Mathematical Methods in Physics-I
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Lecture – 51
Laurent Series and Singularity

Welcome back student to the complex analysis course.

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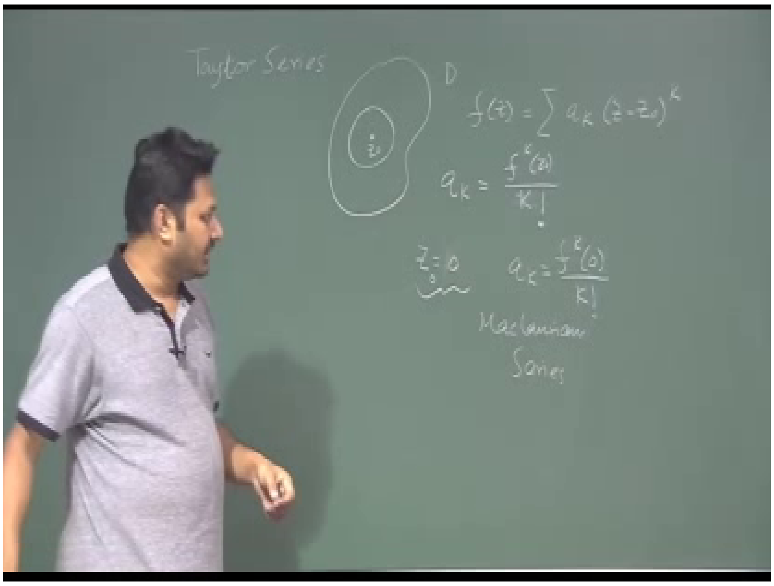

In the last class we did few things like Taylor series and also learnt something about singularity. So, let me once again remind you what is the meaning of Taylor series that if a function is analytic some domain D then I can expand this function f Z which is analytic in some domain D, I can expand this function around some point Z 0 and this expansion can be represented by power series with this form a K Z minus Z 0 to the power K where a K is the kth order derivative; a K was a kth order derivative of this particular function that is given to you and it is represented by f prime f K Z 0 divided by K the factorial K.

Well, if Z is equal to Z 0 is equal to 0; that means, if I want to expand this particular function to a very special point which is 0, then what happen this series will have the same form; obviously, except the fact K is now changed and represented by this Z 0 will just represented by 0 and we call it as Maclaurin series. So, if this is a Taylor series which is written here this is was Maclaurin series well that is a thing we have learnt and all this detail and all this detailing we learnt all the aspects and all these things.

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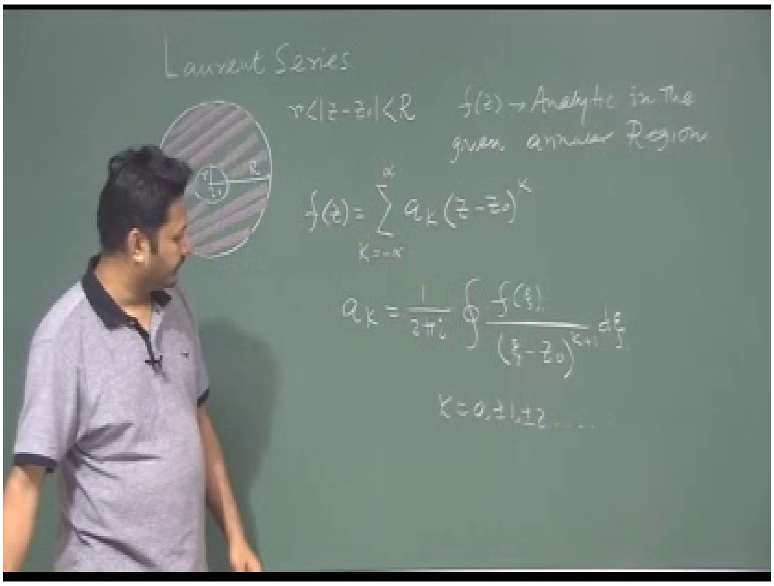

So now today we will going to start something new which is called Laurent series, which is equally important and what is this statement of the Laurent series let us first try to figure out, then we will go to further about the Laurent series.

