Mathematical Methods in Physics-I
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Lecture – 52
Laurent Series Expansion

So, welcome back student to the next of complex analysis, where we are dealing with something called Laurent Series.

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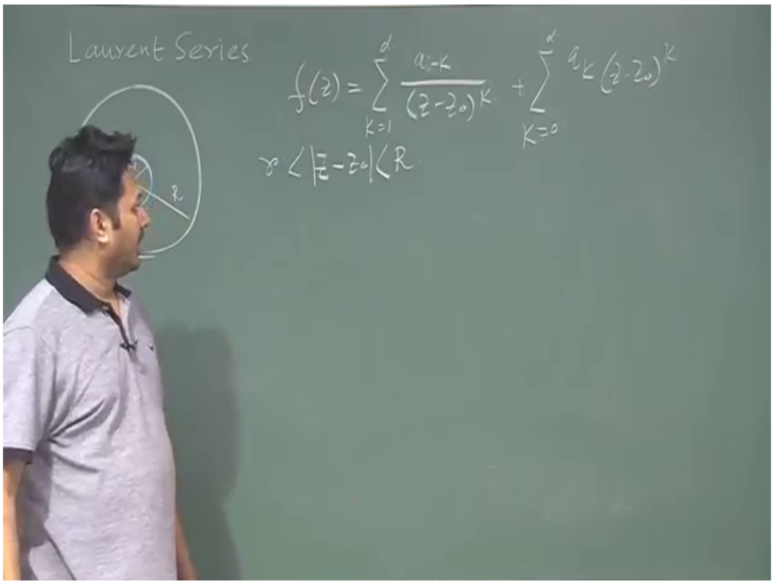

This is an series we described last class that where the function is analytic in some region, annular region defined by this. So, the restriction is something like this. This annular region the function is analytic. And for this region I can expand the function, I can expand this function like this. And this part we mentioned this part is called the principal part and this part is called the analytic part.

So, last day also in the last class also we put some kind of example in front of you, which gives you the idea how the singularity is calculated or how the singularity is coming in the light of this expansion Laurent series expansion. So, today we will give another example where we find that there are simple kind of singularity there is no singularity, removable singularity.

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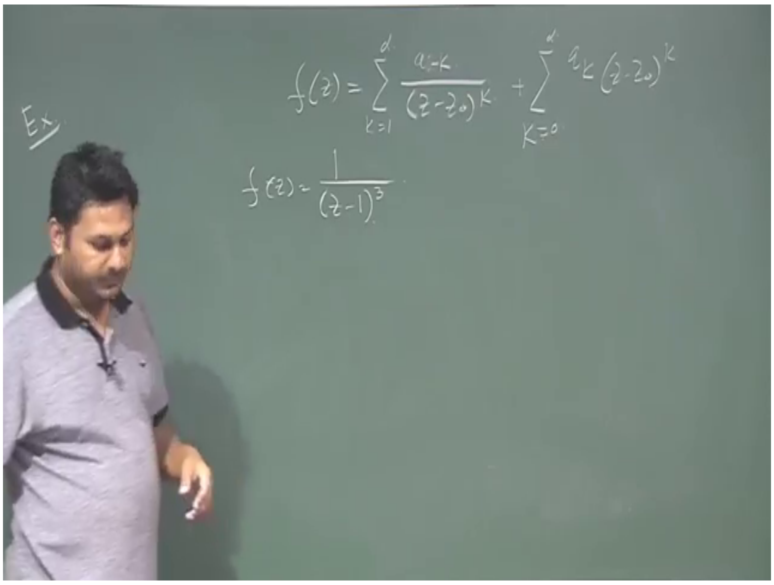

So, all these singularity are isolated here you know note that at z equal to z point only we have some singularity. Like the function f z, this function has isolated singularity at z equal to 1 point. Apart from that this function is analytic. So, this z equal to 1 is called the isolated singularity and this is having a simple pole. But if I multiply here say 3 then this function has a pole of order 3 and so on.

