Mathematical Methods in Physics-I
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Lecture – 05
Dual Space

So, welcome back student for the next class of this linear vector space. In the previous class, if you remember we define 2 very important concept one was spanning set.

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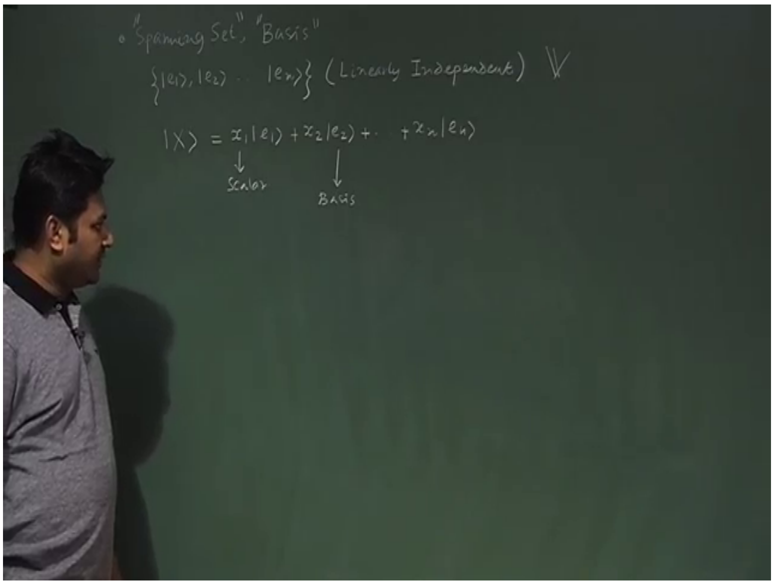

And then, we also defined something called basis. Now if in a vector space what is basis let me recall it. So, say e 1 e 2 e n are the set of n vectors, which are linearly independent to each other they are linearly independent. Then any vector this is a set of vector from a vector space v say. So, in this v we have many other vectors. Any vector x for example, that is in this vector space, can be represented as a linear combination of this like x 1, t 1 plus x 2 e 2 plus x n e n; x 1 x 2 x 3 are essentially the scalars or the components and e 1 e 2 e 3.

Which are linearly dependent set of vector we call it as basis. Now I change the notation this notation has already been introduced to all of you that, this is a very important notation in the context of quantum mechanics. And we call this as a ket notation or sometime it is called dirac notation also.

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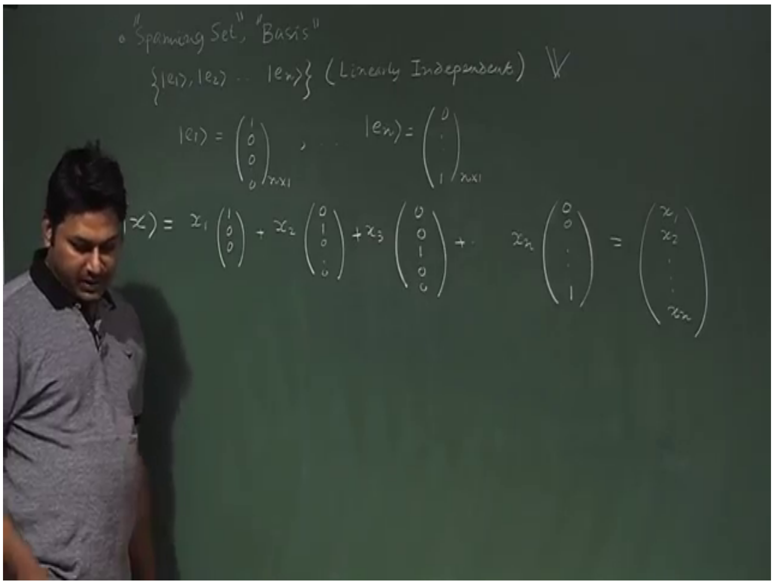

Now, the next question is, also we define something in terms of this n tuple notation say e 1 is if I say this is a n cross 1 column matrix. And in this way if I represent e n which is something like and this is another notation which is the n tuple notation. And any vector if I write the vector in this form then x 1 multiplied by 1 0 0 plus x 2 right hand side I write x is my vector. Then x 2 is 0 1 0 0, then x 3 is 0 0 1 0 0 up to n term, and so on up to x n which is 0 0 1. This is the most natural way to express a vector in the right hand side, if I write the vector the vector will come out to be simply x 1 x 2 x n in a component form. This basis one these are called natural basis lastly we also discuss these things. Now in terms of this natural basis we can expand a vector as well as the matrixes.