So, in the Taylor series the function was defined or analytic in some given domain D. Here we have something different and that is this this is an annular region, this region shaded by this red part this is the annular region the shaded part only where this is r and this is big R say. So, the region is defined as Z minus Z 0 any point this is Z 0 which is less than which is greater than small r and less than big R. This is an annular region that I have already drawn here.

So, the function is now analytic in this annular region. So, f Z is analytic in the given annular region when the f Z is analytic in the given annular region then I can expand this function f Z also some power series having the same form like this where my limit K is something different here which is minus infinity to infinity, that is the new part I am adding here. However, I am not going to prove these things whatever the Laurent series thing is because it is not in the syllabus, I will just going to say about the statement and then do some problem related to Laurent series then things will be clear to you.

Here a K is defined by this this is the definition. So, the definition suggest that if the function is analytic in a bounded region or in a restricted region like this small r here and big R here. So, Z minus Z 0 is a annular region, which is shaded by this red portion where the function is analytic. If the function is analytic in this region I can write this function in some form which is the power essentially the power series, only difference with this and Taylor series is that Taylor series the summation was starting from K equal to 0 to infinity, here I am writing this case minus infinity to infinity, where a K is defined like this, so K here obviously, 0 plus minus 1 plus minus 2 and so on. So, still now we have some doubt what is going on and all these things, but let me do one by one let me do the thing one by one.

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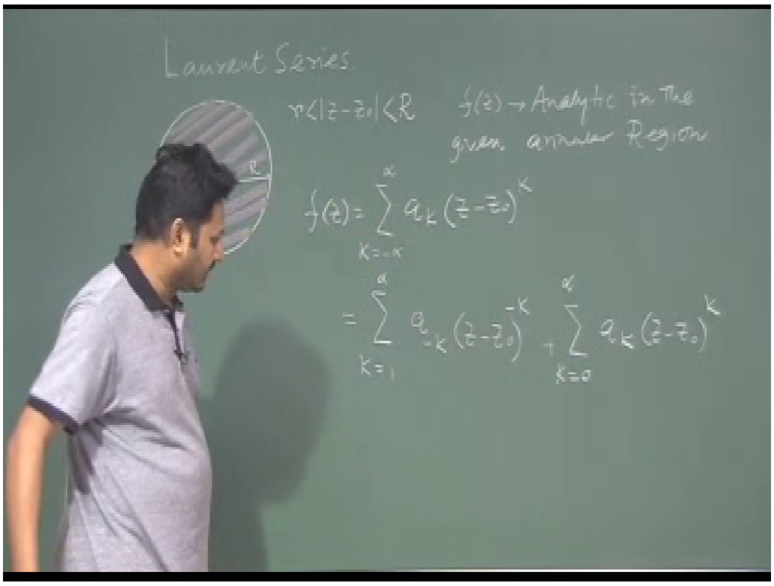

So, if I now write this in 2 particular form in divide into 2 form, say K equal to I just remove this negative sign this minus infinity to infinity. So, I will do that like one to infinity and write a minus K, and Z minus Z 0 to the power of minus K. I write the negative part whatever the negative portions having through this K negative K, I write this in this way. I just change the sign and then I write K equal to 1 to my 1 to infinity and the next part is as usual and I write K 0 to infinity, a K Z minus Z 0 this. I divide the entire stuff into 2 part one is this and another is this this portion if I now write let me use this part because it is not required.

