Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 53
Laurent Series Expansion (Contd.) and Concept of Residue

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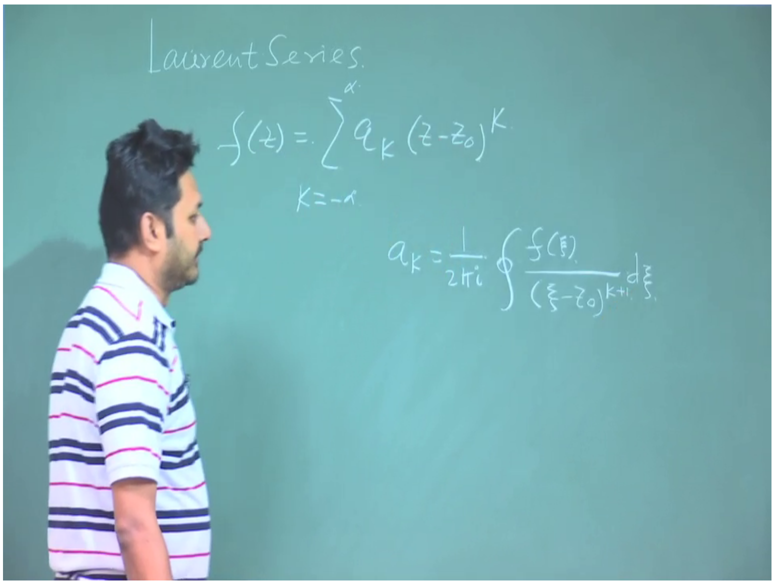

So, welcome back student to the next class of complex analysis. In the last class, we started something called Laurent series. So, let me once again remind you if f z is a function, I can expand this function in this particular power series, where k running from minus infinity-to-infinity; and it is restricted over some annular region. And the coefficient of a k was something like 1 by 2 pi i closed integral f psi psi minus z 0 d psi with k plus 1 here. And then we expanded so that was the statement and then we tried to expand few function for a given in some given region.

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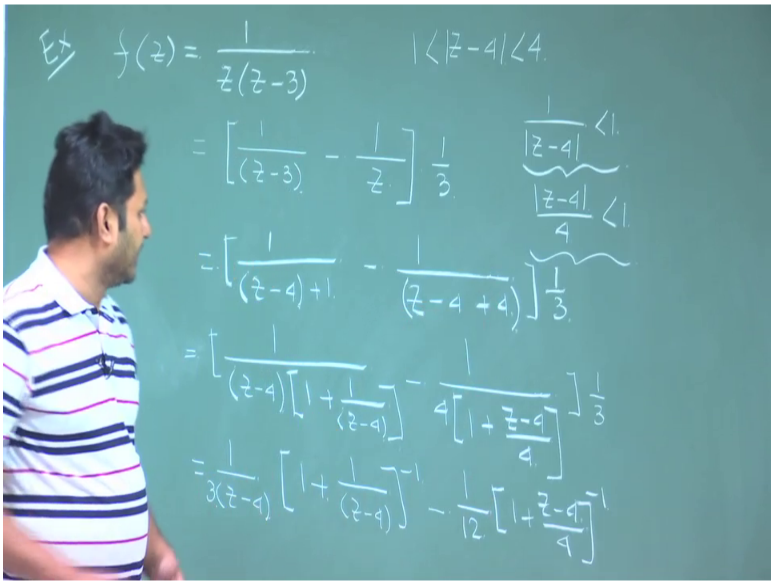

So, we will go with that. So, we need to solve few hours examples also. So, let me go with that for more examples. So, f z, this is example another example of Laurent series. So, it is something like z multiplied by z minus 3 and the region that is given is something like this z minus 4 with this. So, this is a restricted boundary and we need to expand our function with this condition. Now, if I want to do that, we will do the way we have already done this thing so far. So, we will just separate it as say 1 by z minus 3 minus 1 by z; if I do then this z comes here z minus 3 comes here z will cancel out, but one extra three term we will have here. So, this 3, I need to take care with this 1 by 3 term. So, this 1 by 3 is this.

