Mathematical Methods in Physics-I
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Lecture – 54
Classification Of Residue

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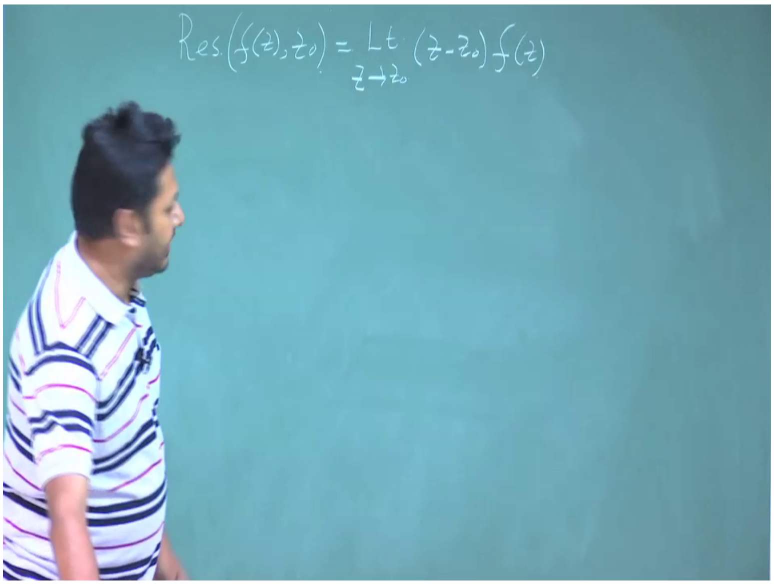

So, welcome back to the next class of complex analysis. In the previous class, we started very important concept called residue. And if a function has a simple pole then we find the residue of that function f z at z 0 can be represented as limit z tends to z 0 z minus z 0 f of z that that was the recipe of that that we know. Now, we extend this and try to consider what happened if the function has singularity of order n or order.

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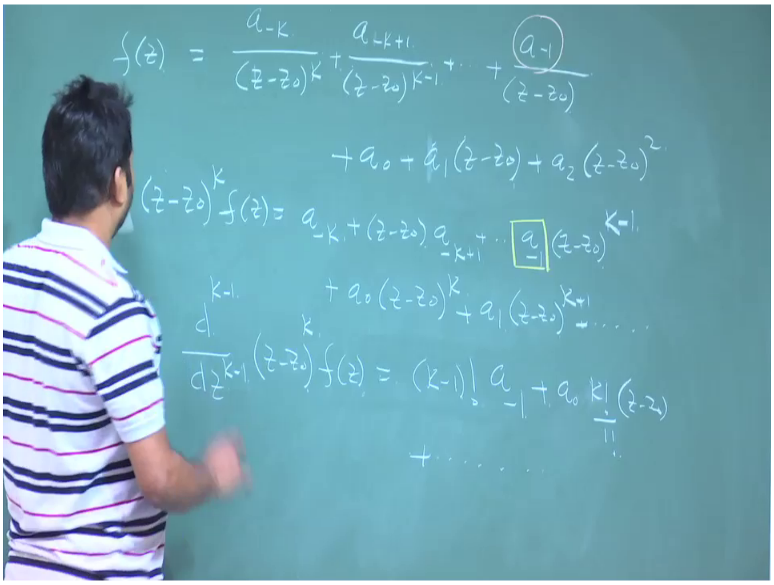

So, if the function has a singularity of order k then this function f z has to be represented in this particular form z minus z 0 to the power of k n plus a minus k plus 1 z minus z 0 k minus 1 plus a minus 1 z minus z 0 this is the principal part. And the analytic part is a, a 1 z minus z 0 plus a 2 z minus z 0 square and so on. So, this is expansion is valid when the function has a singularity at z equal to 0 and the pole is of order k, so that is the pole of order k.

