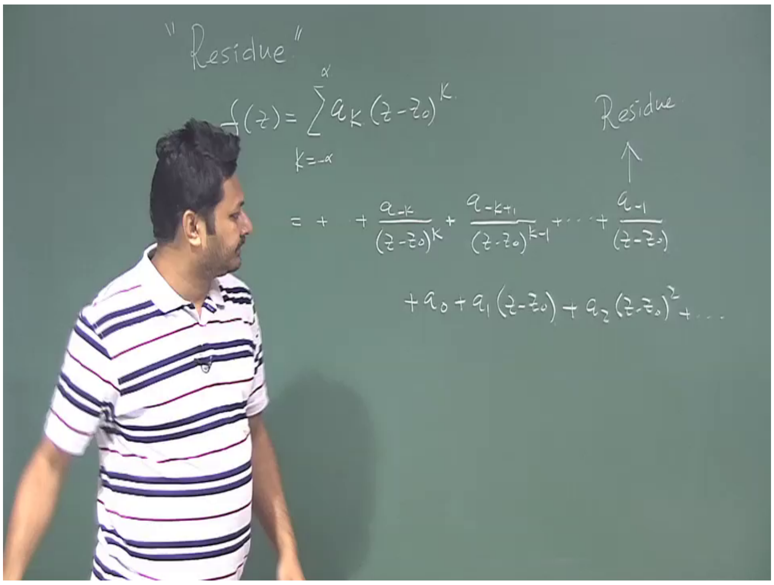
**Mathematical Methods in Physics-I**
**Prof. Samudra Roy**
**Department of Physics**
**Indian Institute of Technology, Kharagpur**

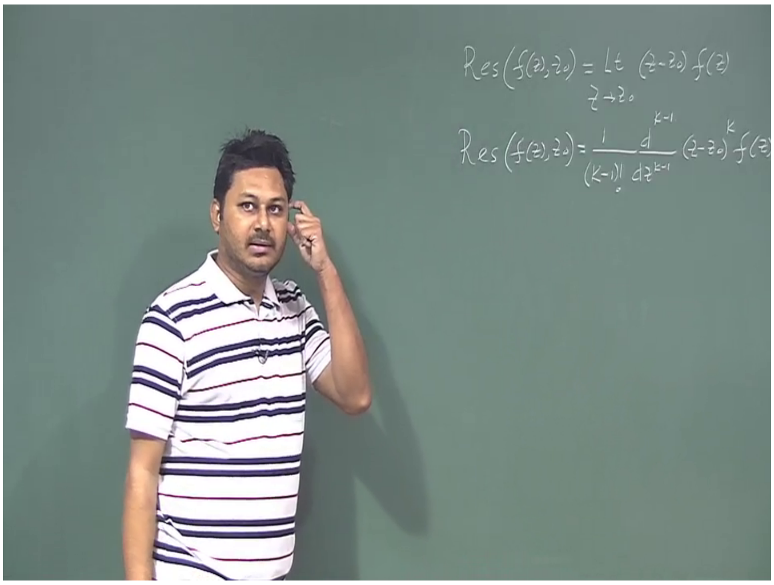
**Lecture – 55**
**Calculation of Residue for Quotient From**

(Refer Slide Time: 00:53)