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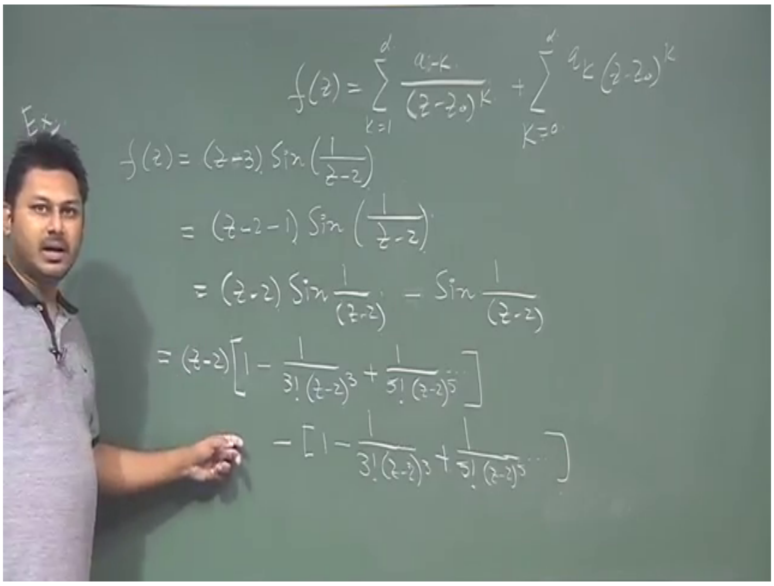

But there are few function also which has some essential singularity. So, for example, the function f z is something like z plus 3 z minus 3 sin of 1 by z minus 2. If this is the function if this function look like this way. So, how to deal with that? So, then I mean there are 2 ways. So, I can write this z minus 3 as z minus 2 plus minus 1, and then sin of 1 by z minus 2. And then if I multiply I will have z minus 2 sin 1 by z minus 2. This is a first part. And second part is minus sin of 1 by z minus 2.

Now, if I expand this function. If I start expanding this function. So, the first part will expanded in this way z minus 2 multiplied, the first term is one. Second term is minus, second term is minus z minus 2 cube divided by factorial 3. Then plus z minus of 2 5 factorial 5 and this. But this function on the other hand is expanded sorry, one I am making one mistake. It is 1 by z minus 2. So, it will be 1 divided by factorial 3 z minus 2 cube plus 1 by factorial 5 z minus 2 5 and so on.

Next part is also represented like this 1 minus 1 by factorial 3 z minus 2 cube plus 1 by factorial 5 z minus 2 5 and so on. So, 2 part I separate out and if I write it here finally.

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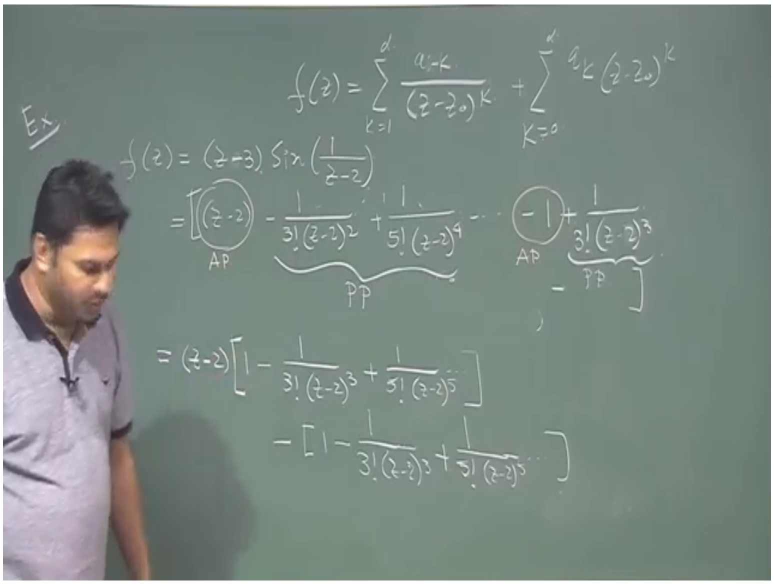

Then this function is represented like first part is z minus 2, then the next part is 1 divided by factorial 3 z minus 2 square. Then plus 1 by factorial 5 z minus 2 4 to the power 4 and so on. Along with that we have this part also.

So, minus 1 plus 1 by factorial 3 z minus 2 cube and so on. So, this is my total function total expansion. So, in total expansion we find that this is one you find that this entire portion this portion, this portion except these 2 point, minus 1 here and z minus 2 are principal part. So, this is your principal part, principal part, this is your analytic part, analytic part. So, we have only 2 analytic part here and infinite number of principal part. So that means, at z equal to 2 point, this function has an essential singularity.