So, now, the question is how to deal with these things how to deal with these things how to find out my goal here is to find out find out this quantity a 0 somehow. So, the very first thing that we can do that we multiply z minus z 0 to the power k to both the side. If I do z minus z 0 to the power k you multiply with a f z then the first term will be a minus k in the right hand side. Second term will be z minus z 0 a minus k plus 1 plus 1 I have a minus 1 multiplied by z minus z 0 to the power k minus 1 and plus a 0 z minus z 0 to the power k plus a 1 z minus z 0 k plus 1 and so on. I will just multiply in the previous case I multiplied z minus z 0. In this case if I have a pole of order k, I just multiply z minus z 0 to the power k to the entire function. And expansion of the function should be something like this. Once I have the expansion of the function then still this quantity is sitting here which I need to find out.

So, what we will do I need to reduce these things. So, in order to reduce these things best way to derive that left hand side to make a derivative of left hand side with respect to z up to k minus 1 time, so that all these things will gone, so that is k minus 1 z k minus 1 right hand side is z minus z 0 to the power k f of z. Once I make a derivative with k time all these term will go away only term that is here should be k minus 1 factorial a minus k:. So, every time when I make a derivative this power will come and for that k minus 1 factorial term will be there, then I will have one term analytic part, I will have a 0 and again this k minus 1 k into k k into k minus 1 and all these term, so here. So, I will have factorial k here divided by factorial 1 z minus z 0 to the power z minus z 0 and so on the right hand side I will have this term. Now, what is the next thing because I already have this term in my hand; now if I put the limit at z tend to z 0, so all the analytic part will go away.

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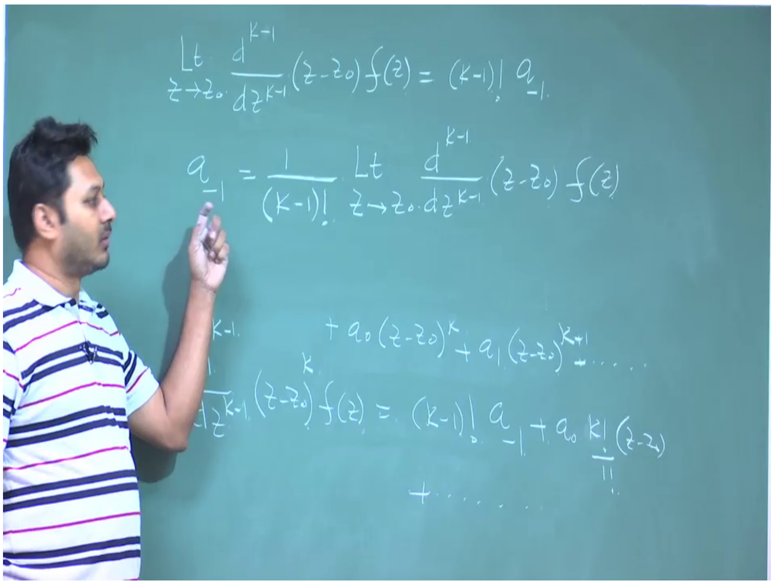

If I do, so let me do it here, so I will do that. So, limit z tends to z 0, this quantity k plus 1 k minus 1 dz k minus 1 z minus z 0 function of z if I do then in the right hand side I only have one term which is k minus 1 factorial a minus 1. From that I can now extract a minus 1. So, a minus 1, which is my residue is now extracted for a function which has so a minus 1 will be how much, 1 divided by k minus 1 factorial limit z tends to z 0 d k minus 1 this is my recipe, this is my recipe to find out what is my a minus 1. Once I know the function has a singularity at z equal to z point with pole of order k. So, I have two formula in my hand; and if I write these two formula side by side, this formula will be something like this.

