Mathematical Methods in Physics-I

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Lecture – 56
Cauchy's Residue Theorem

So, welcome back student to the next class of complex analysis. So, today we will learn. So, in the previous class, we know how to calculate the residue and all these things today we will going something interesting which is called Cauchy's residue theorem.

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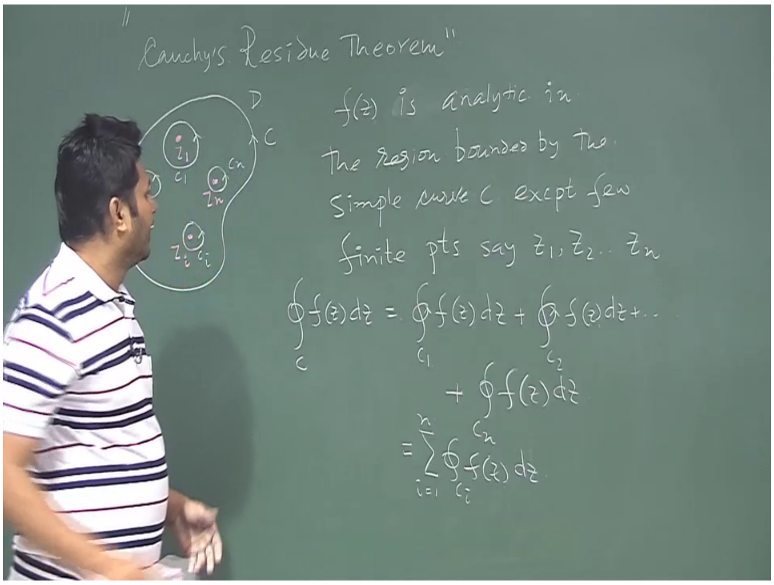

So, this Cauchy's residue theorem that this is not a very new thing to you because I have mentioned these things in the previous few classes; so, let me again give you the; what is the meaning of these things and what is the statement of this theorem. So, if say this is a region D and which is bounded by a simple curve C. So, C is a simple curve which is bounding a region D where the function f z is analytic.

So, f z is a function which is which f z which is a function f z is a analytic in the region bounded by the simple curve or contour C except few finite points; say z 1, z 2, z n. So, first try to understand what is f z is a analytic in the region bounded by a simple curve C except few finite points say z 1, z 2, z n so; that means, it is not entirely analytic in the region there are few points I mentioned these points like this here, here, here, here; this is say Z 1 Z 2 and so on. So, these few points z 1, Z 2 general point Z i and Z n there are n points here; suppose where the function is failed to analytic; that means, these are the region the in this region the function is in analytic except at this particular points and these points are finite.

In that case, if that is the case; I know that if I make a circle around here if I make a circle around here, if I make a circle around here and if I make a circle around here, I can write the closed integral of f z over this region as this is say C 1, this is C 2, this is C i this is C n, this i is a general element. So, I just write this as the summation of this quantity; this, we have already proved in our previous class.

So, I am not going to prove it once again, I will just encircle these things and then it become a multiply connected and for multiply connected this is true; I just cut out these things; cut out these things and then if I add; then I will get these things; this if I write in summation form, it will be look something like this summation of C i i goes to 1 to n f z d z. So, this is the first statement and second, I can write this in this form and what about these things what about these things this is I need to erase these things first then I can write.

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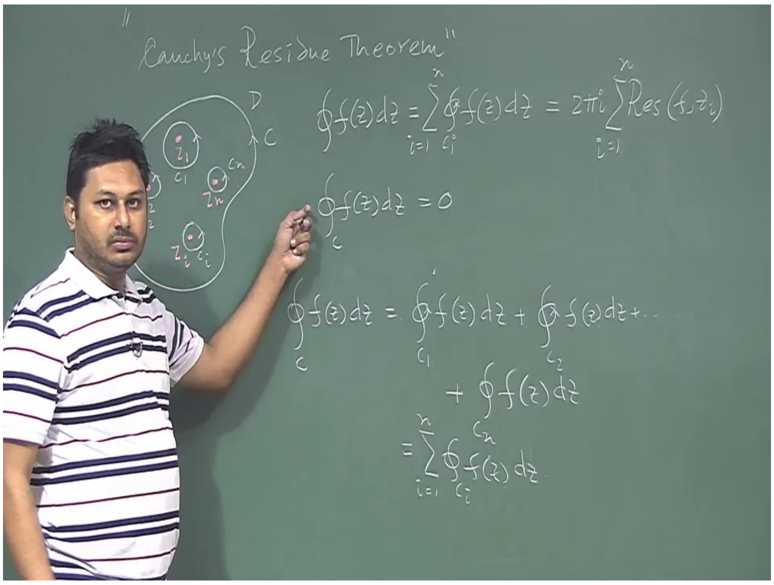