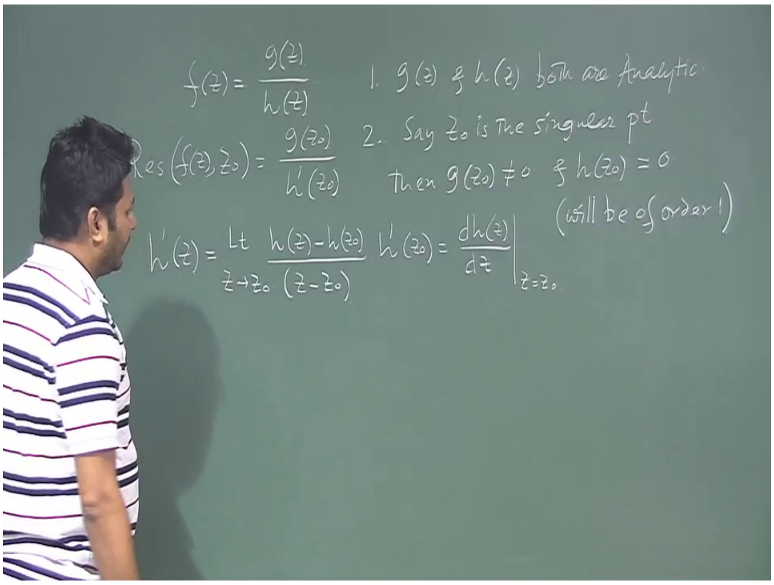

So, welcome back student to the complex analysis course. In the last class, we started something interesting called residue. If a function f z is expanded in terms of Laurent series, there will be many term associated with that so like this a minus k divided by z minus z 0 k a minus k plus 1 z minus z 0 k minus 1 and so on. And one term you will get which will the last term of the principal part which is something like this; up to this, this is a principal part because this z minus z 0 is in the denominator and then we will have some terms like this. So, we are looking for this particular term a minus 1, which we called the residue. This term is of our interest now there are few recipes we used last day.

(Refer Slide Time: 02:31)




So, let me once again write these recipes. So, first is if the function has a simple pole then we normally write this residue in this form say z 0, I want to find the residue. So, it is something like limit z tends to z 0 z minus z 0 function multiplied by the function. So, this is the one form; and second form was something like this. If the function do not have a pole of order 1, singularity of pole simple pole rather it has pole of order n or pole of order k, then it will be something like 1 divided by k minus 1 factorial d k minus 1 d z k minus 1 z minus z 0 to the power k function of z, it was something like this. And we use both the I mean we exploit these both the cases and find some problem. Let me erase this.

(Refer Slide Time: 03:54)

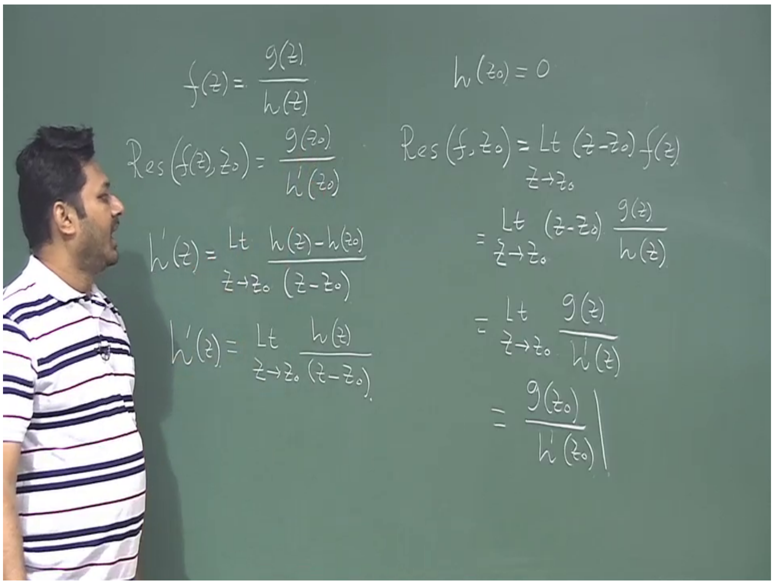



Today we will like to learn another way to calculate this residue say some function of z is gives as g of z h of z. In some special case, I can exploit this, whatever the thing I will going to derive. The special case is 1 g z and h z these two functions f z is a function, which can be represented at the ratio of two functions; and g z and h z both are analytic. Second thing is that the singularity if the singularity at 0, so z 0 says z 0 is a singular point say z 0 is a singular point then the g z 0 should not be 0 and h z 0 should be equal to 0. And this 0 will be of order 1 that means, we should have a simple pole at z equal to 0 for h, so that the function is not analytic at z equal to 0, it is obvious.

So, we have two condition in our hand; if this two condition satisfied then we can derive something from this. Then let me first write the result then residue of this given function at z 0 point should be equal to g of z 0 divided h prime of z 0, where h prime is a derivative h prime is a derivative of h function. So, h prime z 0 is the derivative of h function which is a function of z with respect to z at z equal to z 0 point. So, this is the recipe. So, this recipe is useful, we will see that. Last day I just show one example; so today we will exploit more examples and try to find out how these things will be applicable, but before that we need to show that this is the correct thing. So, how to prove that?

So, in order to prove, so let me first how define how h prime z is given. It is limit z tends to z 0 h of z minus h of z 0 divided by z minus z 0. So, this is the limit. So, let me erase this it is not of any, so let me erase this part also, so that I can have the place to write something. So, this is the derivative I can write in this form. So, now, in the condition we mentioned one thing I just erased that that h of z 0 is equal to 0 that is the one condition that we have.

