, the value of integration is 2 pi i summation of residue if z i. So, here it is 2 pi i the first residue is e divided by 2, second residue is e to the power minus 1 divided by 2 like this I have already got the reason, but I can modify that. So, this quantity e to the power is something. So, 2 pi i e plus e minus 1 divided by 2 this quantity is cos hyperbolic

So, it is 2 pi i cos hyperbolic of 1; this is the result we have this is the most close form we have. So, the function is given the integration we need to do for this function the region is given which is z 2 need to find out where is the point where we have that singularity, then calculate the residue and restricts state forward.

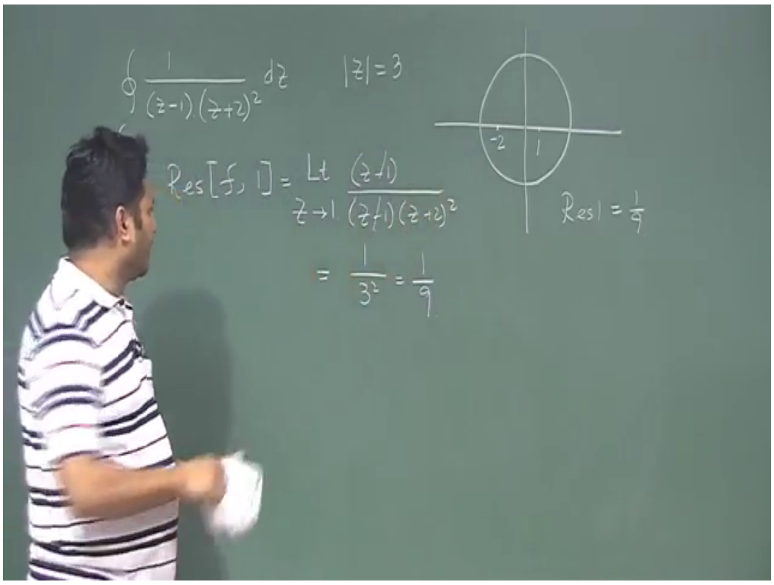
(Refer Slide Time: 07:50)



So, after having this; let us go to some other example. So, another example I have in my hand is integration of 1 divided by z minus 1 and then z plus 2 whole square d z, this is the function I need to integrate over some circle closed loop c and the circle is given as mod of z is equal to 3.

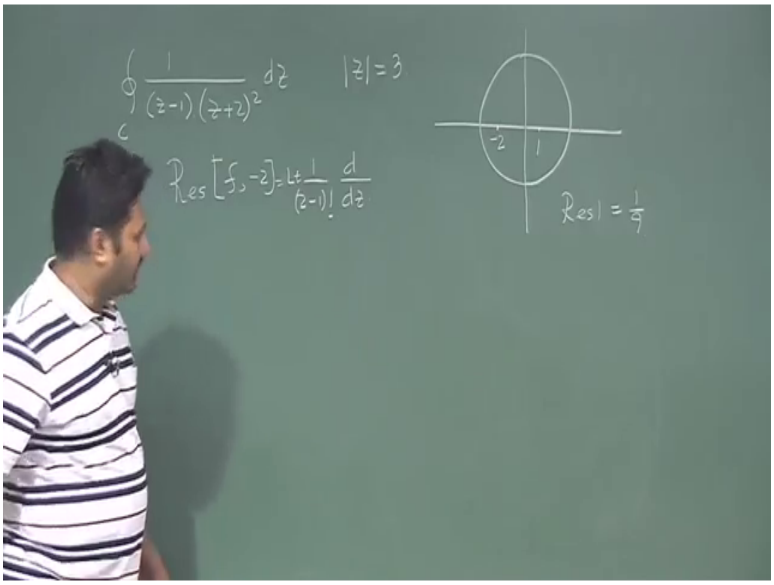
So, mod of j d equal to 3 means I should have a circle with radius 3 this is a circle with radius 3 where the singularity where the singularity. So, singularity first let us find out say z 1 1 singularity is at 1; this is for simple; this is a simple pole singularity z 2 is at z 2 equal to minus 2, but this is of order 2 order to singularity. So, both the singularity are inside because 1 is 1. So, somewhere here and another is 2 somewhere here minus 2 somewhere here. So, 1 point is here another point is here both the thing both the singularities are inside the circle. So, need to calculate the residue for both the singular points.

