Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 50
Classification of Singularity

Welcome back student to the next class of complex analysis. So, in the last class we learnt about the Taylor series.

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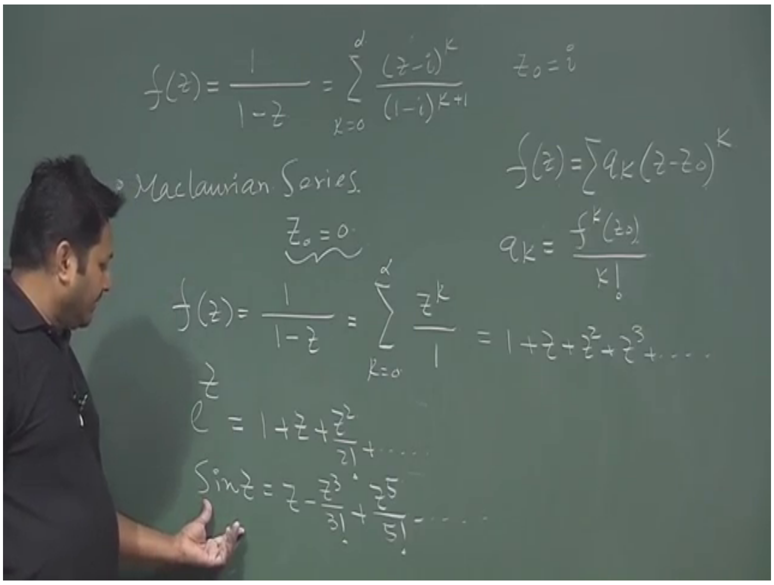

And if a function if I want to make a Taylor series expansion it will essentially be a power series, if it is analytic mean the region some d, and in that region if z 0 is there. I can expand this function in terms of this power series with a k having the form this. So, I have already reach in the expression that we find in our last class, that f z is given to this and we were asked to expand this particular function in terms of Taylor series around the point z 0 is equal to i.

And I did the calculation and my result was something like this. So, I just write it because I need to show more about that, and I am also find out what is the radius of convergence of that particular series and we find the radius of convergence is root 2. So, if I make a circle of that particular point i circle around that particular point i with the radius root 2, then that is the region of convergence of this particular series. At that point also I need to show you one thing that i just mention last class that is Maclaurin series.

So, Maclaurin series is a special form of Fourier series when z 0 is equal to 0 that is the that is the condition I need to put.

So, here what should be the Maclaurin series? The Maclaurin series is simply the Maclaurin series is simply z to the power k divided by 1, or just z to the power k this is the standard result were k running from 0 to infinity k running from 0 to infinity. And this is the standard result that we have already we know, where there is no i mean. So, this series is already known to us. And if I expand this series in terms of Taylor series at that particular point it look something like this, where I am doing the expansion at z 0 equal to 1 i point. But here in the Maclaurin series I am doing the same thing, but my z 0 point is not i rather 0.

So, if I put that then 1 by z this quantity if I expand it will be something like this, k 0 to infinity z to the power k by 1. Now if I expand this it will be just one z square z cube and so on. This series it is already known that i mention. In the similar way you can now if I do this function say e to the power z, e to the power z if I expand in Taylor series at z equal to 0 point it will be just 1 plus z plus z square by factorial 2 and so on. So, this is an ex expression of e to the expansion of e to the power z, which is nothing but the Taylor series at z equal to 0 point. Also the same way I can expand sin of z which is z minus z cube divided by factorial 3 plus z to the power 5 divided by factorial 5 and so on.

Again my sin z expansion is exactly the same expansion you know it is nothing but the Taylor series expansion of sin function at z equal to 0 point. So, this is both the thing it is nothing but the Maclaurin series, it is Maclaurin series which is the standard expression we know it is nothing but the Maclaurin series. You should appreciate this fact. So now, we go to our next few problems where we need to find out the Taylor series expansion and corresponding Maclaurin if it is there.