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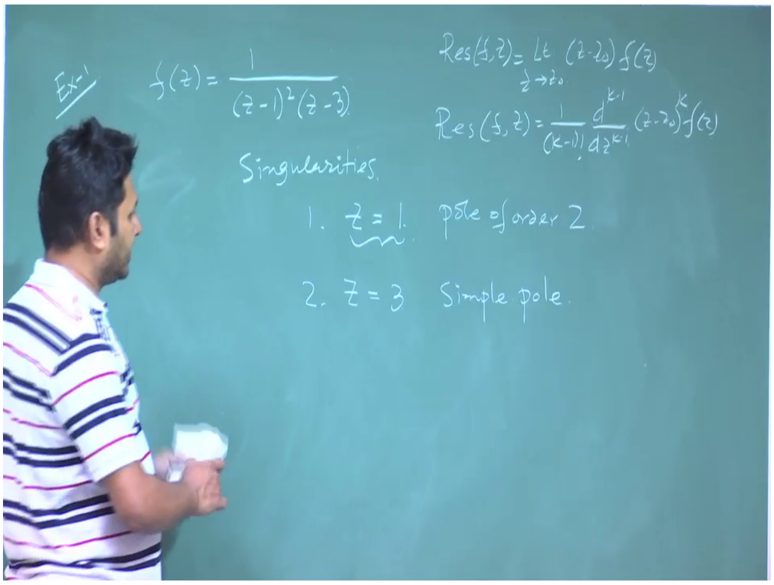

So, let me write it here. So, residue f z is something limit z tends to z 0 z minus z 0 f z this is one recipe. And when we have a simple pole residue f z is 1 divided by k minus 1 factorial d k minus 1 d z k minus 1 z minus z 0 to the power of k f z, this is my recipe two, two recipes in my hand. So, I will going to use to find out the residue. So, now, go directly go to the problem. So, let us start with few simple problem, and then we will try to understand how these things are working.

So, first example - example 1, f z is equal to 1 divided by z minus 1 whole square z minus 3. Suppose this is the thing I need to figure out the residue at z equal to minus 3. So, first how many singularities are there, singularities. So, first I need to find out singularities. One is, z is equal to 1 this is the first singularity pole of order 2. So, here this singularity is not a simple singularity it has a pole of order 2. However, second singularity is at z is equal to 3 point at z equal to 3, but it is a simple pole. The formula suggests that if it is a simple pole, the formula is quite straight forward and simple. So, let us try to find out first the residue at 3.

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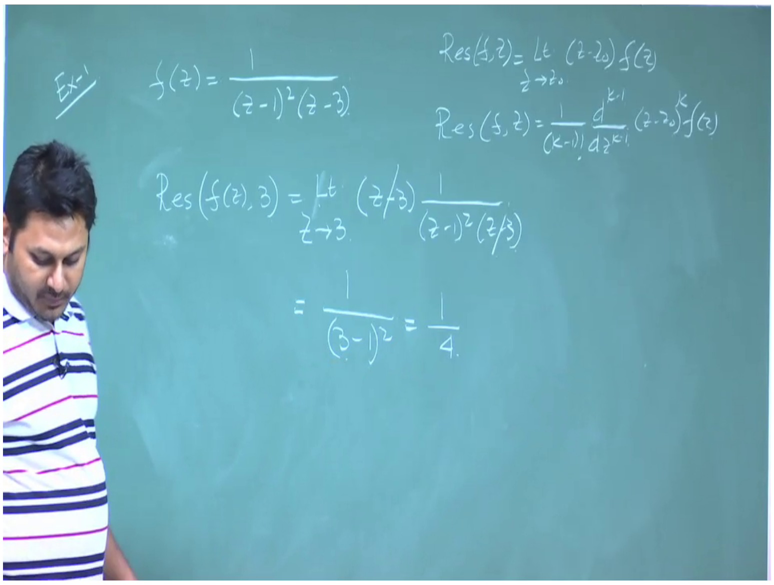

And mathematically it should be written as residue of the given function f z at 3, what is the recipe, I will multiply I will take the limit at z tends to 3, multiply the function z minus 3, the function here is 1 divided by z minus 1 square z minus 3. I will do that and it will cut here and here. And now I put the limit to the function, it will be 1 divided by 3 minus 1 whole square. So, I will find 1 by 4. So, my residue at 3 is comes out to be 1 by 4. What about the other residue because other residue is at z equal to 1, but having a square term associated with that; that means, it is not a simple pole. So, I need to use the second recipe this one.