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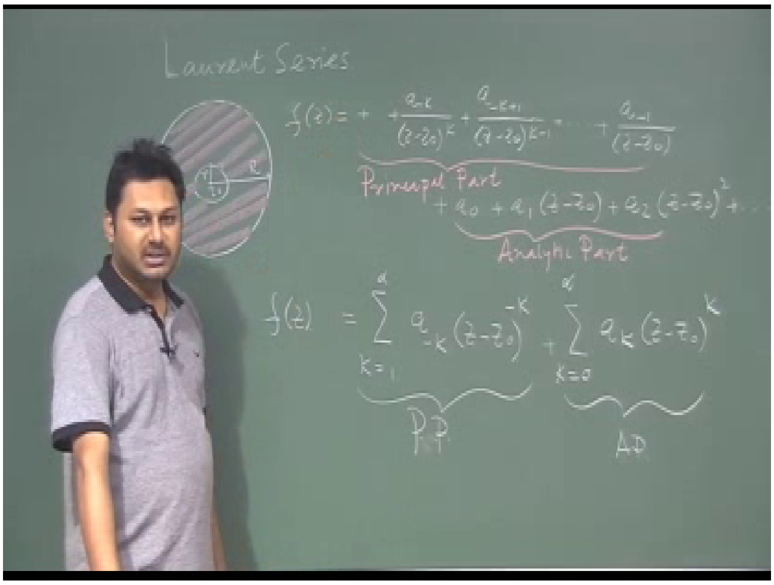

I expand. So, this is my f z, I am expanding this entire part. So, if I expand then f Z will be something a minus a minus K divided by Z minus Z 0, K plus a minus K plus 1 divided by Z minus Z 0 K minus 1 so on plus a minus 1 Z minus Z 0, plus a 0 plus a 1 Z minus Z 0 just expand this thing and try to find out how these things look like how this expansion look like.

So, some portion has 1by Z minus Z 0 term and some portion is Z minus Z 0 term. The portion that is having 1 by Z 0 Z minus Z 0 term is called this portion up to here is called the principal part. And this portion which is a 0 a 1 Z minus Z 0 and all these things is called the analytic part. So, here if I write the thing, this portion is principal part. So, this portion is my say principal part PP and this portion is called the analytic part AP. This portion is principal part this portion is analytic part, this portion is analytic part because Z equal to Z 0 point this function is do not have any singularity, but here you can find that if I expand the function in this particular form; that means, at Z equal to Z 0 we have singularities. So, this singularity we have discussed in the last class again we will go back to that thing and try to find out in the light of this Laurent series expansion.

So, what is the meaning of singularity and all these things, but you should note that at Z equal to Z point, it should have some kind of isolated singularity that is the first case second thing is that it may have some kind of order for example, here we have infinite order. So, it is if this series go on in this side then we should have an essential singularity that we have discussed again today we will discuss all these things as I mentioned in the light of Laurent series, but this portion which is analytic there is no problem know there is no such problem at all at Z equal to Z 0 point that is why it is called the analytic part.

So, let us now give some example of singularities and then we may understand how these things is happening. So, Laurent series I know how the Laurent series has expanded and now I try to find out few series and check.

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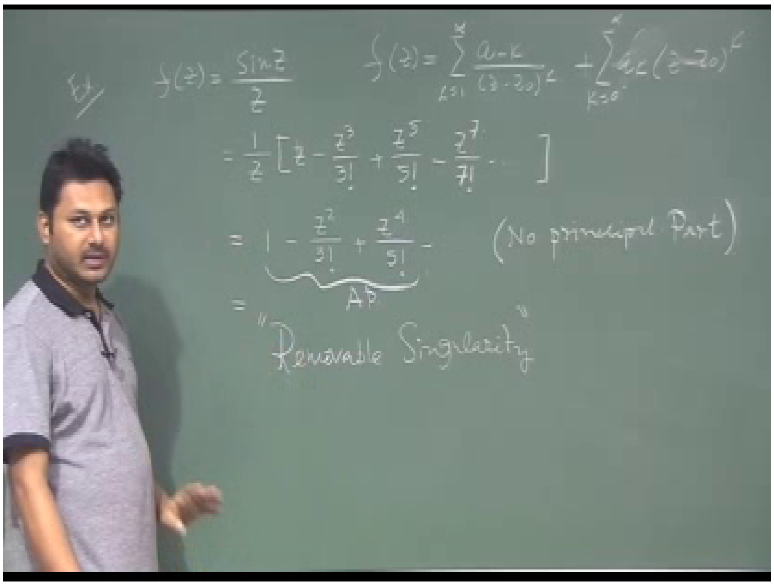