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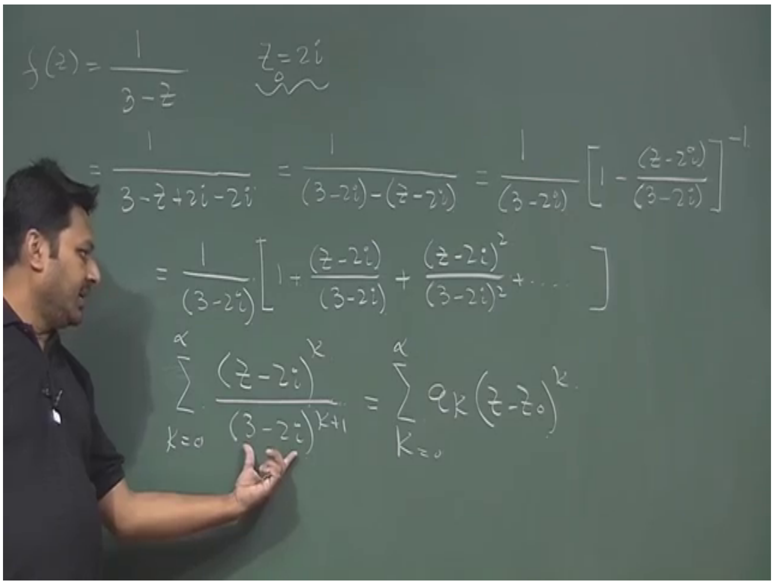

So, let us take another example state forward f z is equal to 1 divided by 3 minus z. I need to expand this function in terms of Taylor series, but I need to know around which I am going to expand this. So, the point is z equal to 2 i z 0 rather z 0 equal to 2 i. And also after expanding we need to mention about the radius of convergence of these things. I can expand, but I need to mention the radius of convergence, which we can find once i put this as a power series form, a k is known then I can do that. So, the procedure is exactly the same that we have been using i add 2 i and subtract 2 i, and write this expression in this way. 3 minus 2 i minus z minus 2 i then take common this and I will have 1 minus z minus 2 i divided by 3 minus 2 i whole to the power of minus 1 ok.

So now if I know right expand this 1 by 3 minus 2 i, then it will be 1 minus 1 plus z minus 2 i divided by 3 minus 2 i plus z minus 2 i square divided by 3 minus 2 i square and so on. I can just expand this considering the fact this is less than 1 and that condition basically gives me the idea how the radius of convergence is there; so in the submission form. So now, I have a series. So, in the submission form I will simply have z minus 2 i whole to the power k divided by 3 minus 2 i k plus 1. This is the series I have for this given function when I am expanding the series at 0 equal to 2 i point.

So, this is the series I have. So, it is equivalent to our old power series as i mentioned a k z minus z 0 to the power k is equivalent to this. Now if I compare then we can readily find z 0 is nothing but 2 i and a k is this quantity. Again I can have the value of l.

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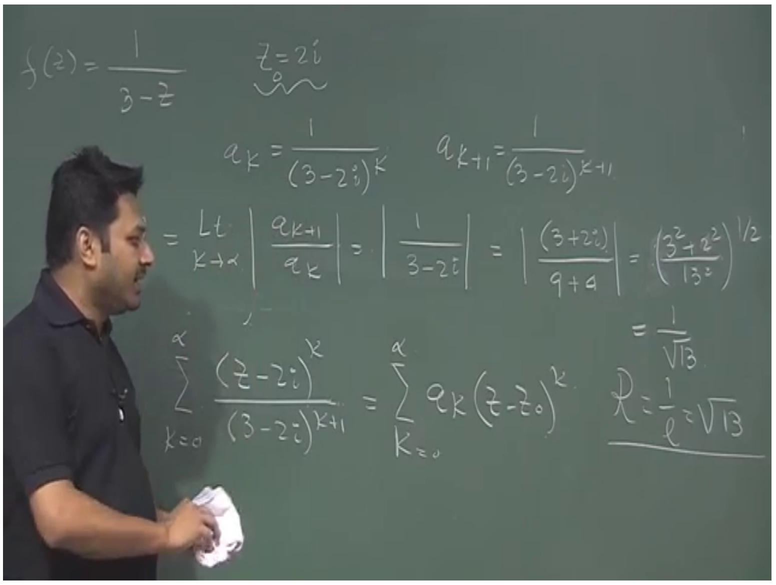

So, a k if I compare with that is given as comes out to be 3 minus 2 i whole to the power of k. So, a k plus 1 is 1 divided by 3 minus 2 k, k plus 1. And my l limit k tends to infinity a k plus 1 divided by a k is straight away I can write 1 divided by 3 minus 2 i, this quantity.