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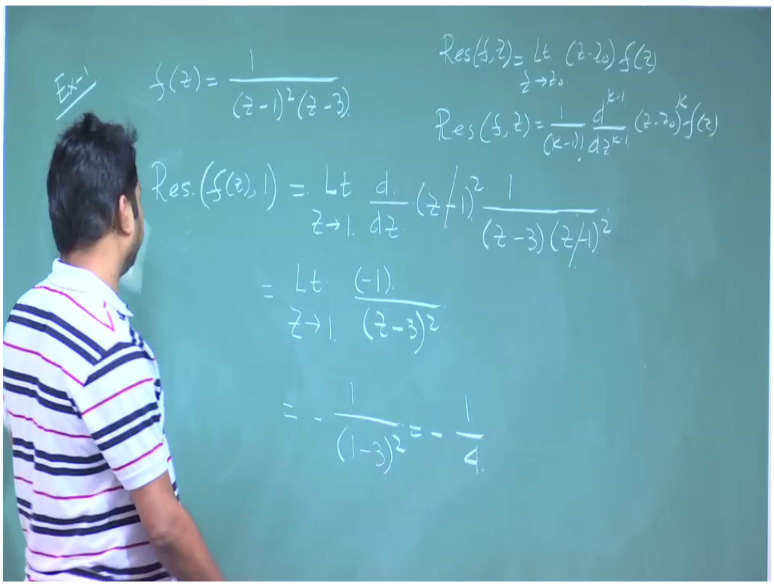

So, again residue of f z at point one is how much I need to take the limit z tends to 1 and then take a derivative first order derivative because since k is equal to 2, 1 divided by 2 minus 1 it is 1 factorial, so this term is one. So, here I will put one, so that is why I dint put anything, but it will be the first order derivative because the derivative of order should be k minus 1 if the singularity has the pole of order k. So, here pole is 2 so that means, I need to do the derivative one less than that. So, first order derivative. So, which quantity I need to derive z minus 1 multiplied by the function, function is this.

So, now, this quantity will cancel out that is why we multiply that and then I need to make this limit, limit z tends to 1 and then derivative of this quantity derivative of this quantity is 1 divided by z minus 3 square. Once I make 1 divided by this thing, so minus 1 term will somewhere come here because of this derivative. So, I will have this. So, now I will put z the value of z here. So, here minus of 1 divided by value of z here is 1 minus 3 square which is minus of 1 by 4. So, first case my residue was 1 by 4; in the second case, the residue comes out to be minus 1 by 4. This is the first example and blindly I am using these things. So, I am getting some results.

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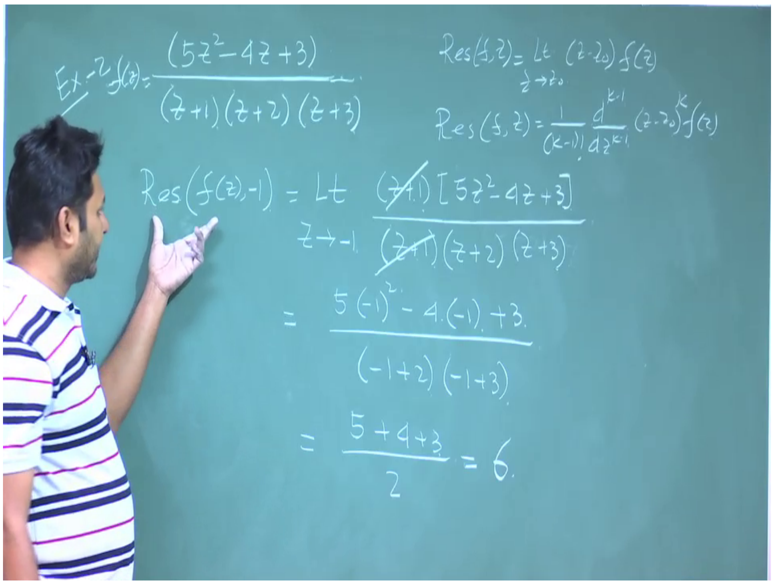