So, first one is the same function that I wrote last day let me start with these function sin Z divided by z. So, this is a function if I now try to expand this function 1 by Z is sin Z I can expand in terms of some series here. So, first term is one, second term is minus Z cube divided by factorial 3 first term is not one first term is z. So, I already divided by z. So, it is not correct. So, it is factorial 5 Z to the power 7 factorial 7 and so on. So, now, if I write the total function total form of this function, then it comes out to be 1 minus Z square divided by factorial 3 plus Z to the power of 4 by factorial 5 and so on.

So, now, if I want to tally this this whatever the expression in is in our hand in terms of the Taylor series that Laurent series expansion. So, Laurent series expansion let me write it here how the Laurent series expansion was given it was something like a K minus 1 divided by Z minus Z 0 to the power K, where K running from 1 to infinity that is the first part or the principal part and the second part was a K divided by a K multiplied by a K multiplied by Z minus Z 0 to the power K, where K is running from 0 to infinity. So, there are 2 parts. So, I am if I want to tally with these things whatever is written here we find that there is no principal part here at all.

So, all these things are analytic part this is just analytic part starting from here. So, there is no principal part. So, no principal part if an expansion do not have any principal part we say this function has a singularity, here we know that at Z equal to 0 seems to be singularity, but this singularity is removable. So, we called it a removable singularity removable singularity. Even its seems that we have a singularity at Z equal to 0 when I expand we find that this expansion which we tally with whatever the expansion Laurent series expansion we have here, then there is no principal are associated with this expansion. So, that is why we say this is a singularity, but this singularity is removable; that means, there is no as such there is no such singularity more examples can be given let me give you some another example it is not the only example one can have.

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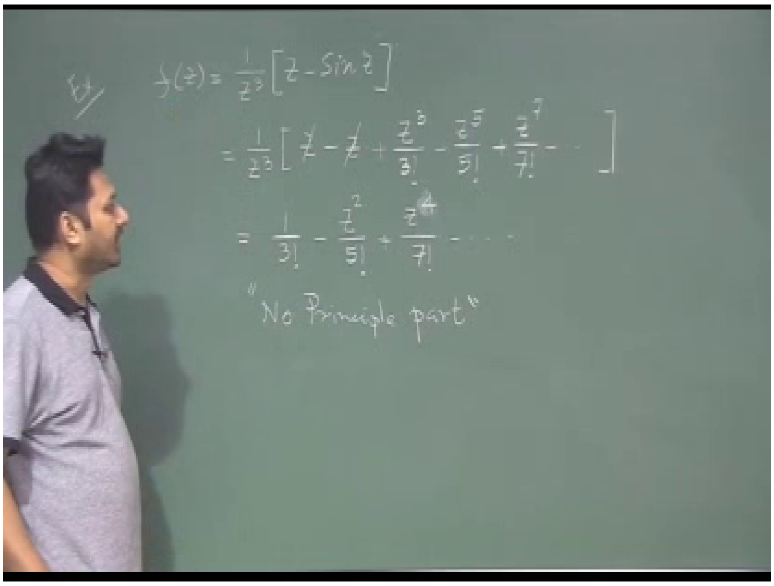

So, 1 by say Z cube Z minus sin Z this is another function and this function also seems that Z equal to at Z equal to 0 we have singularity, not only that this singularity seems to be having a pole of order 3 because last day we were discussing this this kind of things. So, today again we will try to discuss these things in the light of Laurent series, but let me first expand this and try to figure out where we should have really any kind of singularity for this function or if it is there it is removable or not.