So, please note that first this is the function is given and I need to expand this function around the point 2 i and if I expand it will come like this. Now if it is this now I need t o mention about the radius of convergence, and radius of convergence is coming from this fact l value 1 by l should be my R radius of convergence. And this quantity I need to find out. So, it will be simply mod of 3 plus 2 i divided by this a plus b into a minus b a square plus b square. So, 9 plus 4 it should be something like 13 a plus b into a minus b. So, a square it is 9 and b square it is 4; so 9 plus 4 13.

So, the result will be something like 3 square plus 4 square divided by 13 square whole to the power of half 3 square by 2 square, so again 13. So, it will be just 1 by root over of 13. So, R which is 1 by l will be root over of 13 that. So, I find the radius of convergence here for this particular problem and if I want to visualize that which is important every time I suggest you to visualize where this convergence is happening.

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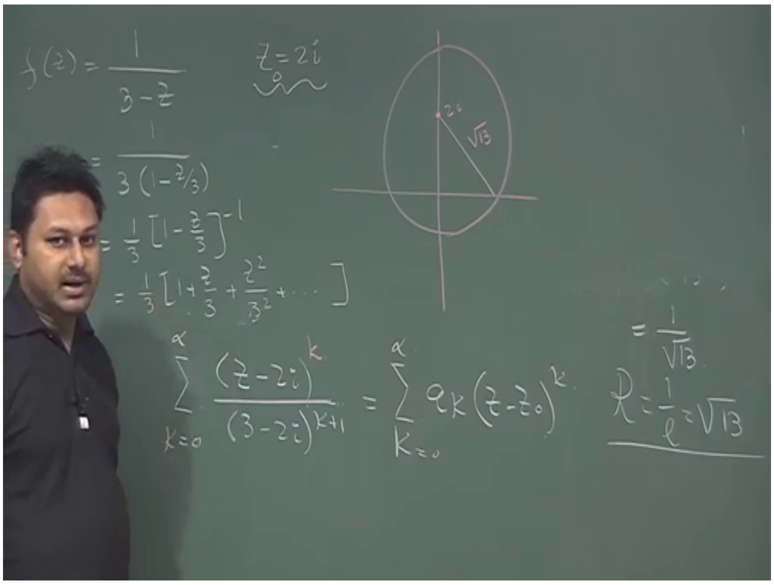

So, it is 2 i point is somewhere here if I make a circle over that with radius root over of 13.

This is the region where I can expand this function. And if you want to find out the Maclaurin series again you just need to put 2 k is equal to 2 i is equal to 0, then you will have z to the power k divided by k to the power 3, k to the power 3 to the power k. So, it will be just 1 by 3 to the power k and the expansion of z. So, it will be the Maclaurin series this because of this 3 term I will have this. So, the Maclaurin series can easily be figure out in this way. If I take 3 common then it will be 1 minus z by 3 and then 1 by 3 1 by z by 3 whole to the power of minus 1; now the expansion gives me that this 1 by z by 3 plus, plus z square by 3 square and so on.

And the radius of convergence should be such that the value of this thing should be less than 3. Because when it is less if it is less than 3 then this quantity what every whatever we have is less than 1 and then this expansion is possible. So, the radius of convergence for this Taylor series is root over of 13. If I do the radius of convergence for this series Maclaurin series is will be something different and it is obvious. Because when I am doing the Taylor series expansion if I put this point higher and higher my region of convergence will be larger. Another result I am having a radius of convergence large. On the other hand in this case if I do the Maclaurin series this series will be something like this, where the radius of convergence comes out to be root over 3 or something like that.

Is there any other problems? Let me check another interesting kind of problem.