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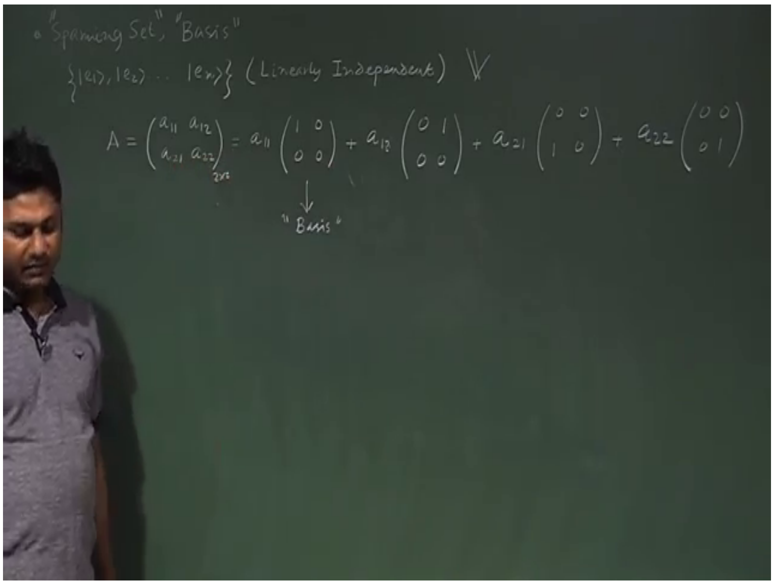

So, let me just briefly describe how a 2 by 2 matrix say A, this is for to note that how the basis is working natural basis is working. So, if this is a 2 by 2 matrix which is also be a element of this vector. So, we can also expand this matrix A in terms of it is basis vector. So, what should be the basis of a matrix it should be something like this. So, I can figure out a few matrices like this 1 0 0 0 then 0 1 0 0 0 0 1 0 and so on.

And this since this is a 2 by 2 matrix. I can define these matrices through which I can decompose the entire matrix elements. These things now work as a basis, and not only that it is a natural basis any matrix I can expand in this form. So, now, I have an idea that how the natural basis is working, but there are more important things related to basis. So, let me go back to our 1 dimensional case 1D that means, I have only 1Direction say x.

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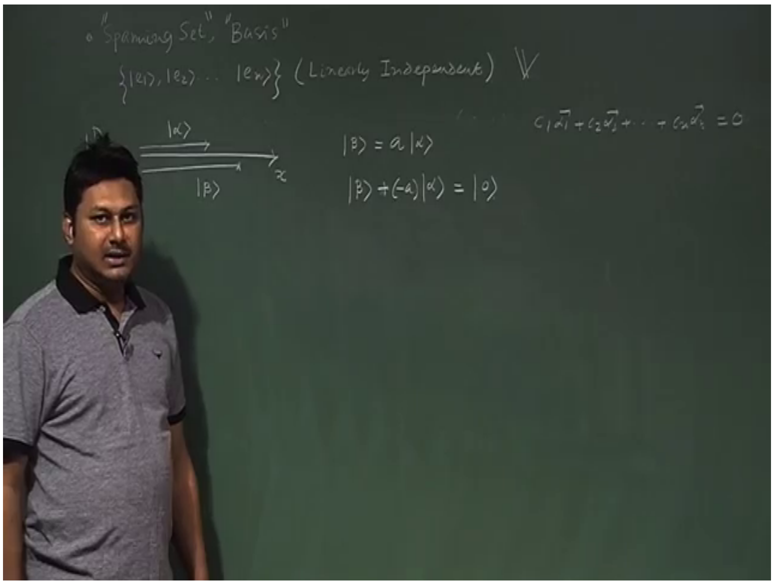

So, now the question is if I have 1 direction, 1dimensional thing then how many linearly independent vector I can find is a very important question that how many linearly independent vector one should expect for 1dimension. For example, if I write a vector like this say this is my alpha which is a vector in this direction.