So, if I do that then variedly I can find that Z the first term and then the series of sin where we have additional Z minus then plus Z cube of factorial 3 minus Z to the power of factorial 5 factorial 5 plus Z to the power of 7 factorial 7 something like this. And now if I do that then I find this Z is basically cancelling out and finally, I am having something like this that is 4 factorial seven and so on. So, we have one series again this is not the same series that we have in the previous example. So, this is however, another series where we have a term few terms few series, but again there is no principal part since there is in this series there is no principal part we can say that this at Z equal to 0 this function this particular function should not have any singularity or it is has a singularity which is removable in nature. So, it is removable singularity because there is no principal part associated with that. So, now, we will have some function which may have some singularity.

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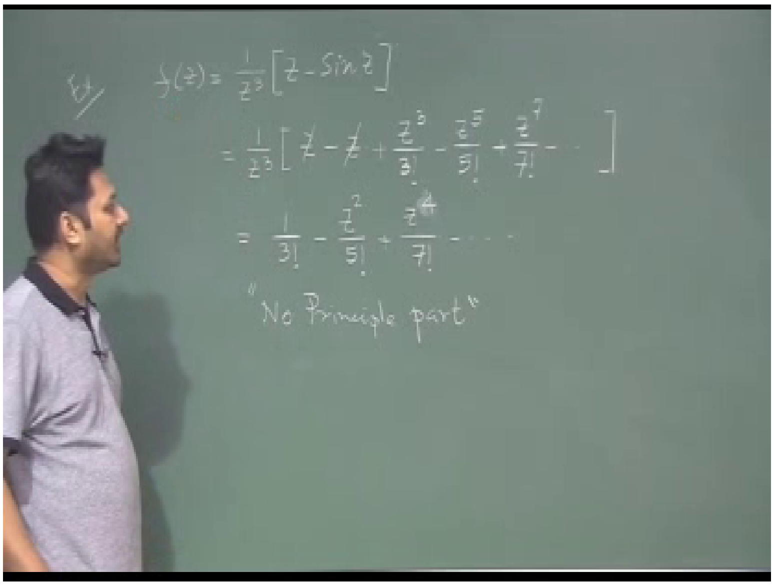

So, let me write f Z in terms of Laurent series once again which is a minus K Z minus Z 0 whole to the power of minus K that is the principal part, plus where K is running from 1 to infinity and then K is equal to 0 to infinity a K Z minus Z 0 whole to the power of K that is the Laurent series that I have in my hand.

So, now I am saying that I have a function say f z, I have a function f Z and the number of principal where we have few terms in the principal part and they said this few term is this a minus K Z minus Z 0 to the power K plus a minus K plus 1, Z minus Z 0 K minus

1 so, on plus a 0 plus a 1 Z minus Z 0, plus a 2 Z minus Z 0 whole to the power 2 and so on. So, here we have a series in my hand where we have this up to here I have few terms which is say principal part. So, I have few terms up to K, I have a few terms which have principal part. So, if we have these things. So, then we have some kind of singularity of this function and order of singularity is K because I have few terms. So, now, let me simplify this is a general form. So, let me simplify this and I say this principal part has only one term, if this principal part has only one term then K is equal to one only and then I have let me write.

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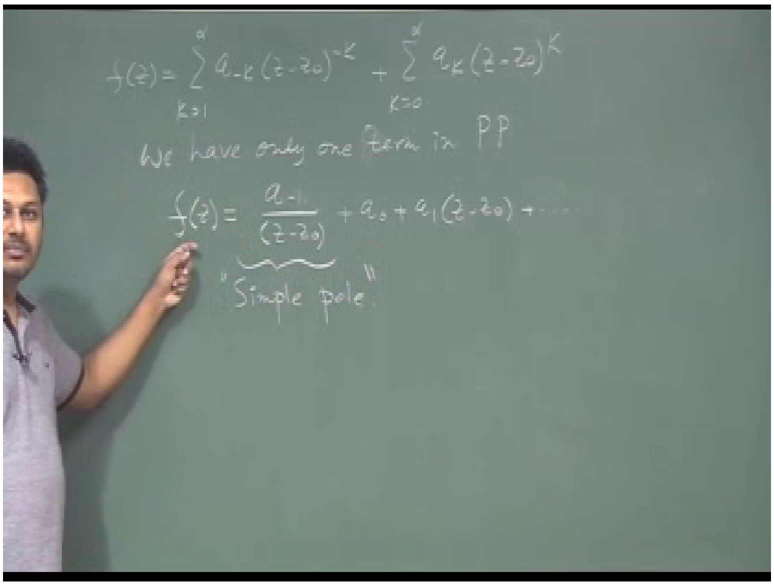