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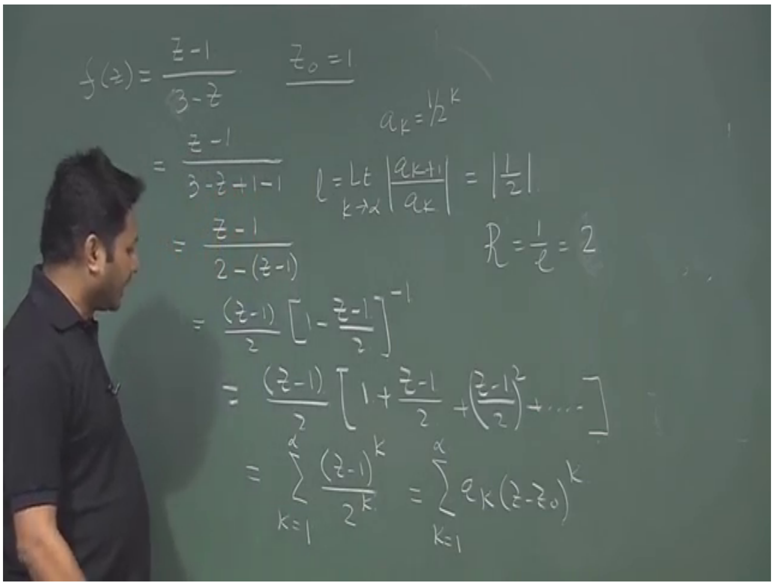

So, f z is given as z minus 1 divided by 3 minus z 3 minus z we need to expand this things around some point z 0 is equal to 1. This is the point where we need to expand my function. So, we will stick our old our old process; that means, I will add one and subtract one, and as a result let me find what we are getting 3 plus 1. So, I am getting like 2 and then 3 minus 1 I am getting 2 and minus say z minus 1. I am getting something like this.

This is the expression I have from here; so 3 minus z plus 1 plus 1 plus 1 minus 1. So, I am writing this things just by introducing the point i here it is something a different kind of a function that is given one z is given up stair. So now, i expand in this way. So, here z minus 1 is there 2 is here, if I take this 2 common and then it will be 1 minus z minus 1 by 2 whole to the power minus 1. The expansion will slightly change and it will be something like this. Now I will expand this z minus 1 by 2, 1 plus z minus 1 by 2 plus z minus 1 by 2 square of that and so on.

So, the expect the expression already I have. So, it is suggest it is sum over z minus 1 k divided by 2 k. K is say one to infinity. This expression i finally, get. So, this is the a series Taylor series expression of this particular function at point z 0 equal to 1. So, again if I tally that with power series then my a k comes out to be 2 to the power k. So, a k plus

1 by a k minus a k mod of that with limit k tends to infinity, which is my l is something like 2 mod of 2 or 2 R seems to be 1 by l is half.

So that means, I can expand this quantity, sorry one thing because it is k is a k is 1 by 2 2 over k, because this is 1 over 2 by k. So, here it will be just mod of 1 by 2 and l will be 2 1 by l it seems to be correct, because the coefficient is here in the denominator. So, a is 1 by 2 k. So, if I now try to find out what is the convergence where it should converge, what is the region?

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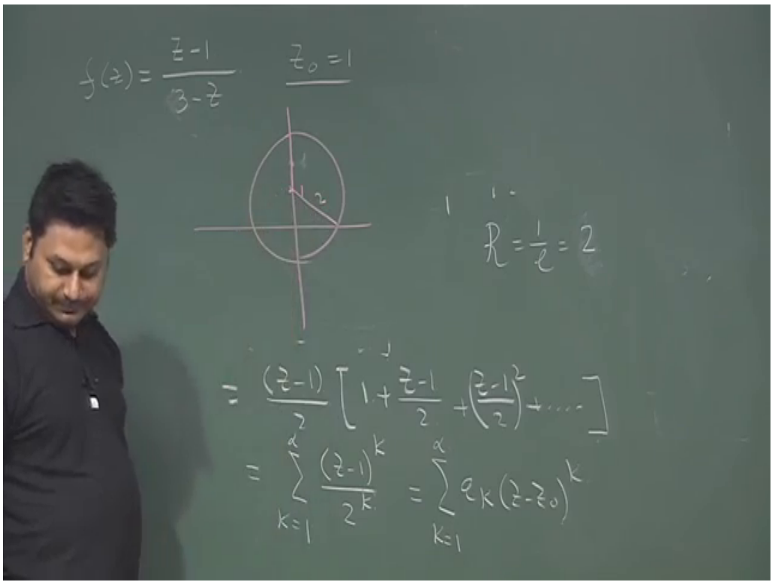

Then if it is 1, this point is 1. If I take a circle, say this point let me put it point here then if I put a circle of radius 2, then I will have something this.