Now, I need to make my problem in such way that I can have this z minus 4 term and this. So, here if you remember that 1 divided by the restriction suggest me that 1 divided by z minus 4 this is less than 1 that is one condition; and another condition is this quantity divided by 4 is less than 1 as well. So, I need to use this two conditions and then expand this, so that is the trick to solve these kind of problems. So, z minus 3 I can write z minus 4 plus 1, I can write this.

Another part I can write which is z, I can write z minus 4 plus 4 like this. Now, I can expand this term because here I can have one divided by z minus 4 less than one. So, if I take 1 by z common anyway 1 by 3 term is already there. So, separately if I do this, this quantity, so let me write here right hand side. So, I am expanding only this term z minus

4 plus 1 taking this I am expanding. So, if I take z minus 4 common then it will be 1 plus 1 divided by z minus 4 this quantity. Now, I can expand that, now I can expand that.

In the similar way, in the right hand side, if I take 4 common minus 1 divided by say if I take 4 common then it will be 1 plus z minus 4 divided by 4. So, once I have 1 divided by z minus 4 because I required these things; and in other case, I have z minus 4 divided by 4 because I required this I want to use this condition from the given two condition. And the 3 term as usual is there. Now, we are done. So, now, we need to just expand this. So, it will be 1 z minus 4, let us put 3 here, and then it will go to up, so 1 plus 1 z minus 4 whole to the power minus 1 this is the first term. And the second term will be minus of 1 by 12 1 plus z minus 4 divided by 4 whole to the power minus 1.

Let us let us check once again everything is correct or not. So, z into z minus 3 I dividing into one by z minus 3 z z goes here z minus 3 goes here. So, z z will cancel out, we will have only 3 term and I divided this by 3, so that this three term cancel out. Then I write this at z minus 3 I write z minus 4 plus 1, so that I can have z minus 4 term in my hand. And here only z is there in order to have z minus 4, I take z minus 4 plus 4, so that the final term is z. Now, I take z minus 4 common out of here then 1 plus 1 by z minus 4 here, and I take it as 2 to the power minus 1. And here also it goes to up stair and I have this. So, let me erase this.

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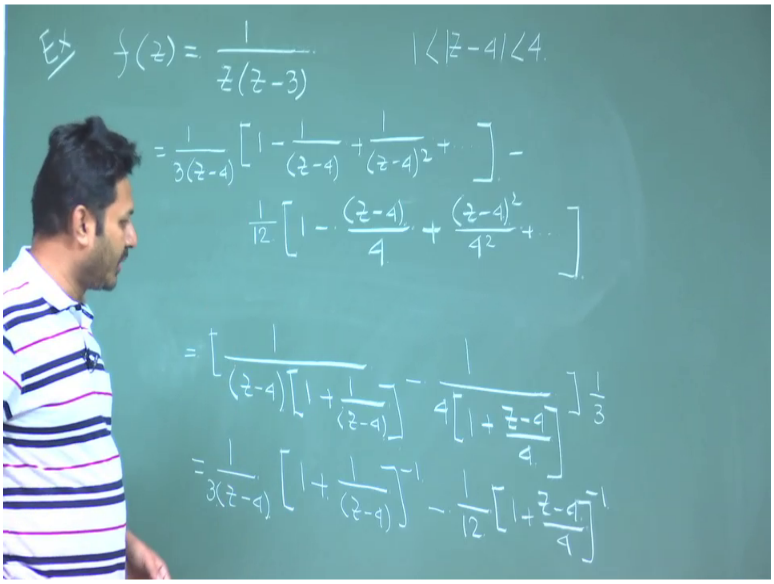

Now, I if I expand it will be something like 1 divided by 3 z minus 4 then this series which is 1 minus 1 by z minus 4 plus 1 by z minus 4 whole square and so on, this is one series. And another series that will have for the other term. So, let me erase this part. It will be minus 1 by 12 and then 1 minus z minus 4 divided by 4 plus z minus 4 square 4 square and so on. So, this is my entire series, this is the Laurent series. And if I use this Laurent series, then the terms will be something like this. So, here the residue is coming like z the residue that is the coefficient of 1 by z minus 0 is comes out 1 by 3. So, we will come to this point later. So, let us first try to understand how a function can be expanded in terms of Laurent series. So, we are doing few problems last class also, today also I am planning to do few problems, so that you are familiar so with this procedure and know how to expand.