(Refer Slide Time: 09:35)



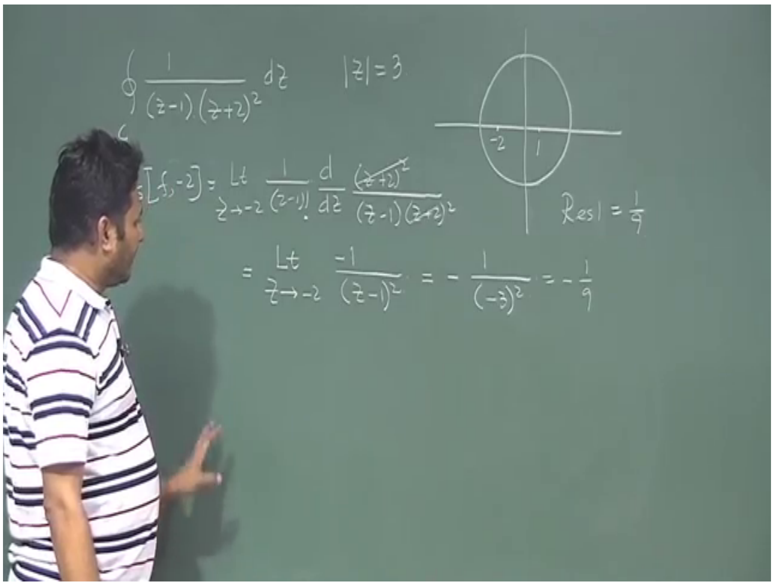
Let us first find out the singularity. So, residue f at z equal to 1 point which is limit z tends to 1 z minus 1 divided by z minus 1 z plus 2 square of that z minus z minus 1 cancel out. So, I will get this quantity if I put z equal to 1. So, it will be just 1 divided by 3 square which is 1 divided by 9. So, first residue I find is 1 by 9. So, let me write it here somewhere. So, residue 1 is 1 by 9. Now I am going to find out the second residue this quantity this is of order 2.

(Refer Slide Time: 10:36)



So, residue of the given function at minus 2 point which is according to our formula; I should write 2 minus 1 factorial d of d z; I need to write the limit also. So, it is under limit space is not sufficient. So, let me write it once again from here.

(Refer Slide Time: 11:09)