(Refer Slide Time: 07:54)

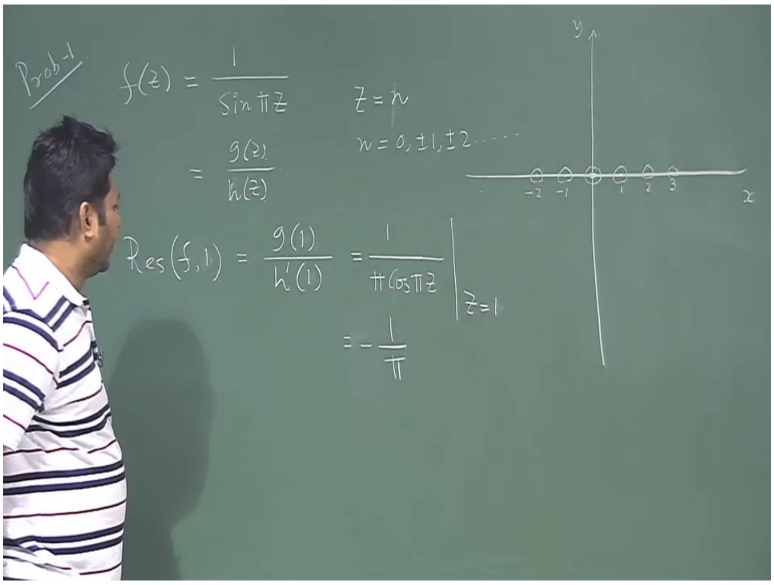



So, here we have something h of z 0, so this quantity going to vanish. So, eventually this is quantity is limit z tends to z 0 h of z divided by z minus z 0, this quantity I have. Now, from the concept of the recipe, so now, I going to use whatever the whatever the recipe we have standard recipe that residue of function at z 0 is nothing but limit z tends to z 0 the same thing I am writing once again which I have just erased few minutes ago. This is blindly if I use the formula then this will be the formula for finding the residue.

Now, I have something extra in my hand and f z is given in some form. So, let me write down the f z, so z minus z 0. And my f z is the ratio of these two functions, so I will going to write that, so g z by this. Now, if you look carefully then you will find that this quantity is already here this quantity is already here with a limit z tends to z 0 h prime z is already there. So, if I replace this, this divided by this from here then it will come like limit z tends to z 0 g of these things divided by h prime of z, I replace this entire thing from here. So, h prime is given by this. So, I just replace this quantity here to this. So, I will have something.

Now, I am allowed to put the limit if I put the limit, it will come like g of z 0 divided by h prime of z 0 which is our recipe and which is already written here. So, this is the same thing that I have written here and I just prove that. This is one another way you can consider to find out the residue, and it is interesting way to find out the residue provided these conditions are satisfied. For example, g of z 0 should not be 0 and h of z 0 should be 0; otherwise it will not going to be applied. After having this form, let us try to use this in some practical problems, and that will be useful.

(Refer Slide Time: 11:33)




So, let us try to find out few problems where we can use it. So, last day we have some problem in our hand So, let me remind the problem. So, the problem was something problem 1 say or say example 1 whatever, f z was given as 1 divided by sin of pi z. This is the function that is given to us. Now, you can readily understand that the singularity there are many singularities associated with these functions. For example, for all z say z equal to 0 will be singularity, and all integers of z is a singular point. So, z equal to plus minus 1 singular point, z equal to plus minus 2 and so on. So, this is my x-axis and this is my y-axis this is a z-plane. In the z-plane, I have a singular point here for this function; if this is 1, I have here, I have some singular point here 2, it is 3, I have here and so on here also minus 1 here I have minus 2 here and so on.