So, the condition is we have only one principal term only one term in principal part I write PP. PP is for principal part, then the function f Z will look like a minus 1 divided by Z minus Z 0, because the value of K is only one here which is one. So, I will have only this term plus a 0 plus a 1 Z minus Z 0 and so on. I have only single term which is in the principal part. So, in this case we can say this is a singularity and singularity with simple pole. So, simple pole we have a singularity here because at Z equal to Z 0 the function can blow up and if the function is blowing up at Z equal to Z 0, we called it as simple pole because the order here is one the order here is one this is the only one term.

What will be the examples of these things?

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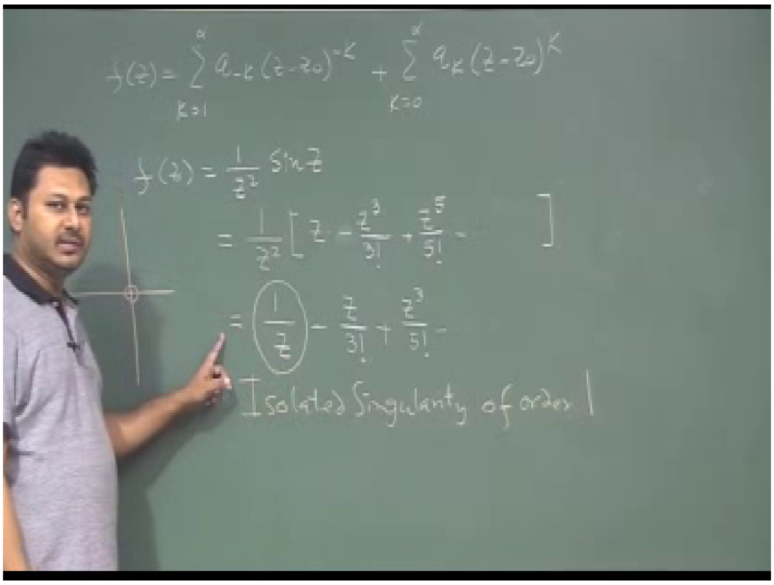

So, say f Z is equal to one divided by Z square sin z, I have a similar form slightly different previously it was sin Z divided by Z now it is sin Z divided by Z square. If I do that then it will be 1 divided by Z square, sin Z if I want to expand this the first term will be Z second term will be minus Z cube divided by factorial 3 and then Z to the power 5 by factorial 5 and so on.

The series is something like this. So, now, if I simplify with this, first time I will have something 1 by z, then my second term will be Z divided by factorial 3, third term is Z cube divided by factorial 5 and so on you should note that every time when I expand this series the coefficient is changing. These things the coefficient associated with whatever the thing is changing, but the important thing here is this term 1 by z. So, 1 by Z term is associated with the principal part because if I expand then I find that only one term is associated with this part. However, here we have many terms are there which is the analytic part, but since it has one term in the principal part we can say it have a singularity of order one or we have a simple singularity or a isolated singularity Z equal to 0 point is also a point here, only here the function has a singularity other in all others point it do not have any kind of singularity. So, that is why it is called the isolated singularity that is first, and order is 1 of order 1 n is 1. So, isolated singularity we have an isolated singularity here of order 1 ok.

So, now we will have some singularity which may not have order 1, but greater than that.