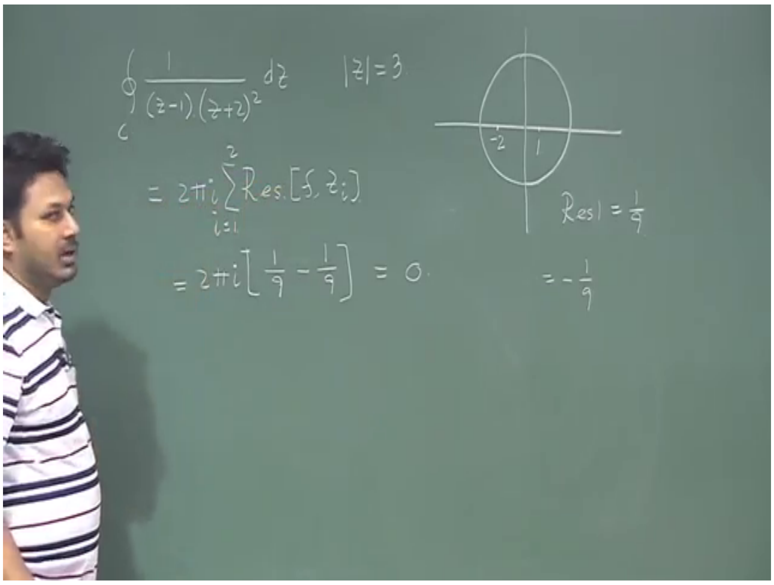


So, residue f at minus 2 is something like limit z tends to minus 2 1 divided by 2 minus 1 factorial which is 1 d d z of z class 2 whole square multiplied by the function which is z minus 1 z plus 2 square up that. So, this quantity; this quantity will cancel out; I need to do the derivative of this quantity see if I do then. So, it is limit z tends to minus 2; I do the derivative, it will be 1 divided by z minus 1 square of that with a negative sign. So, minus of 1.

Now, if I put it here this value if I put, then it comes out to be; then it comes out to be this quantity is minus of 1 divided by z i put minus 2. So, it will be just minus 3 whole square. So, it is minus of 1 by 9 the same value I got here residue 1 plus 1 by 9; in this case; however, I find it is minus 1 by 9. So, now; obviously, you can understand; I believe I am not making any mistake.

So; obviously, we can find that this case integration will going to vanish because the residue up 2 points a exactly same an opposite.

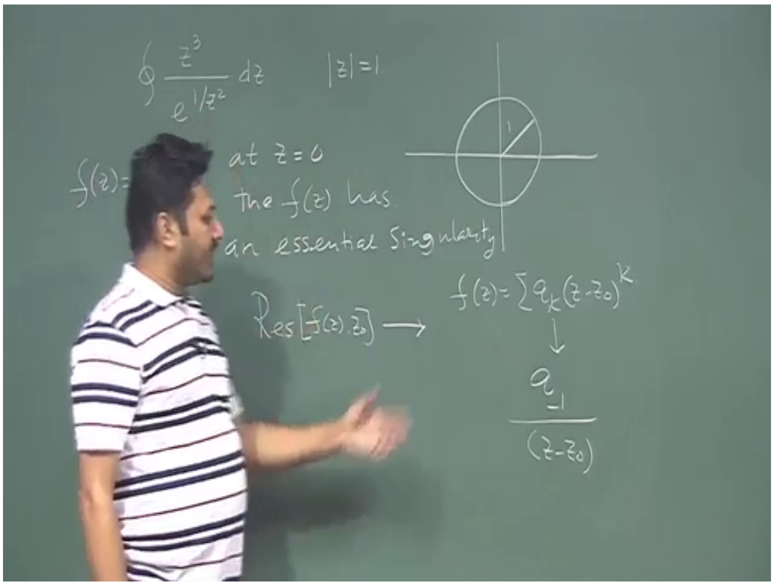
(Refer Slide Time: 13:03)



So; that means, the values of this integration is 2 pi i sum over residue of this function at i here; I want to do just 2 residues it is 2 pi i first residue is 1 by 9 and second residue is minus 1 by 9 and we have a result 0. It may possible to have this result 0 in an; I mean it is not an analytic function in this region that is for sure, but still I can get that for some. So, 1 residue and another residue exactly balanced to each other and 1 can still get 0 result out of that.

So, next example; I should go with another example; it is interesting so.

(Refer Slide Time: 14:05)