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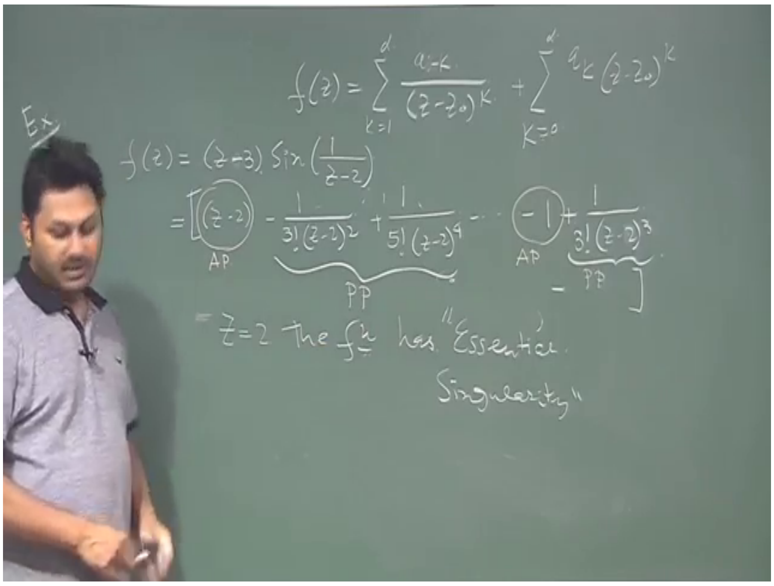

So, the conclusion is at z equal to 2 point at z equal to 2, the function has essential singularity. Essential because we can not remove this singularity at that particular point, whatever the function we have because there we have limited and number of analytic part, but infinite number of principal part. So that means, we should have some singularity which is essential it is nature.

So now we will go tot the next step. Now I believe you have an idea about the singularity and all these things in the light of Laurent series, but now we need to do some problem regarding the Laurent series. So, how a function is giving to you and how you should expand in terms of Laurent series is now the next problem or the next topic that we are going to start. So, let me do with an example because this is the best way to understand the Laurent series is solving some problems so that is why I start with some examples.

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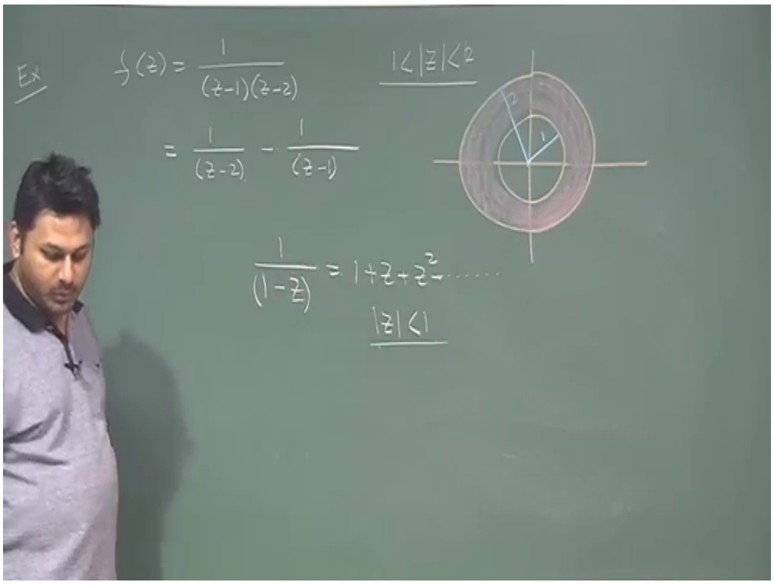

So, the function here is giving to us something like this f z is say 1 divided by z minus 1 z minus 2. We need to expand this particular function. So now, first you should note that this function has a singularity at z equal to 1 and at z equal to 2. Both the singularities are simple pole. So, they this function has a pole of order 1. So, simple pole and singularity at the point 1 and 2.

Now the next thing is that we need to expand this function in the region mod of z less than 2 greater than 1. So, this is the region where we need to expand this function. As soon as the region is given because I know that the function where I need to expand this the region should be where the region is something where the function should be analytic in nature. Here we find that this particular function whatever the function is given is analytic in that region. Because one point and two point both are outside.