So, let us do similar kind of problem few problems, so that you are familiar with whatever the residue you are getting. So, next problem is again quite straight forward. So, example 2, the function is given as let me check once again yeah 5 z square minus 4 z plus 3, the numerator and the denominator the function f z by the way it is this and it is z plus 1 z plus 2 and z plus 3. This problem looks very straightforward because whatever the value are given in the numerator all have the simple pole. Once we have a simple pole, the problem is very, very simple; since there is no derivative is required and all these things. So, let us do that thing and try to find out. So, this is some sort of exercise and it is a good exercise.

So, let us try to find out residue of this function f z at one point. So, where the residue mind it at minus 1, minus 2, minus 3. So, residue at minus 1, first I try to find out. So, my formula suggest, it is z tends to minus 1, I multiply the entire stuff with z minus 1 multiplied by f z square minus 4 z plus 3 divided by z plus 1 here I need to multiply because my residue at minus 1. So, z minus minus 1, so z plus 1 z plus 2 and z plus 3. z minus 1 z plus 1 z plus 1 will cancel out then I need to just evaluate this function at z equal to minus 1 that is what. So, five multiplied by minus 1 square minus 4 multiplied by minus 1 then plus 3 divided by minus 1 plus 2 minus 1 plus 3, this quantity. Next, it is just 5, it seems to be plus 4 plus 3 in the down stair I have 2 minus 1, so it is 1; 3 minus 1, it is 2. So, I have 2. So, up stair it is 5 plus 4, it is 9 plus 3 - 12 divided by 2, so we will have the value 6. So, my first residue at z equal to minus I minus 1 is 6.

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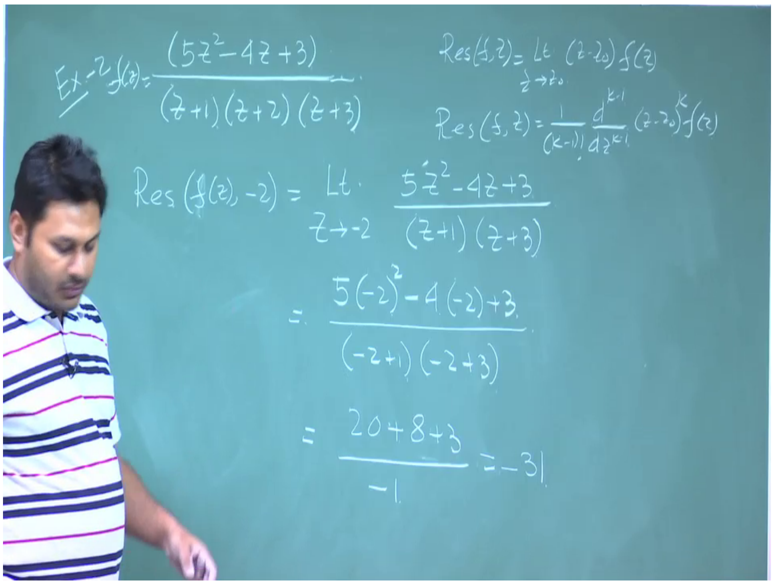

Let me find out the other residue quickly let me find out the other residue quickly. So, residue at residue function of z at minus 2, which is limit z tends to minus 2. Once I multiply z minus 2, z minus 2, z minus 2 will cancel out, I am not doing that part I am just directly writing after removing z plus 2 term which is obvious. So, I just need to evaluate this quantity that is all I need to evaluate this quantity at z equal to minus 2 z plus 2 z plus 2 will cancel out because I multiply z plus 2 and z plus 2 is already there in the down stair.