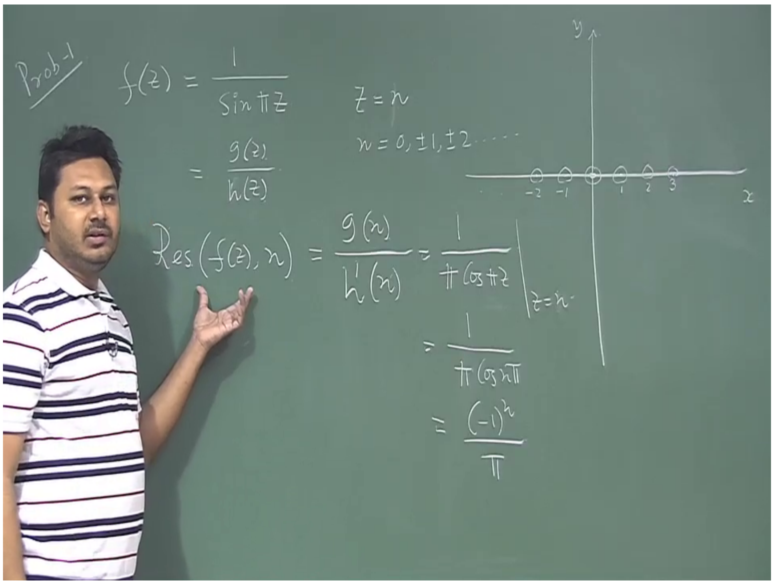
So, we have many discrete singularities here, and if I want to find out what is the singularity at z equal to 0 point for the time being then let us find. So, the function if you look, it is look like g of z divided by h of z, because g of z is 1; h of z is this sin pi z. And the condition suggest that when the singularity is as say singularity I am taking just one point, singularity at z equal to 0. So, if the singularity is z equal to 0, then g of 0 obviously not equal to 0 it is just equal to 1, because the function is constant. What about this h of 0 is 0. So, these two condition is satisfied both the things are analytic in sense there is no problem with the analyticity even at z equal to 0 point sin z, it will be analytic because at z equal to 0 point this function vanishes. So, there is no problem in the analyticity.

So, then I readily can say the residue of this function given function at 0 point, I could not do our recipe it should be g of 0 divided by h prime of 0, because that is the recipe I just derived that at the singularity whatever the function of g at the singularity whatever the derivative of function, h function at that point. So, g of 0 here is one no problem with that h function I know h function is sin pi z. So, if I make a derivative of this function with respect to say z, I will have pi cos of pi z, it should be evaluated at z equal to 0 point. So, when I evaluate at z equal to 0 point, the result that is in my hand is 1 divided by pi so that means, the residue of this particular function seems to be 1 divided by pi.

Now, if I do the same thing for z equal to one point then the result will be something different. So, let me do that also because I am taking I am trying to find out the residue at that particular point, but here also we have some problems. So, here also I have some points where we should have some kind of singularity. So, if I change that so what essentially will change. Now, the singularity the point of singular point is at z is equal to 1. If I still apply this formula then I need to just change this at 1, because now I am evaluating the singularity evaluating the residue at z equal to one point. So, this is 1, this is 1, this has to be 1, and now I am going to evaluate at z equal to one point. If I do that then I will find that it is cos pi which is minus 1. So, the result will change as minus 1 of 1 by pi. So, at z equal to 0, it is 1 by pi; at z equal to 1, it is a minus 1 over pi.

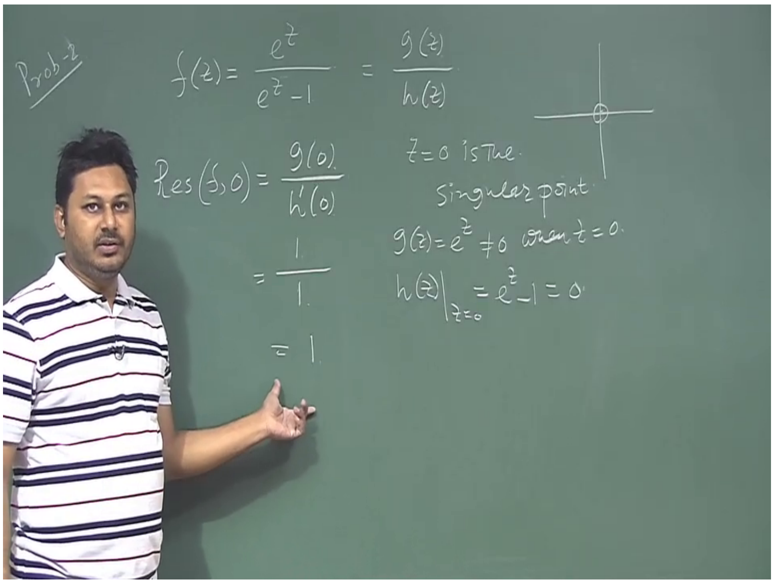
So, we can generalise these things because I know where the singularity lies. The singularity lies at all z equal to n point n is plus minus 1 n is n is plus minus 1, 2, 3, 4 and so on. So, n here is the integer. So, 0 plus minus 1 plus minus 2 and so on. So, this is the general singularity I have, I am taking 1 plus minus 1 and plus minus 2, 0 plus minus 1 plus minus 2 and so on I am taking the general one.