So, let me first define the function first define the region here. First check where the region should be. So, this is my coordinate system mod of z is greater than 1, and less than 2. So, let me draw a circle of radius 2 this is a circle of radius 2 suppose and another circle of radius 1. So, I have drawn a circle of radius 2 here the radius is 2, and here the radius is 1. Mod of z is less than 2 means all the points less than here mod of z is greater than 1; that means, all the point here. So, the region where we try to expand this function essentially this annular region. This annular region I am talking about, where the function has to be expanded in terms of Laurent series or in Laurent series.

So, this is the region where we need to expand the function this annular region, the shaded one. So, here the technique is quite straight forward. So, first in order to do that we can factorise this. So, if I do it is simply comes out to be z minus 2 minus of 1 by z minus 1. If I do then I find it is z minus 1 comes here z minus 2 comes there z z we cancel out minus 1 plus 2. So, plus 1 will stand here and I have this. So, I can write the same thing in this way. Once I write this the next important thing that we need to note that this is the limit here, so once.

So, when I expand this kind of function, what was the result? The result was z plus z square. So, I can expand this particular function in terms of some series. What is the essential condition behind that? Essential condition was mod of z was less than 1. That was the essential condition we always impose and based on that I can write this. So, we will always do the same thing here, and try to find out how the series look like.

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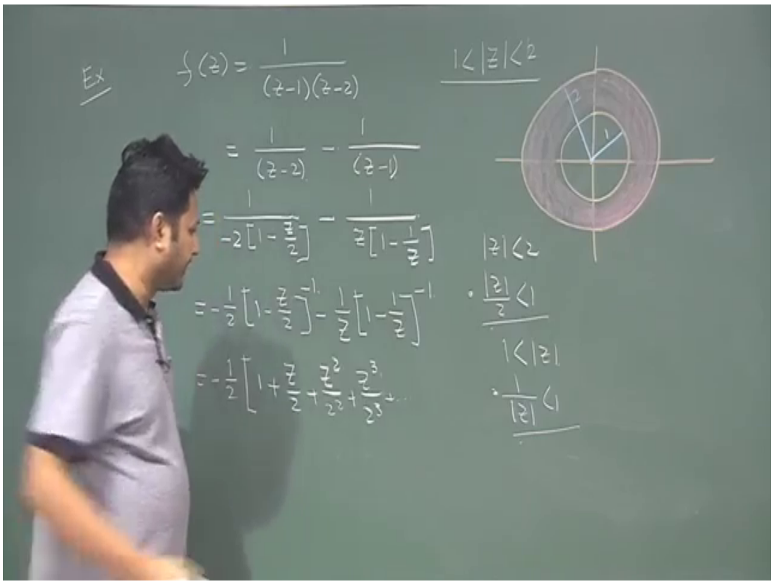

So, here in the first case it suggests in the first case it suggest mod of z is less than 2; that means, mod of z divided by 2 should be less than 1. That is one thing in my hand, and the second thing is 1 is less than mod of z; that means, 1 divided by mod of z is less than 1, that is the second thing in my hand. So, when I expand this 2 condition I need to take care. I need to take care of these 2 condition so that I can expand this particular function n the proper form.

So now if do that it will 1 divided by take a minus common out. So, minus 2 because I need to make z divided by 2. So, if I do I will have 1 minus z by 2, 2 multiplied by z by 2 I will have the same thing. So, I just divide in this so that I can have the term z by 2, because my condition suggest z by 2 is less than 1. So that means, I can have the opportunity to expand this in some series. In the next case also it is 1 by z. So, what I will do that I will take z common and it will be 1 minus 1 by z. Once I take z common out of that then 1 by z will come because 1 by z is the required thing, because 1 by z is less than 1 according to the condition.

So now I can expand. So, if I expand it will be minus of half 1 minus z by 2 whole to the power minus 1, that is the first thing. And the second thing is 1 by z 1 minus 1 by z whole to the power of minus 1. I separate out 2 part and now we are I am I mean we are ready to expand this, because both the condition is in our favour to expand this because z by 2 is less than 1. So that means, I can expand this 1 by z is less than 1 I can also expand this in the series.

So, if I do then I will have 1 by 2 1 plus z by 2 plus z square by 2 square plus z cube by 2 cube and so on, that is the first part. Let me erase this I believe you understand what I am trying to do.