So is the region, this is the region, where this ex expansion is valid. And it will converge for all z’s that are lying in this region. So now, we will go to our next step which is important. So, I believe you now understand how to evaluate a Taylor series at some given particular point. So now, maybe we can go to some another topic which is called singularity and it is classification.

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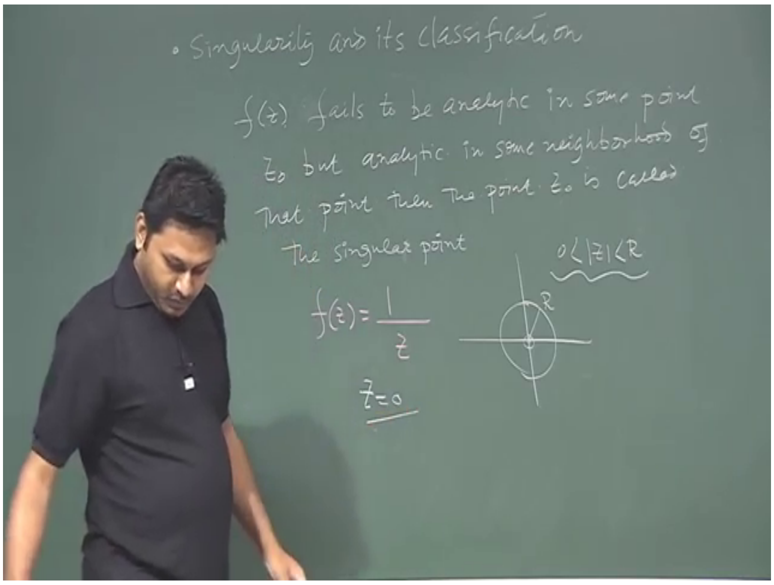

So, singularity we know that, if a function f z fails to be analytic in some point z 0. But analytic the function f z fails to be analytic in some points z 0, but analytic in some neighborhood of that point, then the point z 0 is called the singular point. What is the meaning of that? F z is fail to be analytic some point z 0, but analytic in some neighborhood of that point, then the point z 0 is called the singular point. If you visualize these things it will be very clear. So, suppose a function f z is given like this.

So, at z equal to 0 point the function is analytic. So that means, at z equal to 0 the function is analytic. But if I remove this 0 point, say this is a radius R so that means, mod of z less than R less than 0. So, some deleted neighborhood it is called. So, I am just delete this point 0 point, then this function is analytic in this region in this region mod of z less than or less than 0 the function f z is analytic.

However, at that particular point 0 the function is blowing up. So that means, the function should have singularity. At that point here the singularity is at z equal to 0. Well, I can also have this kind of singularity. Where is the singularity? At z equal to i.

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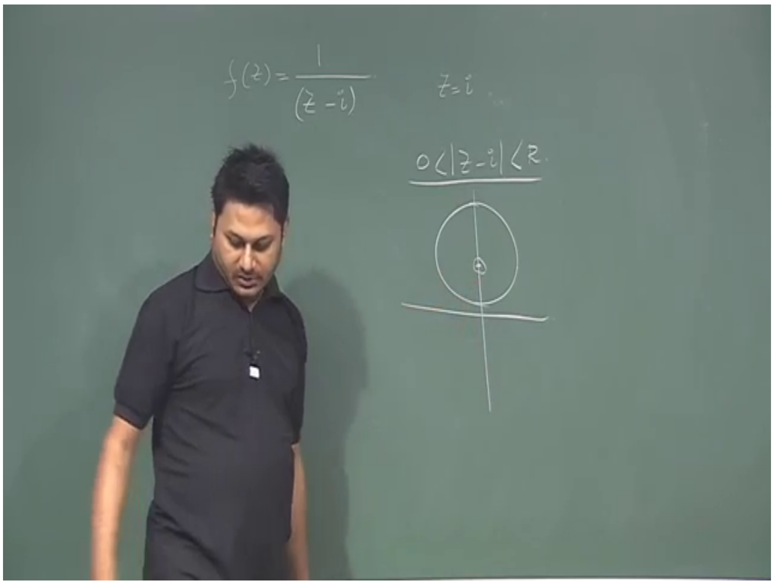