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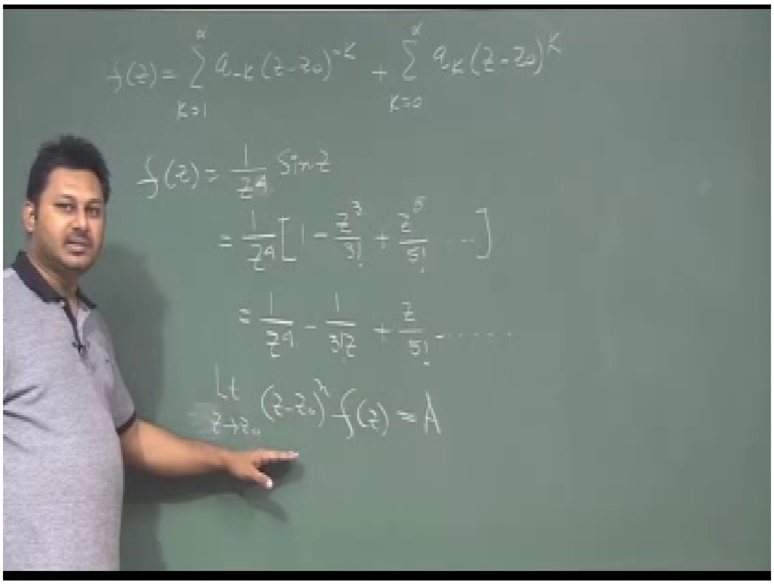

So, the easy example is simply if you do the easy example is say f Z is equal to 1 by Z to the power 4 say 1 by Z to the power 4 sin Z if I expand then obviously, we will find that Z to the power of 4 first term is one, second term is Z cube divided by factorial 3 then the third term is Z to the power 5 factorial 5 and so on. So, first 2 term in the first 2 term you will have one term Z to the power 4, and the second term factorial 3 Z so; that means, these 2 terms are associated with the principal part and the rest term Z by factorial 5 and all these things are associated with analytic part that is this one, but here we have 2 parts and when we have 2 parts 2 terms in this principal part, we can have we can say that this have a singularity not a simple singularity rather we have something where singularity of order 4. Because if you remember last class that if limit Z tends to Z 0, Z minus Z 0 to the power n function of Z is equal to not equal to 0 or some value, then I can say that it is a singularity of order n.

If multiply the entire function to Z to the power 4 and take the limit at Z equal to 0 you will find that it will it will not is not equal to 0 no it is not equal to infinity or something it some have some value a then we will have a fine if the limit exists here then we can say that this this corresponding function has a singularity of order n.

So, here the singularity of order n seems to be 4, quickly give me last example today.

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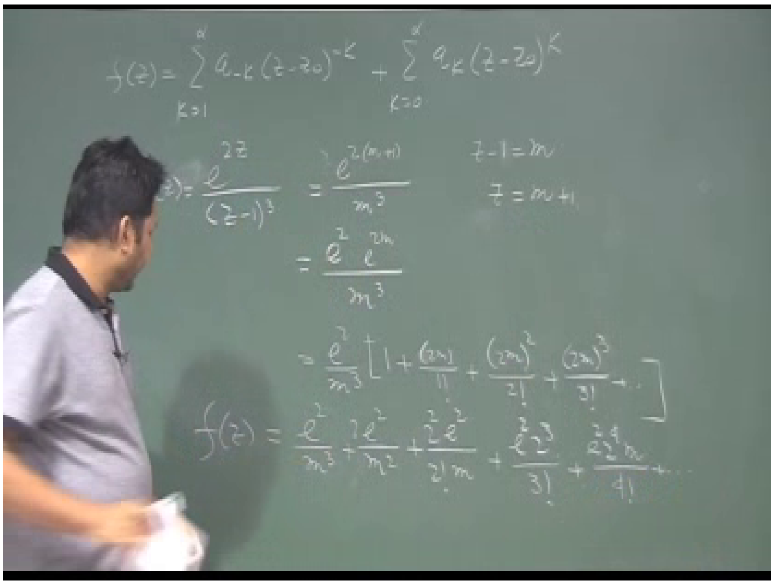

So, let me say Z to the power e to the power function new kind of example it is necessary otherwise things will be monotonous every time the old example. So, it is Z minus 1 whole cube this is the function we have. So, now, I need to find out how many singularities are there. So, we have an expansion. So, here things are slightly tricky. So, I need to I have Z minus 1 here and here z. So, make it same. So, what one can do that one can write Z minus 1 is equal to some say m, then Z will be equal to m plus 1. So, now, if I put here we will e to the power 2 Z is equal to m plus 1. So, m plus 1 2 m plus 1 divided by this. So, I will have m cube m I have because Z minus 1 is m.