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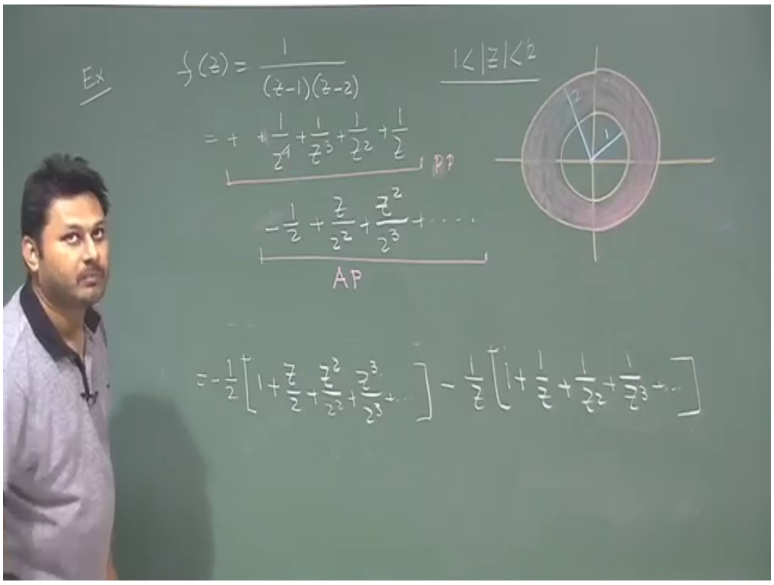

Minus 1 by z this is 1 minus. So, this is plus 1 by z plus 1 by z square plus 1 by z cube and so on. Now I can rearrange these things a slightly so that it looks good. So, let me erase this. So, this function eventually comes out to be something like this. Plus dot, dot, dot plus 1 by z to the power 4 plus 1 by z to the power of, if I take this term this term this term and this term there will 4 terms. So, 4 3 2 and 1 up to this then I start from here because it is. So, plus minus half plus z by 2 square plus z square by 2 cube and so on.

So, when I expand that you can readily understand that this portion from here to here we have this portion is in the expansion is the principal part, and this portion in the expansion is the analytic part. And it is this is basically the desired Laurent series expansion because the Laurent series expansion should be in this form.

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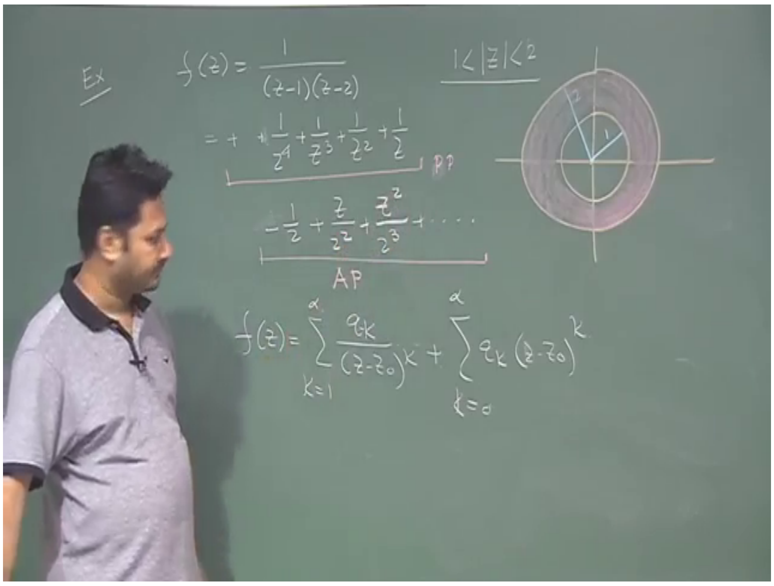

The Laurent series expansion should be in this form a minus k divided by z minus z 0 k, k goes from 1 to infinity, and another term a k z minus z 0 whole to the power of k, k goes to 0 to infinity.

So, there are 2 parts are associated with the function f z. And this is function, and this function is evaluated in this annular region with the condition given like this. Based on that condition if I expand I find the principal part that is this part here, and the analytic part that is the this one, where my a 0 is nothing but minus of half, and the a k is all the cases the a k is turns out to be a 1. And a minus k is all turns to be 1 and a k is coming comes out to be minus half a 0 and first one is 1 by 4 then 1 by 8 and so on. So, this is a very straight forward example of how to expand a given function in terms of Laurent series, when the given region the region is given to you now we go for another example.