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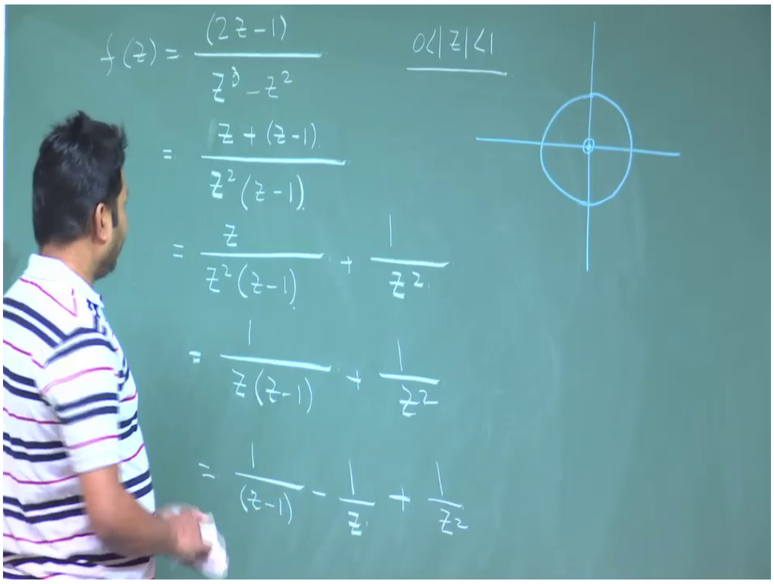

So, next is another problem I have in my hand, which is f z is something like 2 z minus 1 divided by z cube minus z square. So, we have this quantity or this function in my hand and the restriction of this function is mod of z is greater than 0 less than 1. So, it is better that every time you have a check where we need to physically expand this function. So, if I have a circle of radius 1, so this is a circle of radius one. So, the region I should cut this zero point and also one point. So, the region annular region is here in between these two rings whatever the region where we need to expand this. So, process is similar. So, 2 z, I can write it as z plus z minus 1 divided by this quantity, I can write z square multiplied by z minus 1. And again divide the 2 term as z divided by z square z minus 1 plus 1 by z square, I have this quantity plus this.

So, now, if I write 1 divided by z z minus 1 plus 1 by z, I have two term, but this term again I can simplify. And if I simplify it will look like 1 divided by z minus 1 minus 1 by z, this z goes here, this z minus goes here with a negative sign z will cancel out, we will have one. So, I can factorise this quantity in this way plus 1 by z square as usual, it was 1 by z square because I am writing 1 by z square. So, I have three term 1 divided by z minus 1, 1 by z, 1 by z square, but my limit y restriction of z is that z should be less than 1 then it is done.

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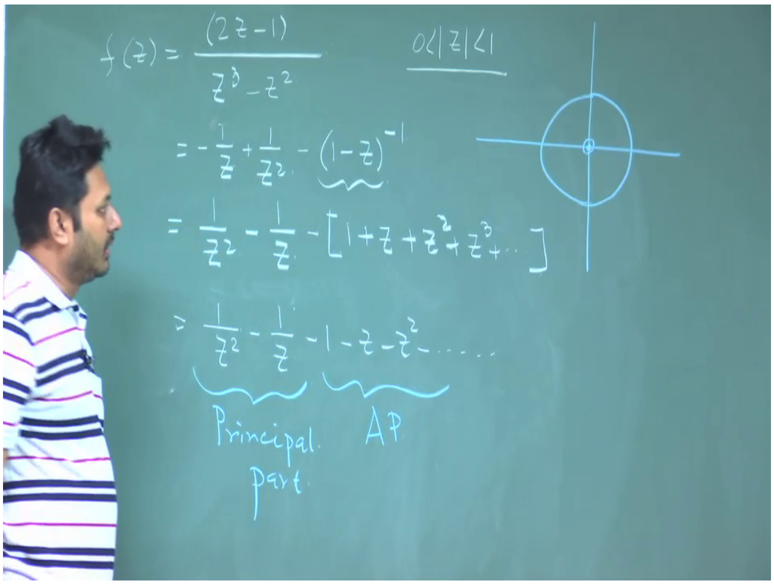