So, now this 5 minus 2 square minus 4 minus 2 plus 3, I have this. If I evaluate this quantity as minus 2, so now, it is minus 2 plus 1 minus 2 plus 3. So, here I have 5 into 4, so 20 plus 8 plus 3 divided by minus 2 plus 1, so minus 1 is here and then 3 plus 3 minus 2 so that means 1 is here. So, I have minus 1 here 1 minus multiplied by minus 1. So, it will be seems to be how much minus 31, 8 3 11 and then 20, it is minus 31, minus thirty one is the value of the residue at function z equal to 2. One is left. So, this is just a lengthy problem, but straightforward you do not need to do the derivative of that

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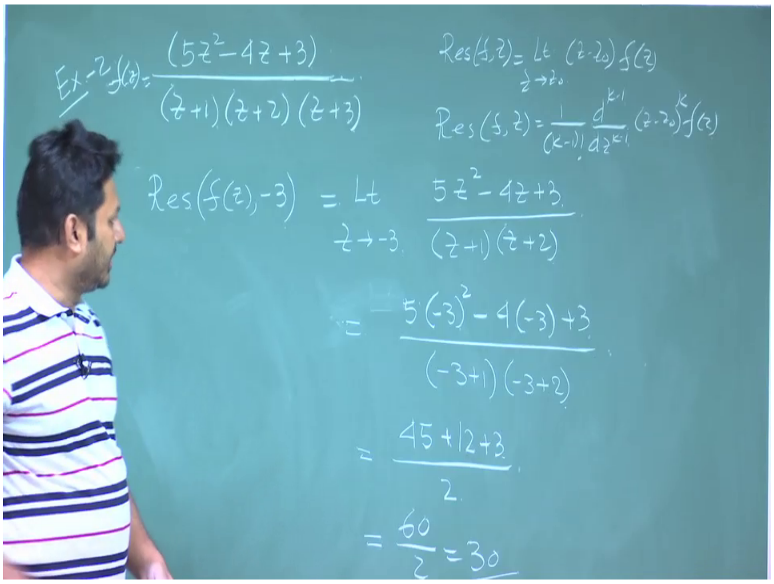

So, residue finally, I have to calculate the residue of f z at minus 3 point. So, again it will be just the limit z tends to minus 3 minus z plus 3 z plus 3 will cancel out. So, I have five function whatever the function is given 5 z square plus 4 z plus 3 divided by z plus 1 and z plus 2; these two terms is in my hand. And then I put the limit directly if I put the limit directly then I have 5 minus 3 square minus 4 minus 3 plus 3 divided by minus 3 plus 1 minus 3 plus 2, so it is 3 9, so 45 minus 12 plus 3 divided by this quantity is minus 1, this quantity is minus 2, minus 2 multiplied by minus 1, so we have two. So, 45 plus these things it is plus so 3 into. So, it seems to be 45. So, it is 60 by 2, seems to be, I will have

30. So, the residue of this quantity at z equal to 3 is z equal to minus 3 is 30. So, we have few straight forward thing.

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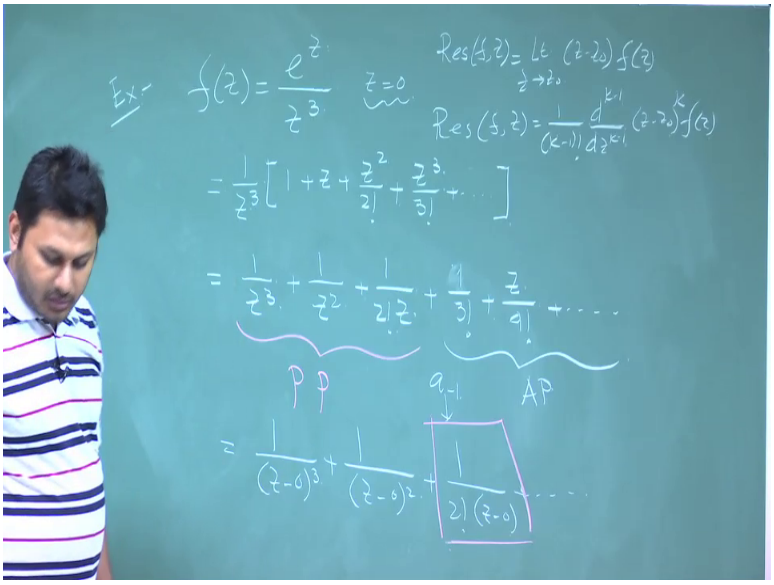