If I want to write another vector in this direction say beta, I can always write beta in terms of alpha beta is nothing, but some multiplication of alpha. This is just increasing the vector beta and if I multiply by suitable value then I can have beta so; that means, alpha and beta are not linearly independent they are dependent to each other. Because if I write this equation in a different way, I will go back to my original linearly I mean linear combinations and I find that a vector if you remember this notation c 1 alpha 1 c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n is 0. And if only solution c 1 c 2 c 3 0, then these vectors are linearly independent vectors, but here we have already find a condition where a is not equal to 0 and there is 1. So, both are not equal to 0.

But still I am having a condition that this is 0; that means, I have a non-trivial solution for c 1 c 2 c 3 where 1 and minus A are 2 values of c 1 and c 2 if it is taken up to 2 2 dimension right 2 components so; that means, I can have a non trivial solution to these vectors are linearly dependent so; that means, in 1D at most I have one linearly independent vectors.

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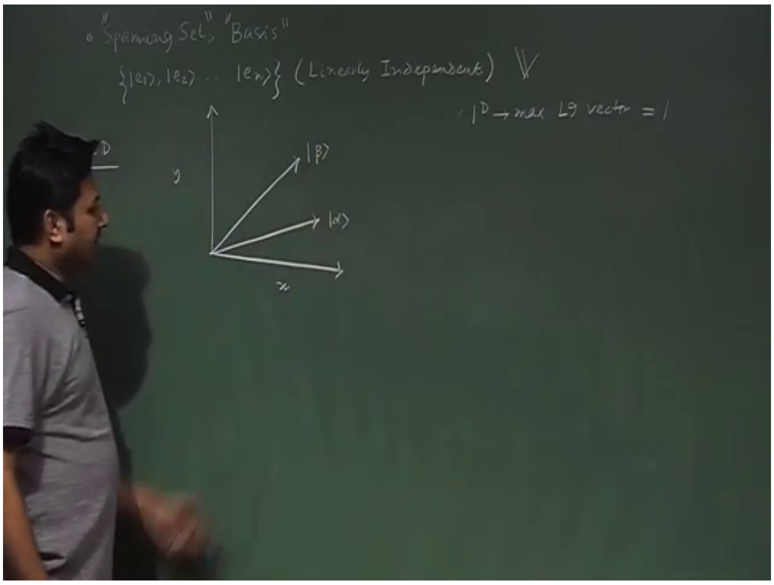

So, 1D I should note that maximum linearly independent vector is equal to 1. Go back to now let us try to find out what is happening in case of 2 D. So, 2 D, in 2D I have x and y this is my 2D plane. In 2 the plane if I have 2 vectors like this and this I call it this vector alpha I call this vector beta. So, there is no way that I can write alpha in terms of beta.

So; that means, essentially alpha and beta these 2 vectors are linearly independent to each other. We can also show that in a more general way. So, let me let me do that. So, now, we have an note that in 2 D, I have at most 2 vectors which are not depend on each other or linearly independent, but I can do that in a more convenient way. So, I will go back to this.