So, let me first complete the statement, then I going to prove, then this quantity which is summation of this which is summation of this will be simply 2 pi i summation of the residue of the function f evaluated at i point with summation which is 1 to n; this is the statement.

So, let me understand. So, let us understand once again. So, we have a region which is bounded by some curve C; the region D is bounded by some simple curve C; the function is analytic except this point. So, these points are singular points, first I have already mentioned that this function is singular point if the singular point is present even, then I can able to evaluate the total integration of that; please remember one thing if there is no singular point here; that means, the residue is 0, then I have the entire region the function was analytic o and I know that if the function is analytic some region closed integral in that region bounded by C will be 0 that I know.

This is Cauchy’s integral formula is this Cauchy Goursat theorem is this rather. So, now, I am saying that f z is not entirely analytic in this region, there are finite points where f z is failed to analytic failed to be analytic then what happened that I can write this as 2 pi i and the residue because this function should have some residue here because these points are singular points and residue I calculate that if function is given there are few residues are associated with that if I able to calculate the residue, then I will just add this residues because here how many singularities are there n number of singularities are there. So, I should have n residues. So, I will just add these n residues multiplied by 2 pi i and I will get the result.

Why this residues is a come in n number of because my integral is now divided into n part 1, 2, 3, 4, 5, 6 up to n C 1 C i goes to n. So, that is why this integration is coming to n parts. So, for each integration; I will have one residue and this residue is add up with this and 2 pi i term will come why it is coming. So, now, let us try to understand that.

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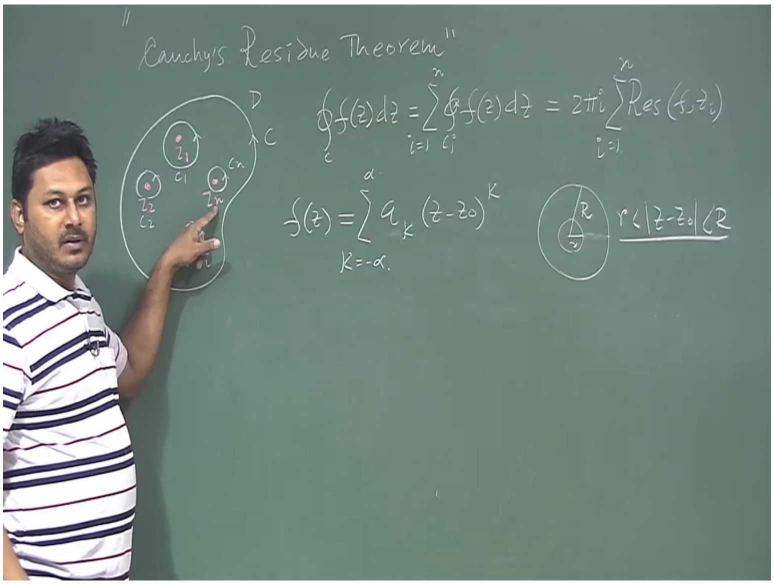

So, statement is quite straight forward that if a function is analytic except this point then if I want to find out the closed integral over that region then I will have something 2 pi i summation of residue. So, now, let us go back to our Laurent series expansion. So, in Laurent series expansion, I have this expansion in my hand and this is valid for when if we some kind of some kind some kind of some kind of some kind of singularities here then this function is bounded this function was in some annular region; for example, this expansion if you remember was for small r to big r. So, there was some restriction for which I can expand this function. So, at z equal to z 0 what happened this function is say z 0 points are nothing, but z 1, z 2, z i and z n. So, around that I am expanding the function around these singular points.