So, z minus i less than some R less than 0, if I take 0 outside this is i take i cut this point and take same neighborhood i find this function is still analytic accept that point here z equal to i. So that means, this is this point is singular point. Next this thing is now i understand; what is the meaning of singular point.

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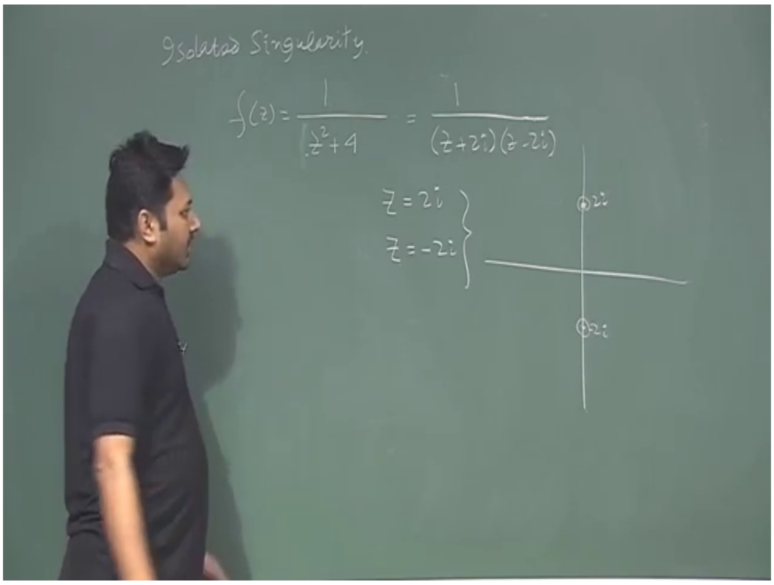

Next is isolated singularity. Say this is the function. Now this function if I write this function in this way, we have already done this in the previous class, but let me now do it move carefully. So, this function is this.

So, we find that z equal to 2 i and z equal to minus 2 i R 2 points where the function again grows up. And these 2 points are called isolated singularity; that means, it is isolated from this point is isolated from other points. So, in general we called it is in isolated singularity. There are many other points i mean all this points are eventually the isolated singularity. This is an example of that.

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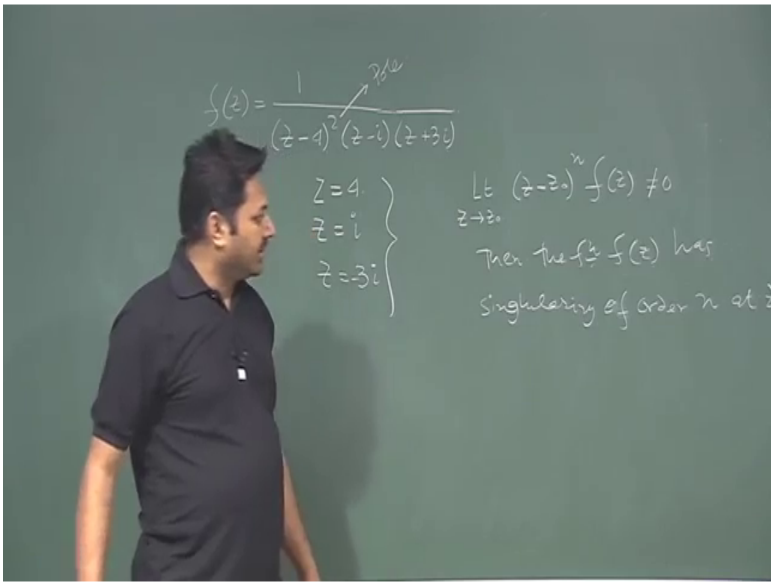

So now again let me put something some function say again write this function in this way. So, how many singularities are there according to our knowledge there is one singularity at z equal to 4, another singularity at z equal to i, and another singularity at z equal to minus i.