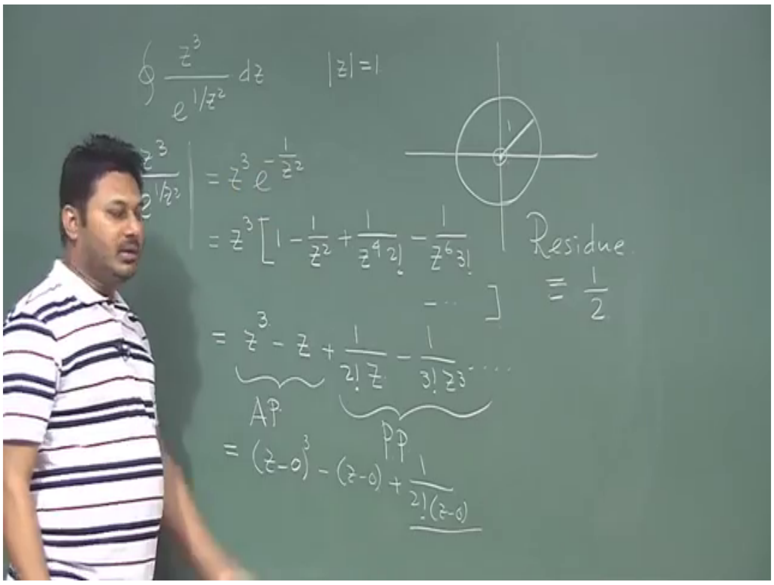
So, the function that is given to us is z cube e to the power of 1 divided by z square d z this is the expression that is given to us and we need to evaluate it in the region mod of z is 1 mod of z is 1. So, mod of z is 1 means I need to evaluate this integration for a circle of radius 1; this is a circle of radius 1 and now you need to first find out where is the singularity where the singularity of this function.

So; obviously, at z equal to 0 the function whatever the function f z where f z is this entire quantity. So, f z is z cube divided e to the power 1 by z square this is the function that is given to us. So, at z equal to 0 this function has singularity and this singularity is essential singularity because if I expand this at z equal to the function f z has an essential singularity. Now if the functional has some essential singularity is difficult to find out the residue in our standard form because no way we mention that what happened if there is a essential singularity.

For example, in this case we have some essential singularity, but our goal is to find out the residue. So, residue is nothing, but residue of a function f z around some point is nothing, but some point z is 0 is nothing, but if I expand this function in laurel series once again I am reminding you this things because it is important that should that should be in your mind that the residue is nothing, but the coefficient of a minus k; that means, the coefficient of z minus z 0 this is my residues, I need to find out this some if somehow I can able to expand this function in laurel serious and take out this coefficient, then I am done, then I can calculate the residue without doing all this recipe without for example, here the recipe will not going to work. So, in that case;

So, let us try to find out; whether we can do a laurel series expansion of this function or not.

(Refer Slide Time: 17:09)

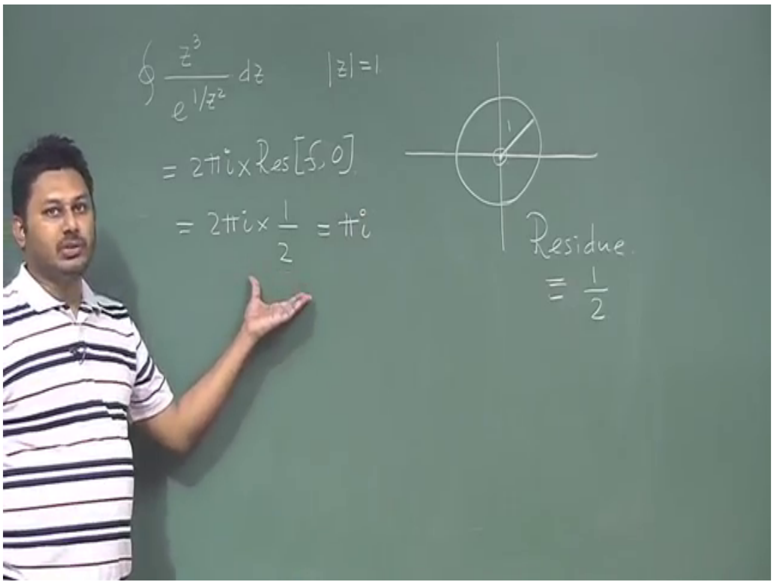


So, this function, I can write it as z cube e to the power of minus 1 by z square the singularity is here sitting here. So, now, I will going to expand this function first term will be 1; second term will be minus of 1 by z square, third term will be plus 1 by z cube it is square; mind it. So, it will be z to the power 4 of factorial 2 third term will be minus 1 divided by the cube of this term. So, it will be z square 2 to the power cube.