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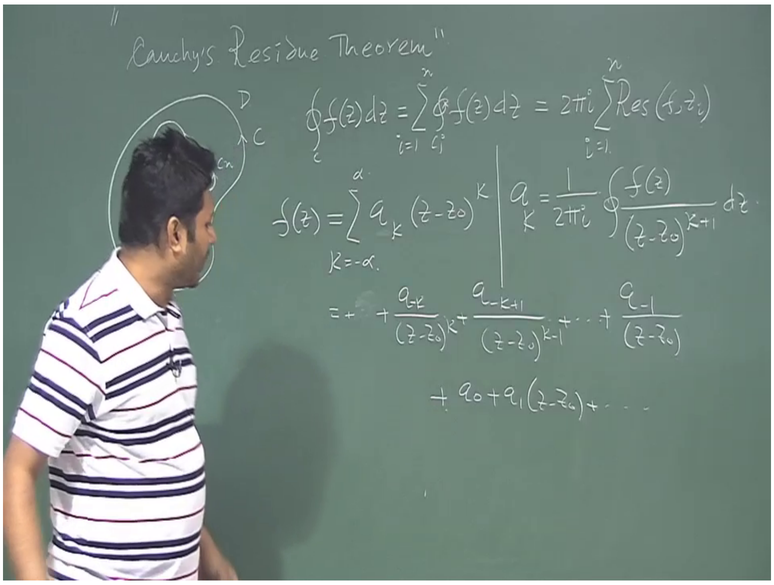

If I do that then I can have several points the old expansion that we have already done minus a minus k plus 1 z minus z 0 as usual thing.

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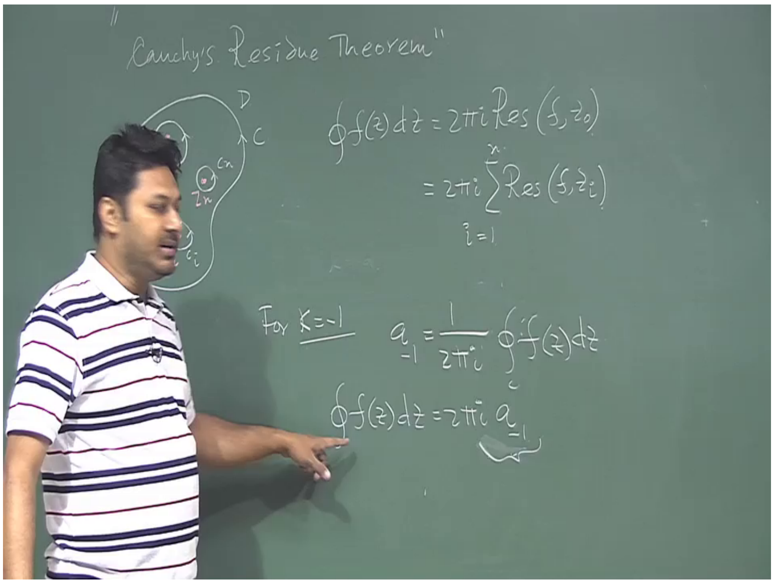

That we have already encountered plus 1 important a minus 1 z minus z 0 and then the analytic part n a 0 plus a 1 z minus z 0 and so on several time; I am writing this. So, that you can aware of these things, but important thing is that what is the value of a k the value of a k; if you remember, I have was one by 2 pi i integration of f z z minus z 0 k plus 1 d k that was the value of a k. So, once I have the value of a k in my hand. So, what

I can do? I can put the value of k. So, for k equal to minus 1 for this special k; what I will get? I will get a minus 1 is equal to 1 divided by 2 pi i integration f of z d z because k is minus one; that means, this term will going to vanish. So, only thing that is in my hand is the integration of f z d z and integration of f z d z is 2 pi i multiplied by a minus 1.

Now, this a minus 1 this a minus one is nothing, but the residue. So, I can rewrite this expression let me erase this. So, I can this rewrite this expression as this. So, 2 pi i residue of this function at that point z 0 where the function is now if instead of one residue if instead of one singularity; if I have many singularities then this will be sum over that. So, 2 pi i sum over residue of f z of f at z i 1 to n which is essentially the expression that I have written in the previous just few minuses ago.

So, now, we have the expression in our hand which suggest how I can evaluate a closed integral f z if the residue is known then I can calculate the closed integration please remember, if there is no singularity then the function is entirely analytic if the function is entirely analytic if you take a closed integral it will give you a 0 because it is a path independent. So, now, we will going to exploit this expression or this theory and evaluate some problems. Now this is a interesting part; how to execute this theory to solve a problem. So, let us take few examples or few problems.