So, now, I have another kind of residue and this residue is something. So, let me check is there any other problem. Let me do once again one example before going to that. So, function of z say e to the power of z divided by z cube this is one function. So, now, I can do one thing that so far we are blindly using this residue structure. So, for this particular problem, we will check because residue is nothing but the coefficient of Laurent series, the coefficient of 1 by z minus z 0 in Laurent series. So, if I evaluate this as a Laurent series then I can have the value of the residue. So, let me do that. So, if I expand this as a Laurent series around z equal to 0 around this point in some limit with z less than 1 or something then it will be z cube and e to the power z is 1 plus z plus z square plus factorial by factorial 2 z cube by factorial 3 and so on. Mind it, singularity is at around z equal to 0.

So, this function e to the power z is a entire function. So, there is no singularity of this. We have a singularity of z here at zero point. So, this is the point where we have a singularity. So, I am excluding this point I am trying to find out how the expansion is. So, excluding this point is entire part the function is analytic; the function is analytic except the point z equal to 0. So, I am expanding these things in this region, if I expand this will look like this. And then I have few terms z cube plus 1 by z square plus 1 by 2 z factorial 2 z rather than 1 by factorial 3 then z by factorial 4 and so on.

So, now, you can see that this is the series where we have the principal part and the analytic part. So, this is my principal part – PP, 1 by z 0 1 by z minus z 0, if I write even this way and around z equal to z 0 the way. So, it is z minus 0 cube plus 1 divided by z minus 0 square plus 1 divided by factorial 2 z minus 0 and so on and this is the analytic part AP. So, in the principal part this is the term which is interesting for us because this is the term where we have a coefficient of z minus z 0 that means, the coefficient is 1 by 2. So, 1 by 2 should be my residue. So, this is the coefficient this is eventually a minus 1 1 by 2. So, without blindly putting this formula by just expanding I can also find out the residue.

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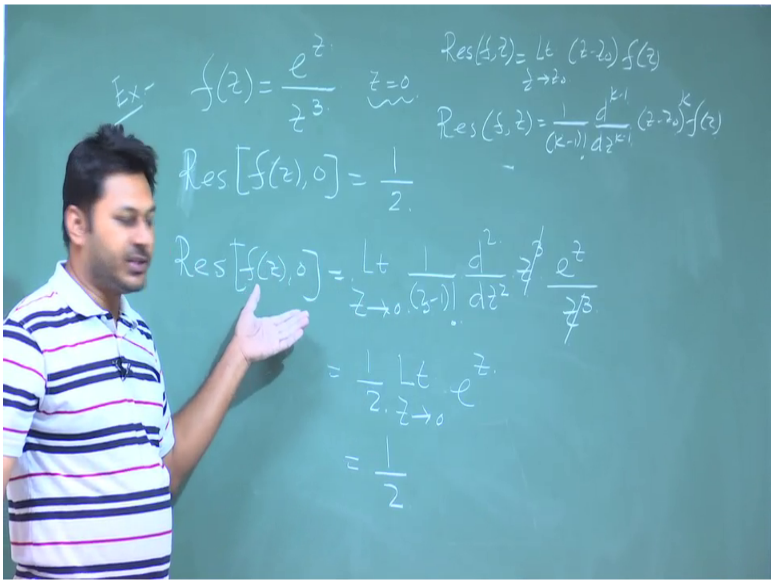