So, it will be six factorial 3 and so on; this will be my series and now if I do the calculation if I now simplify this first time will be z cube because this is the expansion of e to the power 1 by z square that is expansion, but I am trying to find out what is the expansion of the full function f z. So, z cube is the first term second term will be z third term is interesting which will which is 1 by factorial 2 and then it is z and fourth term is minus 1 by factorial 3 and then z cube and so on. So, this is exactly the laurel series expansion that we have this part is your analytical part this is analytic part and this which part is your principal part and we know that the residue is nothing, but the coefficient of z; z minus where the singularity

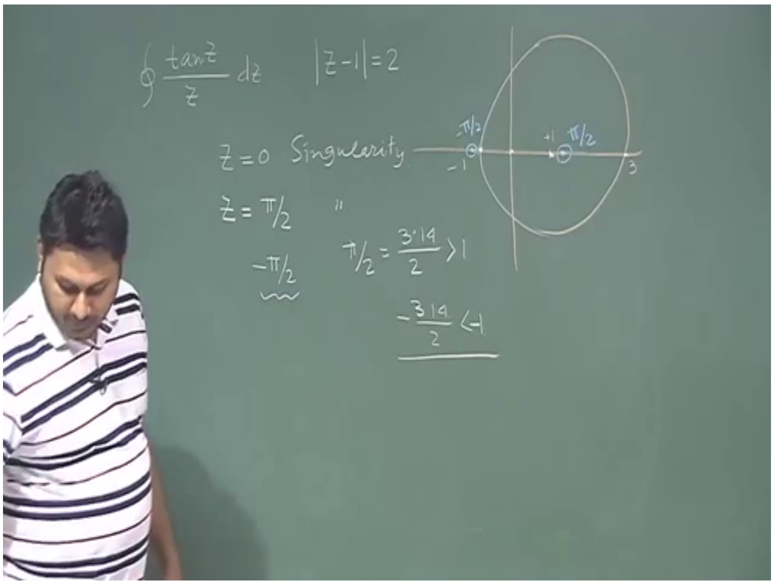
So, here the singularity z equal to z 0 so; that means, I can write this expression in principle in this were z minus 0s cube minus z minus 0 and then 1 by factorial to z minus 0s so; that means, my residue is coming out to be half; so, my residue here for this case residue is coming out to be half. So, once I find out the residue there is the rest part is straight forward.

(Refer Slide Time: 20:12)



So, the rest part is this; that means, the answer the answer of this thing is 2 pi I multiplied by residue of this function at z equal to 0 point which we evaluate there. So, 2 pi i multiplied by 1 by 2 which is pi i. So, this is the solution we have. So, this is a different kind of example and I put this example. So, that even understand the always is it is not necessary that you will use the standard recipe to find out always is it is not necessary that you use the standard recipe to find out your residue rather sometimes; we need to use the expansion and this expansion sometimes helps next problem is something again of interest.

(Refer Slide Time: 21:05)