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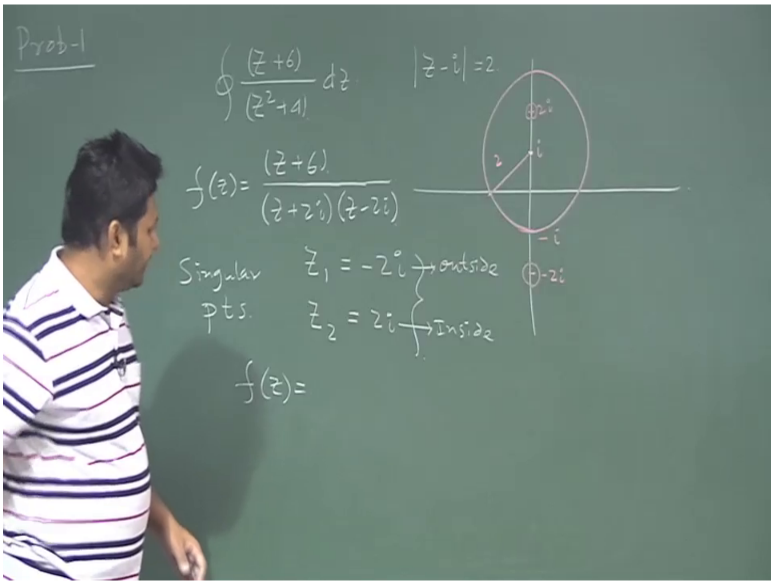

So, what examples; we have first example is one or problem one we will do several problems related to this because this is a very important very important theory that we just show.

So, the function integration that is given to us is this z plus 6 Z square plus 4 d z and you have to evaluate this integration, but I need to know for which region, I am doing this; the region that is given is z minus i is equal to 2. So, this is the function this is the region first I need to identify which region at which region I am evaluating this. So, z minus one z minus i means the origin is at i point and I should have a circle with radius two. So, if I have a circle of radius 2; so, the region.

So, this is 0; this is i. So, I will have a circle like this something like this where this radius is 2 this is 2 this is 2 every place; this is 2. So, now, this is the function. So, I need to find out for this function z plus 2 i z minus 2 i; the function is written in this way I simplify this so; that means, there are singularities are there essentially at say Z 1 is one singularity which is as minus 2 i and Z 2 is another singularity which is at point 2 i this 2 singularity; I have these are the singular points singular points next thing is to check which points is inside; here both are here; both are outside or both are inside or one is inside one is outside.

So, let us try to find out. So, let us locate what is minus 2 i. So, if you remember this was I this point was minus i and minus 2 i is below that somewhere here. So, this point is minus 2 i. So, minus 2 i is outside I should not bother about that. So, this singularity this singular point is outside the region where I am trying to evolve. So, now, this is I this point is 2 i; obviously, the point 2 i is inside the region. So, point 2 i is inside. So, this point is inside and this point is outside, I should not bother about the point outside because this point will not going to have any effect on the integration only thing I need to bother about this point. So, now, I can evaluate this. So, function of f z; I can write simply let me directly do the problem because I know; where the singularities is.

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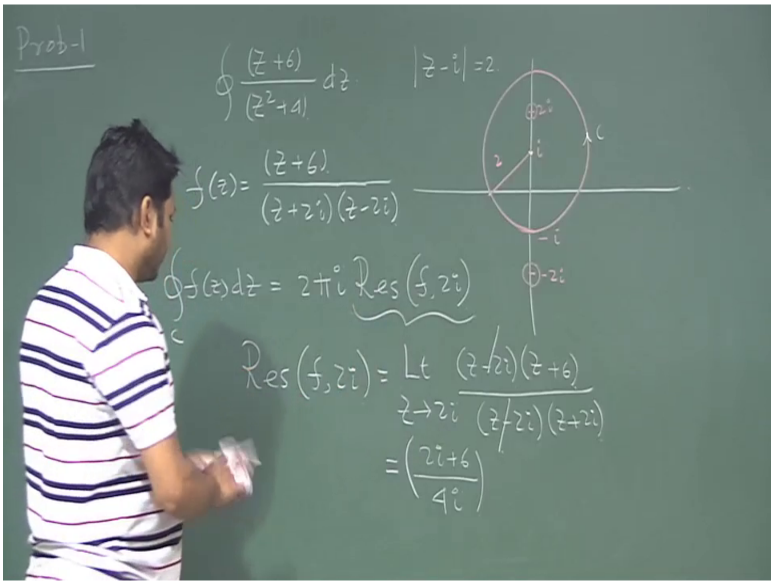