So, once I have this then there is no such problem, because I can expand these things like this way. So, it will be just say first term is 1 by z, I take this because it I have nothing to do with these things; 1 by z is already there plus 1 by z square it is already there. Another term is I write minus 1 minus z to the power minus 1, I take minus common then it will be 1 minus z and I write 1 minus z as 1 minus z to the power minus 1 with a negative sign here. Then the expression is something like 1 by z square minus 1 by z minus this series, this series I know is true for mod of z less than 1, so, the restriction is already given to me. So, I am allowed to expand this series.

If I do then it will be something like 1 by z square z plus z square plus z cube and so on. Now, the totals series is something like this 1 divided by z square minus 1 z minus 1 minus z minus z square and so on. So, you can readily understand that this part this part is your principal part and the rest part starting from minus 1 is your analytic part AP. So, I can just divide this principal part and analytic part, again this 1 by this term is important, I will come through this point again because this is the residue of these things.

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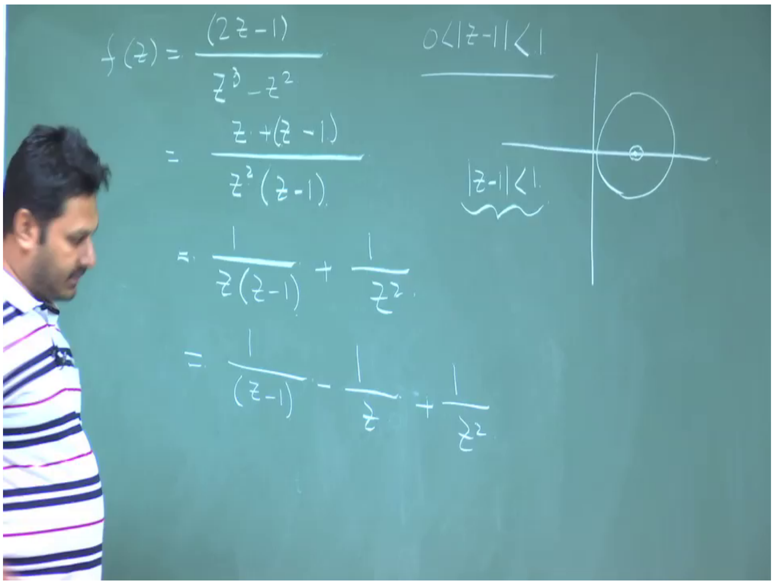

Let us stick with function now I will going to change the region and then try to find out what is happening. So, now next the same function but now the region is been changed, it is z minus 1 mod of z minus 1 greater than 0 less than 1. This is the problem that is in our hand so, that means, if I try to find out the region z equal to one point is here, if make a circle of radius 1, it will be something like, this it will not going to cut the zero point. And another small circle here, because it is greater than 0. So, z minus 1 greater than zero; that means, I will have a small circle here. So, this is the annular region, where I try to expand this function this is annular region where I try to expand this function.

Again we will do the same thing by expanding this particular function by expanding this particular function into some functions which are in this form. So, z plus z minus 1 because z minus 1 z square z minus 1 exactly the same way I am doing I am repeating the problem that I have already done. So, same old calculation and I reached this point. So, this is the same thing that we have done for the previous case, again I am doing the same thing just doing this way.

So, now the question is the problem is I need to use these things z minus 1 mod of z minus 1 is less than 1, I need to use this condition. Here we have z minus 1 here, so I do not need to do anything with these things, because already it is less than 1. What about this z I need to somehow introduce a plus 1 and minus 1, so that I can have this term and then 1 by z square that is interesting. So, we will do that later. So, let us first deal with this thing. Let me erase the function because this function is now written here.