So, let me check what other example is we have in our hand ok.

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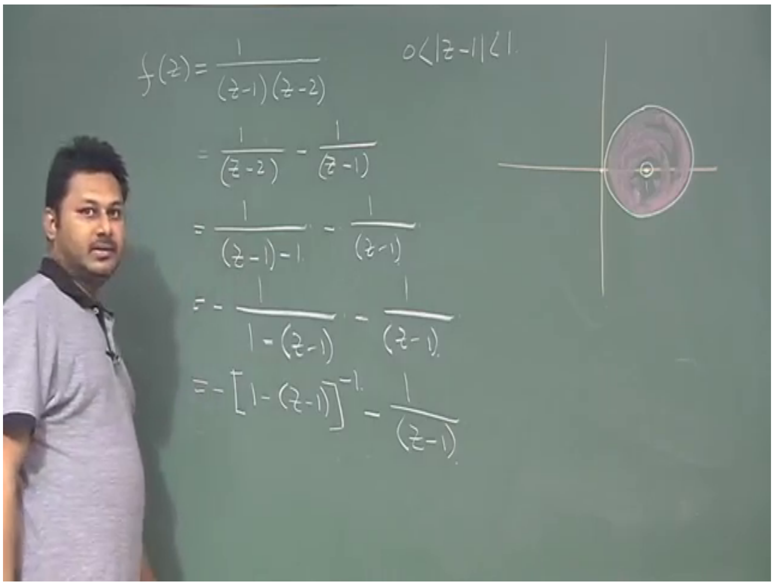

So, the function f z is same. Next example is something similar. The function that is given to you is same. The only difference is the region. So, the region is given as z minus 1 is less than 1. The region that is given is this. So, again let us try to find out which region we are talking about. Z minus 1 means I should have here some point here. So, this is my point here, and having a radius 1 if I make a circle of radius 1 not one less than 1 I should have a circle here some something like this. This is a circle I have. Also note that z minus 1 is greater than 0; that means, also here we have a small circle which is my small region, or because it is not the it is not at z equal to 1 point because z equal to 1, point the function is not analytic you should remember that.

So that means, this is the region where I am trying to expand my functions this is the region where I am expanding trying to expand my function. All the points in this circle except the centre which is at z equal to 1, because it is restricted by this limit. Now again I will try to do the same thing. So, the first thing is to write the expression in this factorised form. If I do that then again I will have in the previous way the old function here. Now one thing here z minus 1 seems to be less than 1. So, 1 by z minus 1 term should be intact here this quantity is z minus 2 I need to deal with this quantity.

So, let me do that in this way. Z minus 2 I can write z minus 1 minus 1 write. So, if I write this then I am done actually. Because now if I take a minus common here. So, it seems to be 1 minus z minus 1, 1 minus z minus 1 because take a minus common. So, it will 1 minus z with a negative sign. So, it will be something like this minus 1 minus z minus 1. So, z minus 1 is less than 1. So, z minus 1 is less than 1. So, no problem with that I can expand this whatever the function is in my hand which is minus 1 minus z minus 1 minus this.

Now, once I have I can now allow to expand this, because z minus 1 is less than 1 this is the condition I already have.

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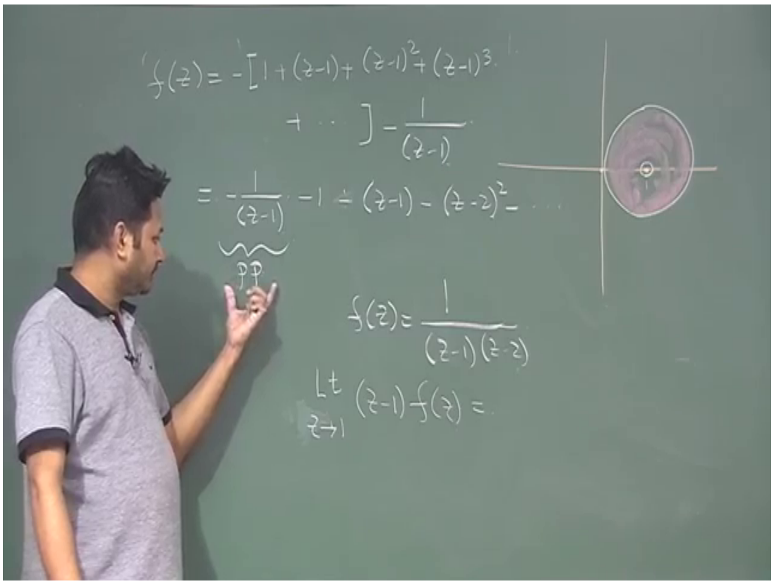