So, what should be the result of these things what is the formula the formula is closed integral f z d z will be this close C is this one what is given here. So, this is C this c. So, if I integrate this over this region, then what I will get? I will 2 pi i and residue summation of residue, but here we have only one point that is singular. So, we do not need to put this summation sign here just one point is there. So, just at that point the residue will be enough. So, residue of the function f at which point at which point at 2 i point. So, let us put 2 i here.

So, only thing I need to evaluate is this quantity which is easy. So, let me just try to find out how the residue one can find if 2 i is limit z tends to 2 i z minus 2 i will be multiplied by the function z plus 6 divided by z minus 2 i z plus 2 i. So, z minus 2 i z minus 2 i will cancel out and then if I put the limit that will be the result of the residue. So, now, 2 i if I put it will be 2 i plus 6 in the down stair it should be 4 I here 2 i plus 2 i. So, 4 i; so, this whatever the value I have this is my residue. So, now, the result is 2 pi i multiplied by the residue. So, I will need to multiply this quantity with a 2 pi i to get the full result.

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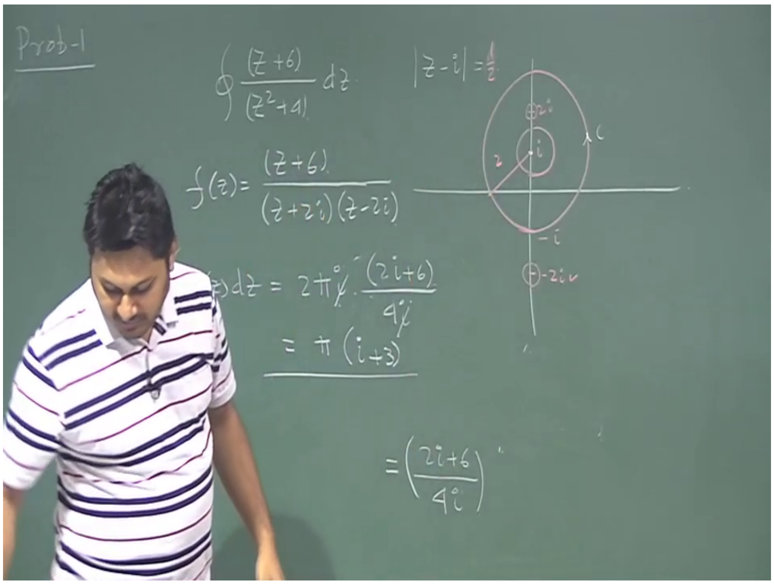

So, if I do then I will have 2 i plus 6 multiplied by 4 I or if I take 2 common here this 4 will cancel out. So, pi this I will cancel out here and if I take 2 common here then this 4 will cancel out. So, pi it seems to be pi i plus 3. So, that is the result that is the result I am looking for. So, this value of interaction the complex integration in the region where we have singularities, but one singularity is outside and if I calculate the residue is comes out to this I just put 2 pi i multiplied by summation of residue since one residue is here. So, I just calculate this and find this result.

But please mind it if the region is changed for example, if it is say 5 or 6 then this radius will be bigger than this then this 2 i point will be included and you need to calculate for this 2 i point also if the region is changed so; obviously, it will be depending on the region if this region is changed to half for example, then both the both the residue is outside both the singular point is outside, it will be something like this. So, inside that there is no singularity the function value of the this integration will be 0 then so; that means, the value of the integration is mainly depend on the region that is given to you if the region is changing, then you will have different-different result in each cases after that we will go to more examples because there are few examples are there; we need to cover which is important.