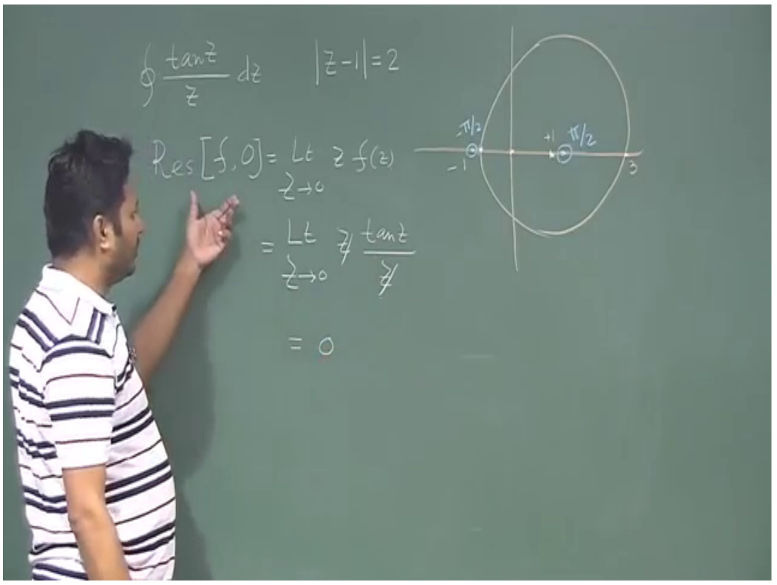
So, the function is given like this; this is the function and I need to evaluate at this point 2, right. So, again we need to find out where is the region; where is my region. So, usual process to find out the region; so, this z minus 1 means I have point 1 here and having radius 2 i; if I draw a circle of radius 2, then it will be something like this . So, this point is my minus 1 this point is plus 1, this point is minus 1. So, that the radius is 2 and this point is 3.

So, this is my centre this point is 3 this point is minus 1; I know the region; now I need to find out where the singularity lies; obviously, this first singularity like z equal to 0 point it is a singularity z equal to 0 point; we have singularity apart from that any other point where you have the singularity f at z equal to pi by 2; we have singularity at least which is inside this. So, pi by 2 also minus also minus pi by I should have an singularity right minus pi by 2 also; we have singularity am checking where I am getting the singularity minus pi by 2 pi by 2 and or the multiple of all these quantities seems to be the singular points.

Now, the question is pi by 2 and minus pi by 2 where they are situated in this. So, pi by 2 is around 3.14 divided by 2. So, it is roughly 1 point something it is greater than 1; obviously, it is greater than 1, but; obviously, less than 3. So, it is if it is greater than 1 and less than 3 so; obviously, location pi by 2 is situated some were here. So, this is my pi by 2 point which is inside which is inside what about minus pi by 2 minus pi by 2; obviously, we will have a minus here. So, it will be something 3.1 by 4 by 2 and which is which is again less than minus 1.

So; that means, this point is just outside here; this point is just outside here minus pi by 2 just here. So, I can exclude this minus pi by 2 point. So, now, I know where is my singularities and all the other points will be outside at least for this given a; now we are ready to calculate the residue.

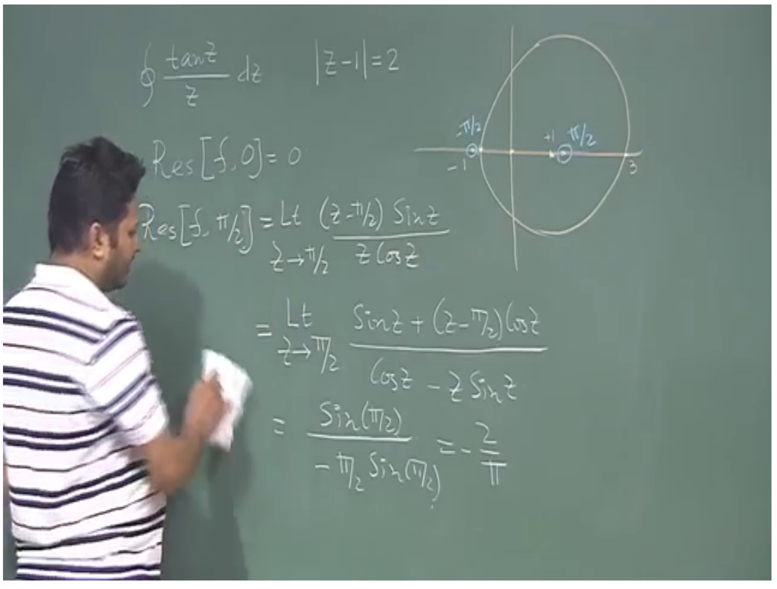
(Refer Slide Time: 24:46)



So, let us first calculate the residue at z equal to 0 point; what is the residual, we will have. So, residue of this function at 0 point is limit z tends to 0 z multiplied by the function which is limit z tends to 0 z multiplied by tan z divided by z.