And now I can expand this function, and this function will be something like this. F z will be minus of 1 plus z minus 1 plus z minus 1 square plus z minus 1 cube dot minus 1 by z minus 1. It will be something like this.

So, you can see again that I have a principal part here, I have an analytic part here, and if I write in this form. So, it will something like minus of 1 by z minus 1 then minus of 1 minus of z minus 1 minus of z minus 2 square and so on. So, this particular form seems to be 2 part. One is this part we called it as principal part only one term in this principal part. And if you remember that this is a simple pole because my function was z minus 1 z minus 2. So, it should have 2 poles, but if I take the limit at z tends to 1 multiplied by z minus 1 function of z. I will have a quantity which is not equal to 0 or I have a finite limit here. Once I have a finite limit; that means, it is it is having a pole of order one. And when I expand this in terms of Laurent series again we are having the same thing and that is this quantity. So, we have one principal term here and that suggests that it should have one pole in this expression.

So now I should conclude the class here because in the next class we will start again with the Laurent series. And try to understand something quite important which is called the residue.

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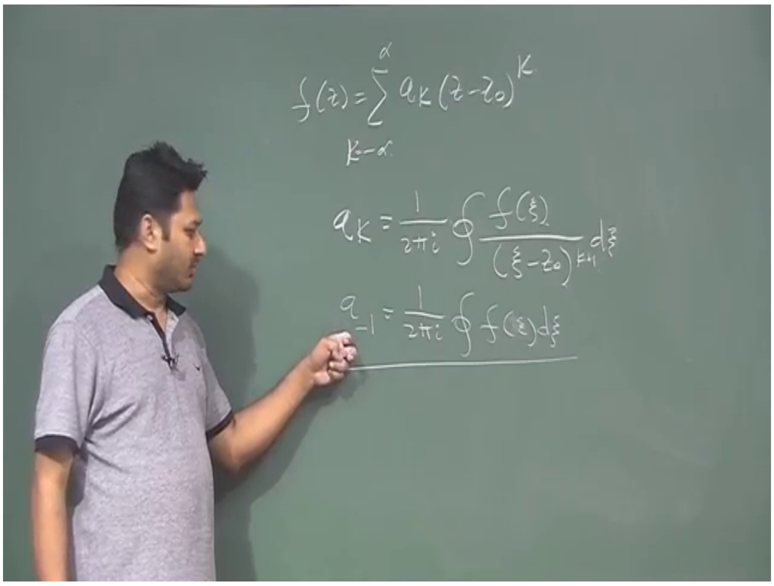

So, today we will not have we did not discussed anything about the coefficient that is coming from the Laurent series, which is a k, a k, a, a k and a minus k. And this coefficient have some significance, because let me give you the idea if you remember the first part I write f z in total was represented by this, when I expand this in Laurent series. Minus infinity to k is equal minus infinity to infinity. Then we mentioned a k as 1 by 2 pi I we did not use this however, but we will going to use in the next class, what is the significant here, k plus 1.

So, now if I put k equal to minus 1, then this equation gives me something quite interesting and that is this. So, I can eventually have the result of this function. If even if this function have kind of some kind of singularity, I can remove this singularity and that region and I can have the expression which is nothing but 2 pi I multiplied by a minus 1. So, a minus 1 is nothing but it is a coefficient and this coefficient is called this residue. And from the residue theorem emerge and using that we can calculate many integration and all these things. So, we will in the next class we will start from here. We may go to few other examples of Laurent series expansion. And then directly go to this residue, and how to calculate the residue what is the way to how many different kind of residue you have and all these things.

So, with this note let me conclude the class here. So, see you in the next class where we learn more about this residue, and also do some kind of problem related Laurent series.

So, thank you for your attention. See you in the next class.