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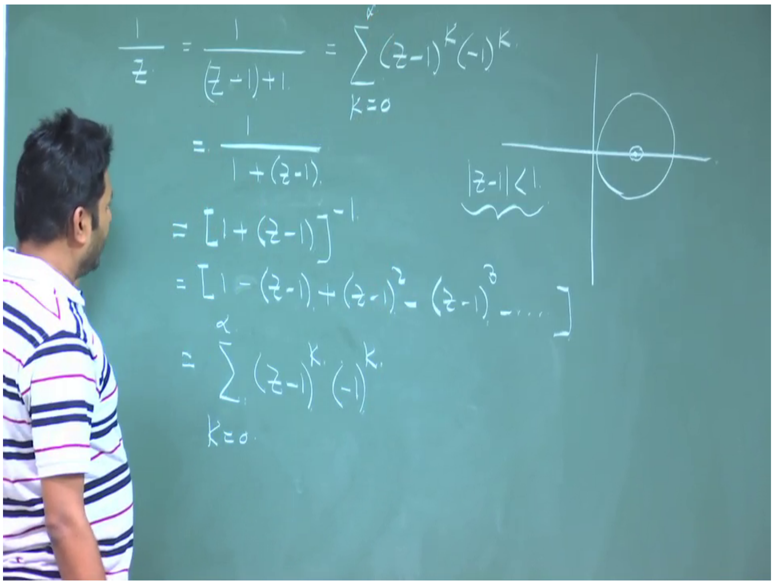

So, if I want to expand this 1 by z with this limit, how to do that, I will take this function as this 1 plus 1 minus 1 plus 1 z minus 1 plus 1, it is something like this. So, then how to deal with that, so 1 divided by 1 plus z minus 1. And once I write this then I am done if I do that then I can write it as 1 plus z minus 1 whole to the power minus 1, I can now expand because my mod of z minus 1 is less than 1. So, I can expand this series and it will simply comes out to be 1 minus z minus 1 plus z minus 1 square minus z minus 1 to the power 3 and so on this is the functional form we have. So, let me erase these things here I should write the function somewhere. So, this is the form I will write the function again once again. So, this form is nothing but if I write in summation form, it will be z minus 1 whole to the power k minus 1 whole to the power of k, k goes to zero to infinity.

So, this is the form the summation form I have 1 by z I calculate and I find. Now, that sign is repeating, so since sign is repeating when k equal to 0, I have this is 0, this is 1; when k equal to 1, this is minus 1; when k equal to 2, this is one and so on. So, now, once

I have z then the question is so let me write it here what value I am getting here in the summation form. In the summation form, I am getting z minus 1 whole to the power of k minus 1 whole to the power of k, k goes to 0 to infinity. Next thing I will do that once I have this value in my hand then I do not need to expand my one because 1 by z square is another term that I need to expand.

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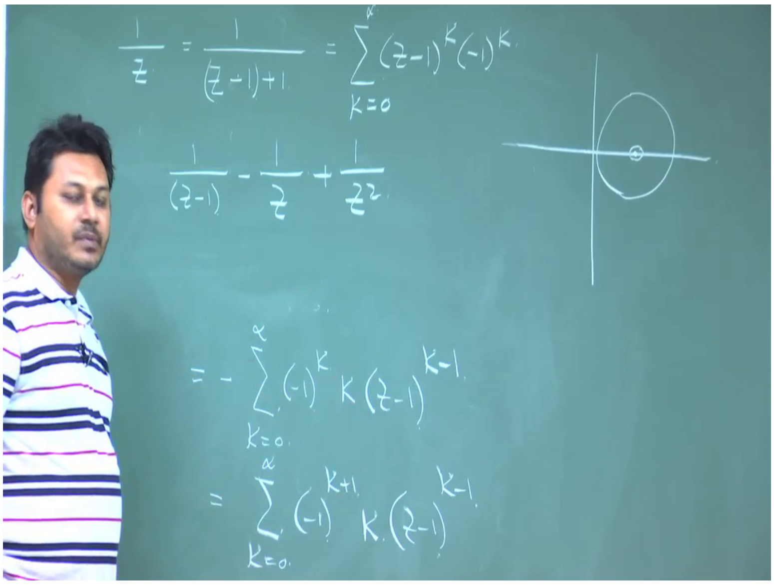