(Refer Slide Time: 17:52)




So, in that case how to tackle these things the same thing. So, here the residue will be f z at n point and the function is g of n divided by h prime of n which is 1 divided by h prime is pi cos of pi z, z will be evaluated at pi point at n point. So, now if I pit it will be 1 divided by pi cos of n pi. And we know this cos of n pi it is minus 1 whole to the power n divided by pi. So, it is the most general form of finding the residue, because there are many residue infinite number of singular points are there. If I take all point together then this will be my result. I hope you understand how this kind of problem will be tackled and how beautiful this form is and you can exploit many things with that.

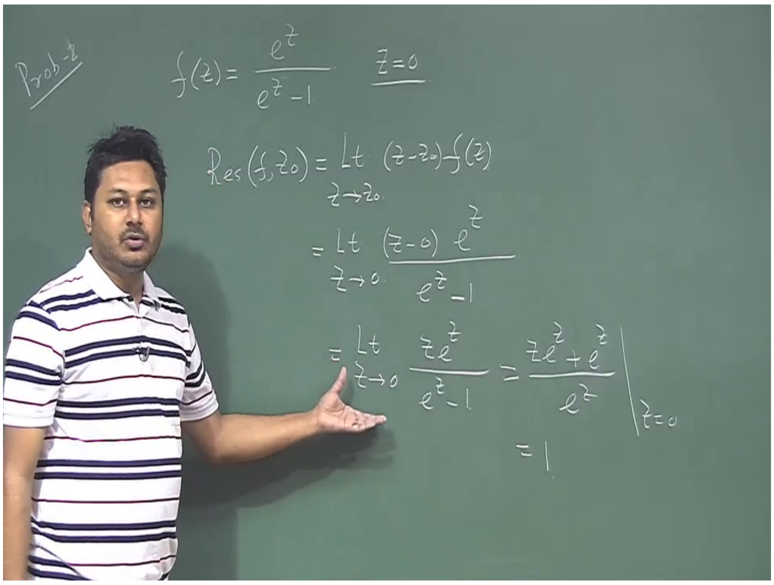
(Refer Slide Time: 19:15)



So, we will go with few other problems, for example, this one - problem 2, f z is equal to let us see what problem is in our hand e to the power z divided by e to the power z minus 1. This is the problem that is in our hand. Again if you see I can write this problem as g function of z and h function of z that is the first thing. Second thing is I have the singularity at z equal to 0 point. So, though both functions are analytic no problem with that rather they are entire function because they have the singularity only at z tends to infinity point, so that is why these functions are both the functions are entire functions, no problem absolutely no problem with that.

So, the next thing is this singularity. So, z equal to 0 point is the singular point so that means, at this z equal to 0 means at here the function is blowing up and I should have a singularity only in that point. Also you should note that g of z which is e to the power of z is not equal to 0, when z is equal to 0. So, my first condition satisfied that g of 0 is not equal to 0. And also h of z at z equal to 0 e which is e to the power z minus 1 is 0 that is my second condition is also satisfied. So, that means, I can apply my standard whatever the recipe I have and this recipe suggest it will be residue of this function at 0 point is nothing but g of 0 divided by h prime of 0. If I apply then g of 0 is 1 h prime when I make h prime then it should be e to the power z only this one term will vanish and if I now put then it will be 1. So, my final result will be 1. So, the residue comes to be 1. Also you can think this problem in a different way and this different way is this, I mean let me just give you the essence of that.

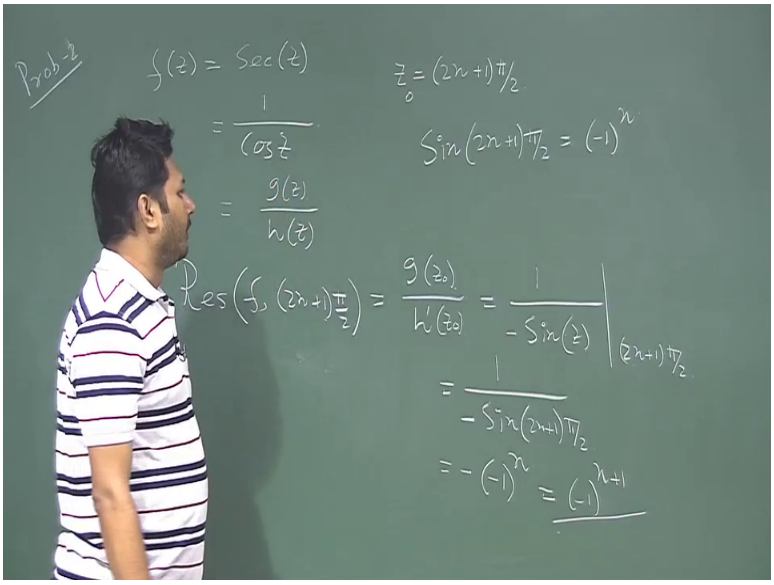
(Refer Slide Time: 22:04)