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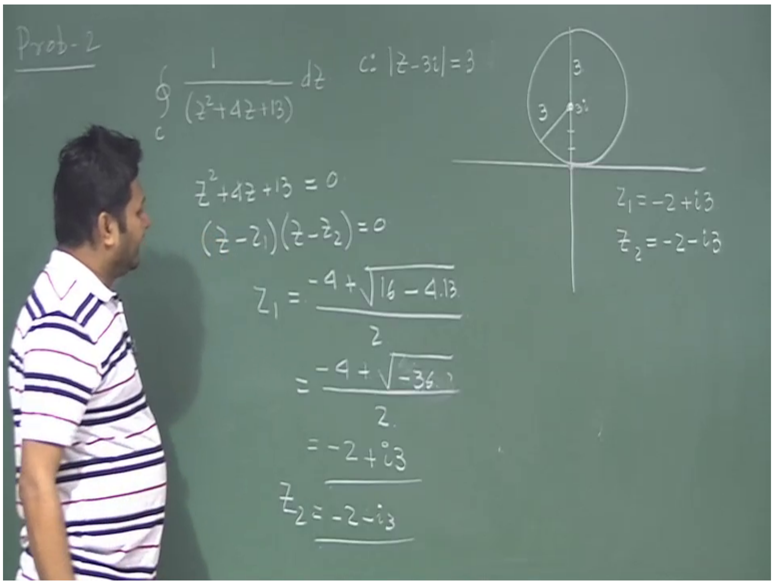

So, let us go to my problem 2; problem 2. So, in problem 2, we have something like this it is given as one divided by Z square plus 4 z plus 13; this is the expression, I have in my hand, but I need to mention also what is what is the region here the region is given as Z; the C contour is as z minus 3 i z minus 3 i mod of these things is equal to 3. So, first we should be very careful about the region and then find out whether this singularity is; obviously, this function has some singularities. So, first let us check where the region is.

So, Z minus 3 i; so, the point 3 i minus 3 i point is somewhere here. So, if this is 1 i; 2 i, 3 i is somewhere here. So, this is 3 i point and I am taking a radius taking a circle of radius 3 so; that means, it will cut to 0 point and then go up here. So, it will be something like this. So, this is the region where I need to this will be in the middle of the circle and this; this is 3, this is 3; the radius is 3 the region is no; understand the next thing is that to evaluate.

So, Z square plus 4 z plus 13, this quantity where it is 0 we will have the singularity that is for sure; so, this quantity if I write z minus Z 1 and z minus Z 2 equal to 0 then Z 1 and Z 2 will be the 2 solution of these things this quadratic equation which we know how to solve. So, Z 1 is minus b plus root over of b square; that means 16 minus 4 a C 4 a c. So, 4 into 13 divided by 2, here we have minus 4 plus root over of this seems to be 4; 16 minus 52; 52 minus 16 is seems to be 36 minus of 36. So, straight away; I am writing this it is minus of 36 divided by 2 divided by 2. So, this quantity is minus of 2 plus if I take 36 out and take i as root over. So, I will be here because root over minus 1 is here 36 out; it will be 6 and divided by. So, it will be like minus 2 plus minus 2 plus i 3 Z 1 is my minus 2 plus i 3.

We know that if we have a complex root of a quadratic equation then if one root is this the other root must be the complex conjugate of that. So, Z 2 I am not going to evaluate that is minus 2 minus i 3. So, Z 1 and Z 2 i evaluate. So, let me write it here somewhere my Z 1 is minus 2 plus i 3 and Z 2 is minus 2 minus i 3; these 2 points I have. So, once I have this 2 point, then I need to now; I need to consider whether these 2 point which of this point are inside and which of this point is outside.

So, you can readily find which point. So, it is minus 2 say Z 2 point is minus 2 minus i 3 what is the coordinate of this point minus 2 minus i i 3. So, minus 2 is somewhere here and minus i 3 is somewhere here somewhere here so; that means, this point seems to be outside and other point I should write it here other point is minus 2 plus i by 3 minus 2 plus i by 3 plus i by 3 is here and 2.