So, z; z is cancel out. So, what will get if I put the limit tan 0 and 0 is 0. So, my first residue gives me the values 0 my first residue gives me the value 0 what about the second residue.

(Refer Slide Time: 25:31)

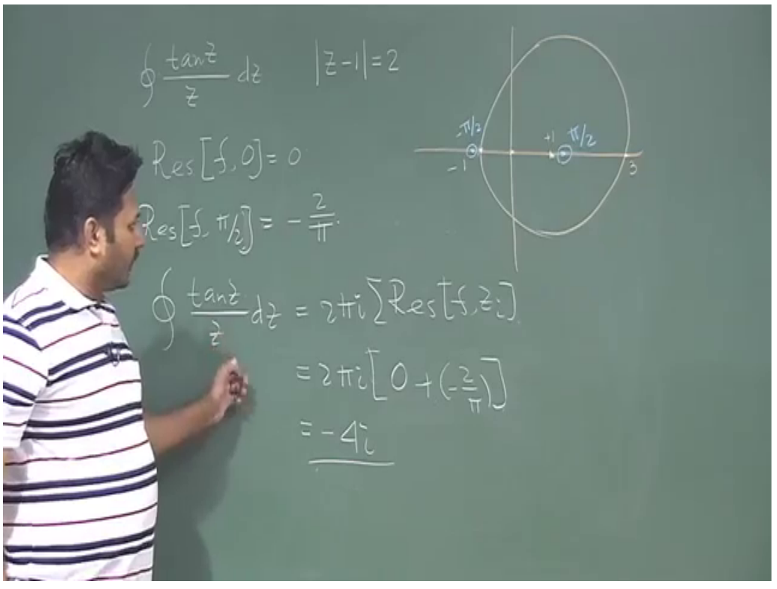


So, residue of this function at pi by 2 point; so, I need to multiply limit z goes to pi by 2 i multiplied z minus pi by 2 and then this function this function; I write as sine z divided z cos z to make it convenient.

Now, add z equal to pi by 2 here we can find at pi by 2 when I could z equal to pi by 2 this portion 0 and the down stairs is also 0 by 0 forms again; I will use this law Lopital rule and if I use this law; Lopital rule, then I need to; so, derivative of this quantity. So, first derivative this quantity; that means, it is 1. So, sine of z plus z minus pi by 2 cos of z and down stair we have first is cos of z and second is minus z sine of z. Now, if I put the limit there will be no problem if I put z equal to pi by to this quantity vanishes, but this quantity stands.

So, I will finally, have sine pi by 2 divided by cos pi by 2 which is 0, but minus pi by 2 sign pi by 2. So, eventually I will have minus of 2 by pi; this is my residue. So, first residue is 0 and my second residue comes of to be minus 2 by pi.

(Refer Slide Time: 27:43)



So, this residue is minus 2 by pi. So, what will be the value of this integration this integration is 2 pi i summation of residues of functions evaluated at different. So, here we have 2 pi i first residue is 0, but second residue; I have some value which is minus 2 by pi. So, it is eventually minus of 4 i; this is the result. So, pi; pi will cancel out, 2 2 is there minus x minus of 4.

So, student I will like 2 stop the class here; today, we will mainly cover different kind of integral problems and the find that Cauchy's form; Cauchy's residue formula can be used to figure out the function if the contoh is given to you and you can evaluate different kind of integration without doing that just using this residue theorem you can evaluate that.

In the next class, we will show some consequence of this Cauchy's integral formula where by exploiting this Cauchy's integral formula; you can evaluate even the real integral that is most interesting the real integral will going to; we will going to solve by using Cauchy's integral formula. So, with that note let me conclude the class here see you in the next class. In the next class, we will be treated with the fact that we will use Cauchy's integral formula to solve a real integration.

So, thank you for your attention see you in the next class.