So, for example, residue of function some z 0 point I know it is represented by limit z tends to z 0 z minus z 0 multiplied by the function f z, this is our standard recipe that I have. If I still apply this here then still I am getting some result. So, let us find out. Limit the singular point is at z equal to 0, this is my singular point. So, limit z tends to 0, I multiply z minus 0 to the function whatever the function is given to me it is e to the power z divided by e to the power of z minus 1. Now, this limit I need to evaluate that is all. So, eventually I have z e to the power of z divided by e to the power of z minus 1. Now, this is a 0 by 0 form, when z tends to 0 this term will go into vanish; when z tends to 0 this lower term is also going to vanish.

So, in order to evaluate these things we normally use this L-Hospital rule. So, this L-Hospital rule suggest that I can derive this and derive this and then put a limit I hope all of you are aware of this rule in the 0 by 0 form. So, if I do then the derivative of these things will be at z e to the power z plus e to the power of z divided by e to the power of z evaluated at z equal to 0. If I do then I will still get the same result. So, z equal to z 0, if I if you put this term will going to vanish, e to the power z, z equal to 0 will be 1, e to the power z, it will be 1, so just one in our hand. So, the residue of these things comes out to be 1. Now, go to the next problem it is interesting this form.

(Refer Slide Time: 24:25)




So, let me do few problems, so that you are familiar with this kind of. So, the next problem which is in our hand is something f z is similar almost similar kind of problem is sec z this simply this. So, now, we need to find out the singularity of this function first to evaluate the residue. So, this function make it simple cos z. So, where are the z points where we have zeros, where are the z points. So, the z essentially be 2 n plus 1 pi by 2 these are the points where this function is going to vanish. So, if you have remember the function cos function, it will be something like this. So, 0 here n pi by 2 then here it is pi by 2 then pi and then again 0, so it is 3 into pi by 2 and so on so that means odd for every odd multiply of pi by 2 gives you the 0 of cos. Normally, I remember in this way; I do not remember in the general way. Anyway I just you show where the function diverges. So, this is the point where the functions would diverge.

Now, again we have the same problem in our hand. So, if I try to find out what is the recipe what is the residue of these things I can consider this function to be as g z divided by h z, again g z at this point is nothing but 1, so it will not be 0. And this is 0 because we are doing the singularity at that point it is singularity. So, the recipe is residue f I am trying to evaluate that point 2 n plus 1 pi by 2 this is my point where I am trying to find out the residue is equal to if I apply the formula that we derived today is a g of z 0 divided by h prime of z 0. Where h prime z 0 z 0 is this point singularity I have derived I write this singular point as z 0 point.

So, now, if I do then I find this is 1, and derivative of these things is minus of sin of z evaluated at the point 2 n plus 1 pi by 2. So, this quantity 1 divided by minus of sin 2 n plus 1 pi by 2. So, I need to find what is the value of sin 2 n plus 1 pi by 2 odd of pi by 2. So, we know that odd of pi by 2 is minus 1 to the power n for sin when sin pi by 2 n equal to 0 then I have 1; when n equal to 1, then sin 3 pi by 2 which is minus 1 and so on. So, this quantity is minus of minus 1 to the power n or minus 1 to the power n plus 1. So, this is the result we have by deriving this particular form.

So, here I should stop. So, today we have learnt a very interesting thing that previous we know there are two recipes that will help you to find out the residue. If a function is given, you first need to find out where is the singularity and then you calculate accordingly. You can even calculate the residue by expanding the function in the Laurent series that will going to use I will going to use that in the may be in the next class to show that also we have done in the previous class either. Today we have learnt that if the function is given in this form; and if z h z has a simple pole, then you can also derive the residue in other way which can be useful in some typical functions. So, we show few problems as an example and with that I should stop the class here. And in the next class, we will start something quite interesting which is called Cauchy's residue theorem from where you can calculate the integration and all these things. With this note let me conclude the class here, see you in the next class.

And thanks for your attention.