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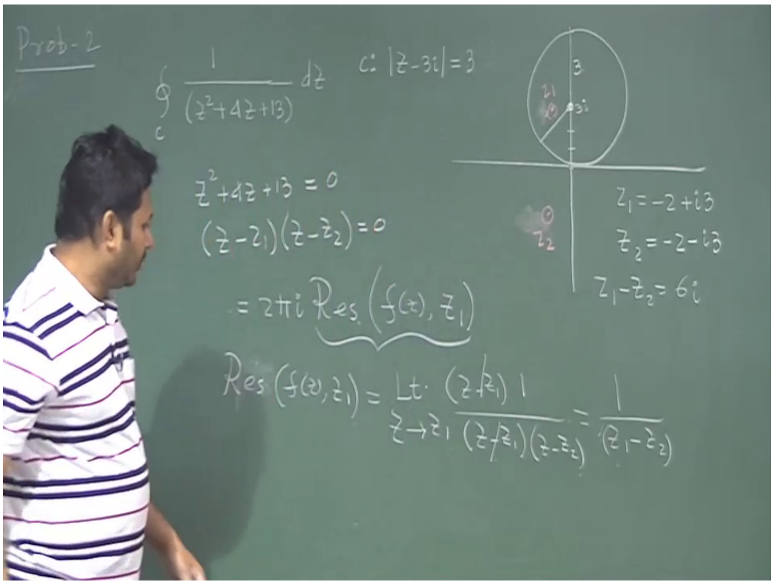

So, somewhere here; so, this is the mirror point of each other. So, it should be something here anyway. So, one point; so, this is my Z 1 point, this point is my Z 1 point and this 2 is Z 2 point. So, again I find that Z 2 point is outside Z 1 is inside since Z 2 is outside I should not bother about Z 2 point, I should only think about Z 1 point and then this result I can directly write 2 pi i residue because the result I now evaluate function of z the residue at the point Z 1 because Z 1 is inside. So, I need to figure out what is this now. So, residue of Z 1 is how much limit. So, residue of f z; Z 1 is limit z tends to Z 1 I multiplied z minus Z 1 with the function which is this. So, one divided by z minus Z 1 z minus Z 2 this term will cancel out now I put this here.

So, I will eventually have one divided by Z 1 minus Z 2 this is my residue. So, what is my Z 1 minus Z 2 if I. So, Z 1 minus Z 2 from here is seems to be 6 of i; 6 of i. Now I am just put the value here 6 of I and I am done. So, residue I figure out it is 6 of i. So, 1 by 6 of I rather one divided by 6 of i.

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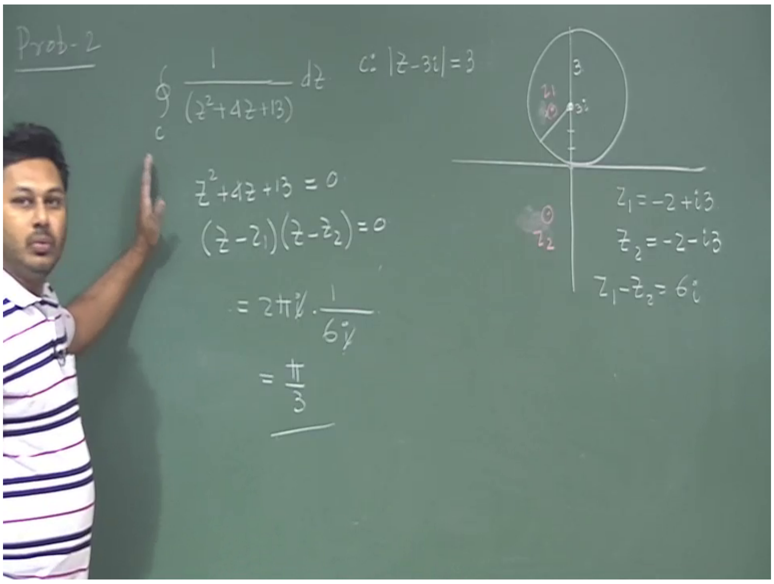

So, one divided by 6 of i is i; i-i will cancel out. So, it should be something like pi by 3 it seems to be something like pi by 3. So, we find the value of this quantity this integration pi by 3. So, we should not have I mean calculate the point at z equal to 0 as I mentioned this is outside, but if the radius is somewhere bigger which is which include this Z 2 point then we need to take care of Z 2 point also. So, here I will like to stop my class.

So, in the next class, we go on with that there are merely 2 examples; I showed there are few other examples. So, I will invest more time in solving different problems and try to try to find out; how you calculate integration by using this Cauchy’s residue theorem which is important and cover as much as possible different kind of problems with the different kind of problems. With this note, let me conclude the class here. In the next class, we will go on with this, we do more problems and till then have a good day.

Thank you for your attention.