So, there are 3 difference singularity isolated singularity are there. But is there any difference between these 2 singularities i these things and these things, we have something here which is called power. So, this is called the pole this is called the pole of singularity pole. So, by definition say limit z tends to z 0. if I multiply z minus z 0 to the power n to the functions this, and if gives if it is not equal to 0 give some value then we called I have a similarity at z 0. I multiply z minus z 0 whole to the power n to the function, if it is not equal to 0 and then put limit z tends to z 0. Then the function f z has singularity of order n of order n at z 0 point.

So that means, I have a function f z i multiply z 0 is a singularity i know, i multiply z minus z 0 whole to the power n. And if it is not equal to then i put the limit if it is not equal to 0, then I can say this z 0 at z 0 point the order of this singularity pole of the singularity is n. if I know apply here, in this particular problem.

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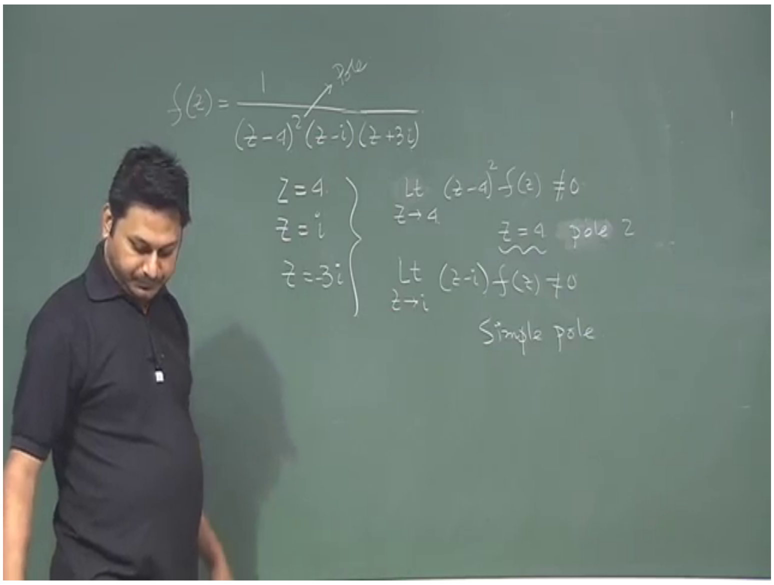

Then I need to apply this limit z tends to 4, z minus 4 square f z which is not equal to 0. When you multiply z this term will cancel out, and then if I put z equal to 4 we have some value of this quantity. So, here z equal to 4 this is a singularity of this is the pole of with pole of order 2.

In the similar way limit z tends to i z minus i f z is not equal to 0, same as this here n value is 1. So, it is called simple pole, this is called simple pole. So, we have a singularity here of order 2, and this is order one this is order one which we called the simple pole.

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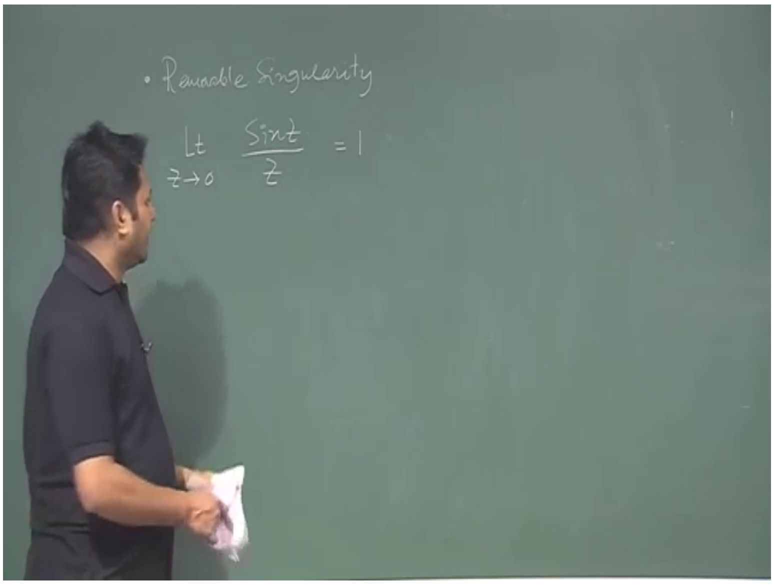