So, then I will have e to the power of 2 multiplied by e to the power of 2 m divided by m cube then I can expand. So, e to the power of 2 divided by m cube and I expand for this. So, first term will be one second term will be e to the power 2 m divided by factorial one, third term is 2 m square by factorial 2, then 2 m cube by factorial 3 and so on. This now if I write e to the power 2 the first term is e to the power 2 every time. So, it is function of Z finally, comes out to be e 2 e square divided by m cube, then e square 2 e square divided by m square plus 2 square divided by factorial 2 m and finally, I have e square then 2 cube divided by factorial 3 and then here we have some term which do not have any kind of m here we have something which is which have m, but in the denominator and then whatever we have is in the numerator.

So, e to the power 2, 2 to the power 4 m factorial or 4 factorial 4 and so on. So, now if I replace this m I will finally, have the expression something like this this is my.

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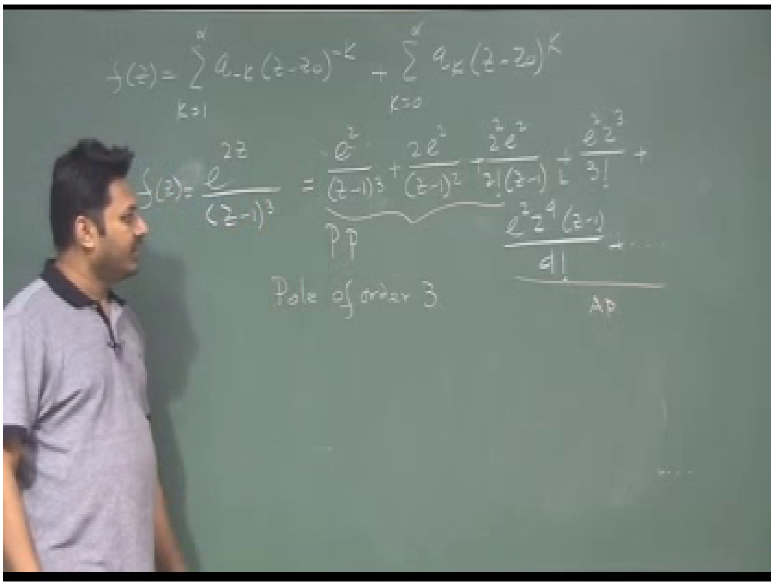

So, my expression will be e square first time is m was represented by Z minus 1 was represented by m. So, it will be this first term second term was 2 e square Z minus 1 square, third term was 2 square e square divided by factorial 2 Z minus 1, fourth term was Z independent. So, it is e square 2 cube divided by factorial 3 which is essentially my a 0 term from where the analytical part then e square 2 to the power 4, Z minus 1 divided by factorial 4 and so on. So, my series is something like this and in this series you can find that there are 3 terms associated with Z minus 1 which is sitting in the denominator, this is associated with principal part and the rest of these things are analytical part and in principal part I have a singularity at Z equal to one and the order or pole of this singularity is 3.

So, we have a pole of order 3 here for this particular example. So, right now I should stop here. So, in the next class. So, today we will learn in the Laurent series expansion in the light of Laurent series expansion how the singularity can be understood and we find that there are few examples where we have no singularity. So, this is called the removable singularity or the function is analytic at that particular point, but it may not be necessary for other functions where we have some kind of singularity which is finite in nature that it seems that it should have a pole of some finite order, but in the next class we will show some example also we do not have any finite singularity. So, it is called the essential singularity and also learn how to expand a function in terms of Laurent series. So, with this note let me conclude the class here.

Thank you for the attention, see you in the next class.