So, 1 by z square I can write in this way. This is interesting d d z of 1 by z; if I do then I will have 1 by z square with a negative sign so that means, it will be minus of this. So, minus of this quantity is nothing but 1 by z square, but 1 by z I already have the expansion in my hand. If I elaborate that, so it will be minus d d z the expansion is this and I need to just make a derivative with respect to z. If I do negative sign summation over k 0 to infinity minus 1 k will remain and this quantity will be k z minus 1 k minus 1. So, I again I have a series. You can also absorb this minus 1, you can put this minus 1 inside. And if we do then you will have minus 1 whole to the power k plus 1 multiplied by k z minus 1 whole to the power k minus 1 k 0 to infinity.

So, once you have this you can have the expansion of 1 by z 1 by z square expansion is in your hand. And if you remember your function was something like 1 divided by z minus 1 minus 1 by z plus 1 by z square something like this. So, 1 by z, I have a expansion 1 by z square, I have a expansion and 1 divided by 1 by z I do not need to do anything with that, so it will stay idle. So, I have the expansion of the entire part. So, this is a the expansion Taylor this Laurent series expansion of a form given function if the limit is given to us or the region is given to us. Right now I should stop here regarding the Laurent series.

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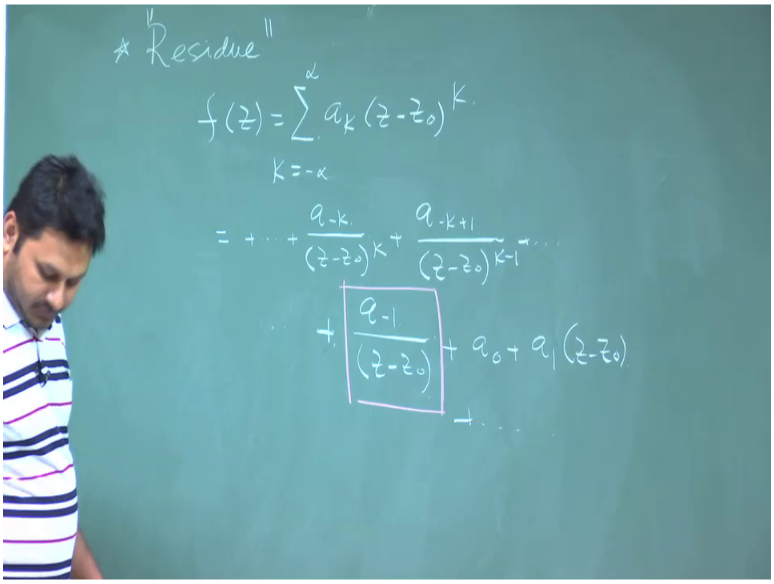

So, we will now we will like to do something new, which is called the residue, which is much important. So, let us directly go to that part. So, what is residue and all these things let me explain you first. So, next important topic is residue, I have already mentioned this term, this terminology what is the meaning of residue. So, if this is the function and this function, if I expand this function in Laurent series, it will be my form several time I have written this form. So, now, by that time you have already familiar with these things, so minus infinity to infinity. So, if I write these things then it will be plus dot dot dot plus a minus k z minus z 1 to the power of k plus a minus k plus 1 divided by z minus z 1 k minus 1 and so on. So, on one term I will have a minus 1 z minus z 1 plus a 0 plus then a 1 z minus z 1 and so on, this is the entire expression that we have if I expand these things, but you should only careful about these things this term. So, residue is nothing but a minus k. So, what is the definition of residue. So, let me erase this.