After that quickly we will write to singularities. One is called removable singularity, which is say like sin z divided by z. if I take the what is the meaning of removable singularity, if I take limit here z tends to 0 where a suppose, the function supposed to have a singularity z equal to 0, but if I take limit of these things it will come to be 1.

So that means, we have a singularity here at z equal to 0, but if the limit if I put the limit here then I will remove the singularity that is why it is called the removable singularity. So, I can do that by expansion of z sin z sin z is say z plus minus z cube by factorial 3 plus z 2 the power by factorial 5 and so on, since 1 by z is here. So, if I put it here it will be 1 minus the other term when i put the limit this term will going to vanish and I will have one in my hand. So, the way we normally use the limit I will do the same thing here, but here the only thing is that since z was original in singularity, but when i put the limit then I will find it is one; that means, it is removable in nature.

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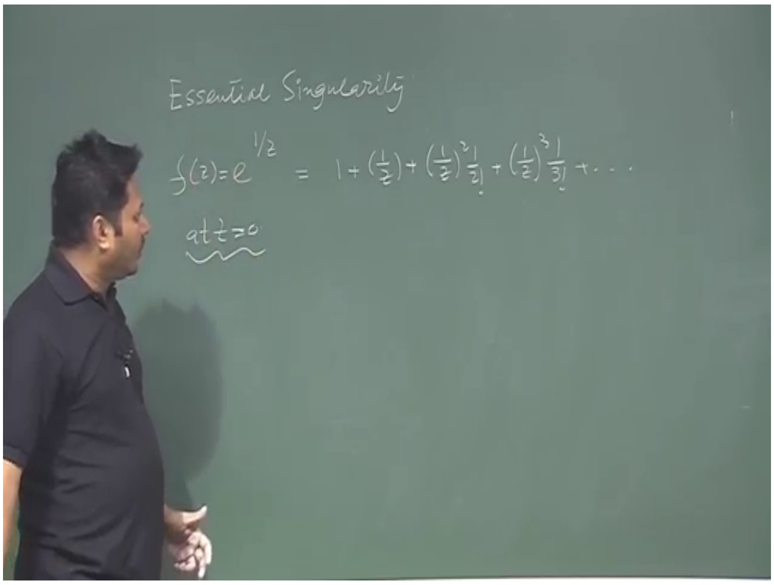

Another kind of singularity with that I will going to intend in the next class we will go more about singularity today we do not have much time it is called essential singularity. Essential singularity is the singularity which we cannot remove. For example, the expression e to the power 1 by z the function something like this; at z equal to 0 point this function has essential singularity. So, the singularity is there you cannot remove anyway.

So, if I expand this function it will be just 1 plus 1 by z plus 1 by z square 1 by factorial 2 1 by z cube 1 by factorial 3 and so on. Now if I put the limit z equal to 0 at z equal to 0 this power of z is going up. So, you never have a process to remove this singularity. That is why it is called the essential singularity. So, today we will we have learn.

So, let me conclude this class here. Today we have learned initially how the Taylor series can be expanded. And next will learn very briefly about the singularity, in the next class again we will we will recap all the singularity is and gives you more examples of different kind of singularities which is important. But in the next class we will start something related to series which is called the Laurent series. So, far we will learn how to expand and analytical function in a region where it is analytic. But if the region is bounded by something say some annular region is there, if I want to expand that function for that annular region, how to expand this function is basically given by this Laurent series. And from that we will have important few important outcomes like the residue and all these things.

So, with that note let me conclude the class here. In the next class we start with the singularity and try to find out; what is the meaning of Laurent series, and how to expand a function in a boundary or in some annular region.

Thank you for your attention. So, see you in the next class. Have a good day.