And here in this case the residue of this function residue of function z at zero point comes out to be half. Now, if I try to blindly put this formula, this formula should be here, it is written. So, it has a pole at z equal to 0, and the order of pole is 3. So, now, I am going use the formula and check whether both the cases I am getting the same result or not. So, residue using the formula the residue of function z at 0 is comes out to be something like limit z tends to 0 1 divided by 3 minus 1 factorial d 2 dz square z cube and the function e to the power z divided by z cube. So, this is the expression if I blindly write this and try to find out the residue. So, z cube, z cube will cancel out. If I make a double derivative, so 1 divided by 3 minus 1 factorial is nothing but 2, then limit z tends to 0, this function if I do e to the power z, if I make a derivative first derivative minus second derivative this function will be simply e to the power z. Then if I put the limit then I will have e to the power 0, so e to the power zero is 1. So, eventually I am having the same residue that I am getting here; so half, here half.

In the first case, I calculate the entire stuff by expanding these things; and in the second case I am trying to find out the same thing, but here I am getting the same result and exploiting the expression exploiting the expression that is written in that side of board; both the cases I am getting the same results. So, with this note, I would like conclude here. So, so far we know that how to calculate the residue if the function is given and if function have some finite singularity and this singularity has some order and all these things, but apart from that there are few other functions where this procedure may not be suitable.

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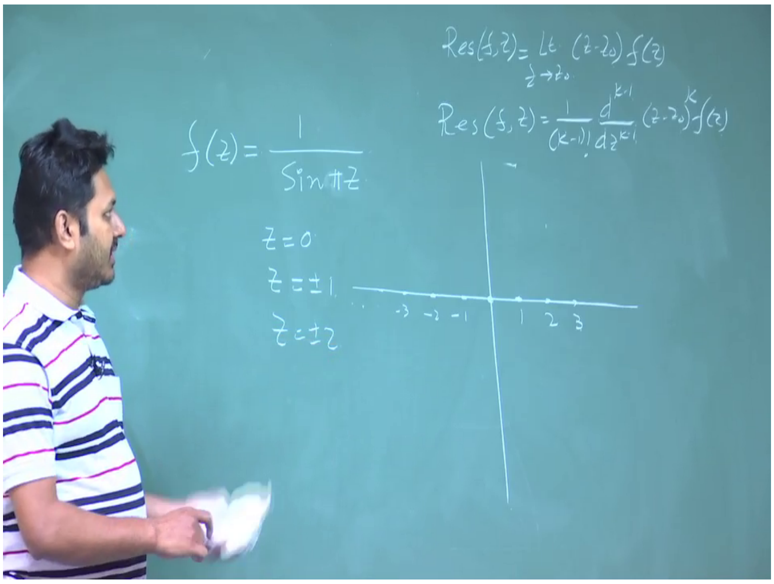

And in that case we need to use something different and for example, a function f z I am just giving you the functional form may be next step I will calculate that 1 divided by sin say sin n pi z. So, here we have a singularity at z equal to any integer if you find. So, z equal to 0, we have a singularity; z equal to plus minus 1, you have a singularity; z equal to plus minus 2, you have a singularity. So, all the singularity are here in this point 0 here, 1, 2, 3, 1, 2, 3 minus 1, 2, 3 minus 1, minus 2, minus 3 and so on. So, the function is analytic in this intermediate region and apart from that this function has an infinite number of singularity. Now, if I try to find out what is the residue of this particular function in any of this singular point then how to figure out, so that is an interesting problem.

So, I will try to do that in the next class because today we do not have much time to do that; in the next class we will start from here. And then we try to understand why the residue is so important in this particular case; that means, in complex analysis. We try to find out why it is. So, important because it gives me directly solution of a closed integral even if some singular point is there I can exclude this singular point and I can find out what is the value of the integration without doing any kind of so called integration just using some formula. So, with that note so let us conclude here, see you in the next class what we start where we start the residue once again and try to find out how to calculate this special kind of function and then try to understand what is residue and how the residue is important in calculating integration.

So, thank you very much for your attention. So, see you in the next class.