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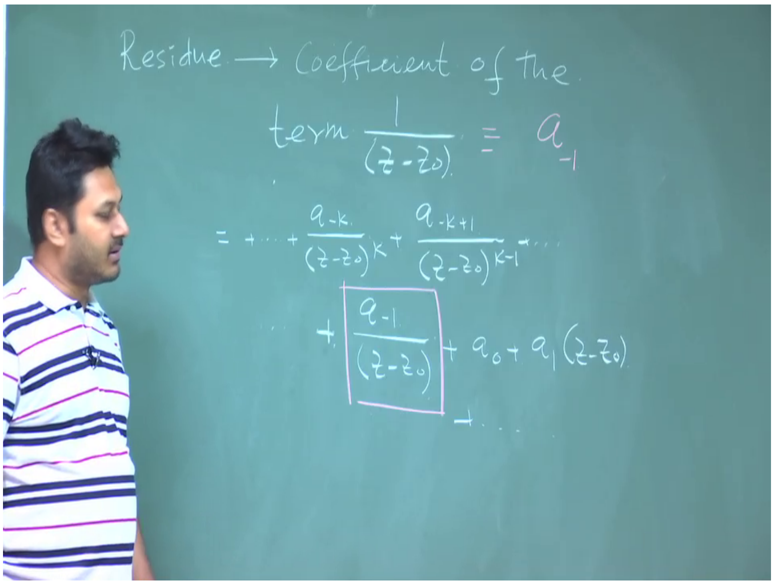

So, residue is nothing but the coefficient of the term 1 divided by z minus z 1. So, if you note that the coefficient of 1 divided by z minus z 0 is a minus 1 according to our notation. So, here it is equivalent to a minus 1, this is called this coefficient is termed as residue, this coefficient is termed as residue. So, now, I know what is the residue I defined, what is the residue. It is nothing but the coefficient of 1 divided by z minus z 1 this is just if I expand a series in terms of Laurent series then there will be analytic part there will be principal part. So, the last principal term the last term of the principal part which contain one divided by z minus z 1 is the residue the corresponding coefficient is the residue.

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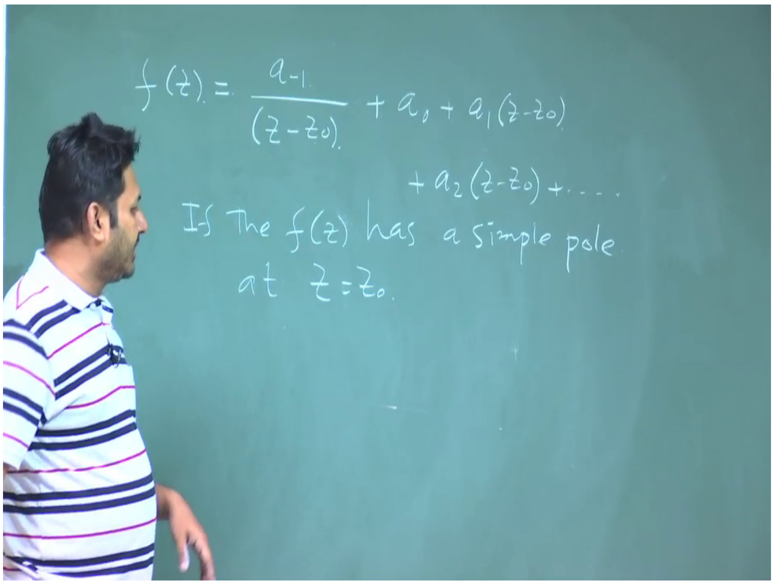

So, now we have a Laurent series in this form. So, I have a Laurent series in this form only one term only one term as my principle part that we normally then 1 z minus z 0 plus a 2 z minus z 0 and so on. So, now the next question is how to calculate the next question is how to calculate this a 1 term. So, I have an expansion, this is my expansion a 0 term is sitting here. So, if a 0 term is sitting here; that means I have the function that means, the function has some singularity as z equal to 0 and this is the simple pole. So, if the function f z has a simple pole at z is equal to z 0, then I can expand this function in this form, because I have I know that the only one term should be there. Because this is the simple pole of order one, that is why the pole is one here pole of order one; that means, it is a simple pole. I can expand this function in this way. Once I expand this function in this way, then it is easier for me to calculate a minus 1.

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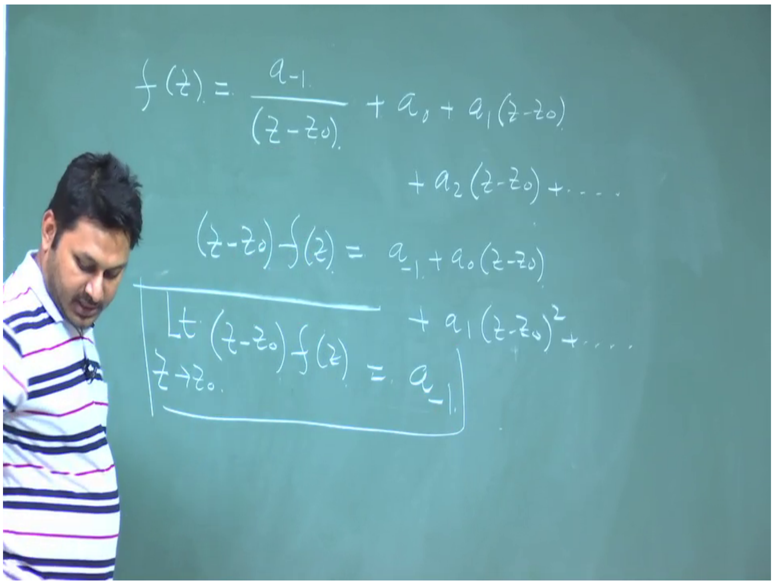

How to calculate a minus 1 I will just multiply the quantity z minus z 0 to the entire function. If I do the left hand side the right hand side will be readily a minus 1 plus a 0 z minus z 0 plus a 1 z minus z 0 square and so on. So, I can extract a 0, but apart from that I have also this term in my hands. So, in order to vanish this term what we need to do we need to just take a limit z tends to z 0 z minus z 0 function of z if I do if I take this limit z tends to z 0, z minus z 0 f z if I take this limit readily. In the right hand side I will have a minus 1 term in my hand because the other term since there is no problem putting the limit at z equal to z 0 it is an analytic part. So, at z equal to z 0 this part is not blowing up. So, I can put this limit and once I put this limit I will have this in my hand. So, this is the formula or finding the residue if the function has simple pole. So, let me write down here.

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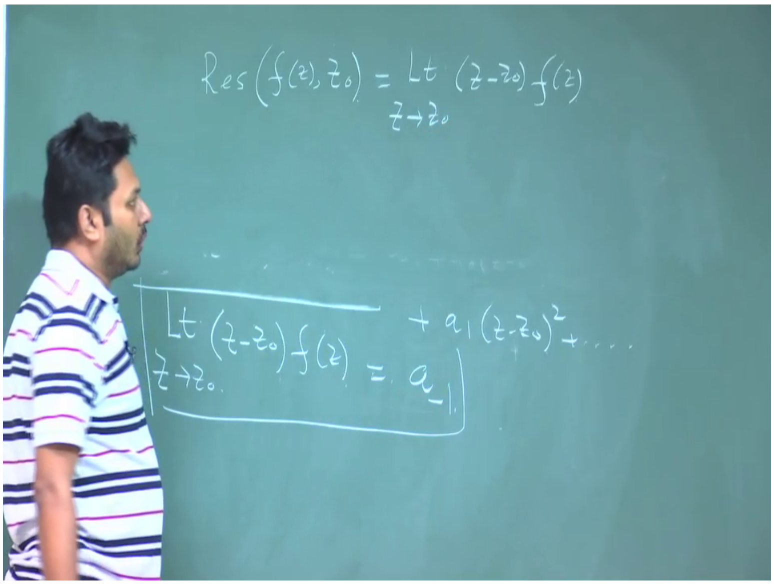

So, how the residue is defined? So, the residue is normally defined as residue of the given function f z at the singularity z 0 is limit z tends to z 0 z minus z 0 function of z that is all. This is the recipe to find out the residue once I have my functional form and know that this is a first order this, this is the singularity of first order. With this note, I will like to conclude the class here. In the next class, I will like to mention more about the residue; if the function do not have the first order singularity or singularity simple pole if I have more than simple pole which is the n order pole, how to tackle with that and how to calculate the residue that we will discuss in the next class. So, with that note I would like to conclude here.

Thanks for your attention and see you in the next class.