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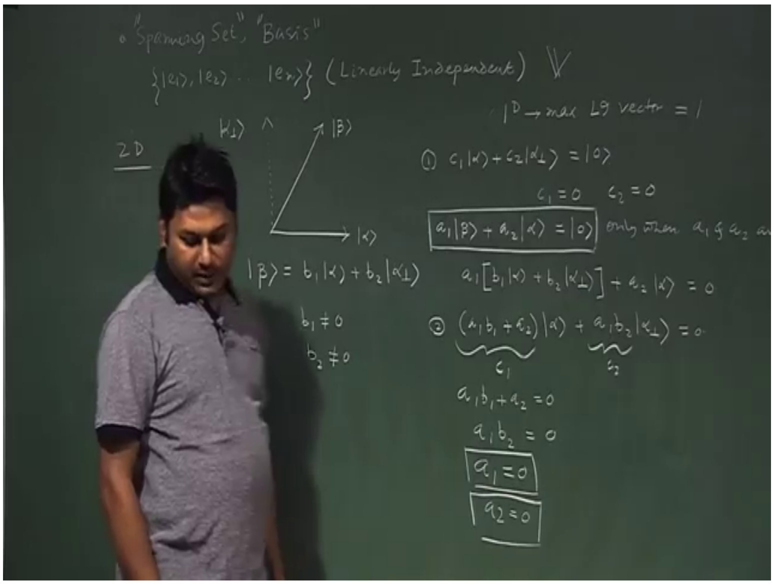

So, this is say my beta vector, this is say my alpha vector which are in 2 dimensional plane. Now what I will do that I will make a vector parallel to perpendicular to this vector alpha, I called it alpha perpendicular vector. When I make perpendicular to alpha vector then I can always write this equation c 1 alpha plus c 2 alpha perpendicular, if it is 0. I am using entirely I am using the ket notation to make you familiar with this notation, you can do that in the same way by putting the normal vector sign it is not going to make any difference, but it is better that you should you should learn this this notation also.

So, now, if I have this expression in my hand, since these and these vectors are perpendicular to each other; that means, no way I can write this vector in terms of this vector. So, only solution is c 1 is equal to 0 and c 2 is equal to 0. This is the only solution; that means, the vectors are this vectors are linearly independent vector that is for no doubt, but what about beta I can now, now I need to find out this condition a 1 beta plus a 2 alpha, if it is 0 my next question is it possible to show that a 1.

And a 2 is also 0, because I am claiming that these 2 vectors are linearly independent vectors because these 2 vectors cannot be written in terms of each other, but I need to show that. So, after considering this equation I can show that it may be possible. So, let me do that in this way. So, beta I know that this is a vector along this direction can be divided into 2 components, these and perpendicular components. If I do that say I b 1 alpha plus b 2 alpha prime these are the 2 components, I can divide subdivide this beta that I always do after having the value of beta in this form. The next thing is I can put this here. So, a 1 I have this equation, now I am using this equation to this equation a 1 beta is this much.

So, let me write it beta 1 alpha plus beta 2 b 2 alpha perpendicular plus a 2 alpha is equal to 0 in terms of alpha and alpha perpendicular, I write entire equation which is this now I can also write this equation by rearranging the term in this way. Now if I look this equation say this is equation 2 and my first equation 1, this essentially both the equation are same equations c 1 some constant multiplied by alpha c 2 some constant multiplied by alpha perpendicular here, also I have something multiplied by alpha.

So, I can consider this as my c 1 I can consider this as my c 2. Here one thing also you should note that since I am subdividing beta into 2 parts. So, this b 1 should be not equal to 0 and b 2 should be not equal to 0. Both are not equal to 0 simultaneously unless beta is a null vector I am not dealing with the null vector. Right now so; that means, this equation c 1 and c 2 are 0 already we figure out.

So, from that I can say from first equation a 1 b 1 plus a 2 is equal to 0. And a 1 b 2 is equal to 0 because a 1 b 2 is nothing, but c 2 a 1 b 1 plus a 2 is nothing, but c 1 both are 0 I have already figured out. So, now, this is 0 with the condition b 1 and b 2 not equal to 0 I can also say that if this is not equal to 0 a has to be 0 to satisfy these things. So, a 1 is 0 one condition, I figure out if I put a 1 is 0 then naturally it comes that a 2 also has to be 0. So, now, a 1 and a 2 both are 0, it is the only condition that we find.

So, here if I find this equation which I try to find out we can readily see that this is possible only when a 1 and a 2 are 0. And what is essentially means essentially means beta and alpha are linearly independent vectors. So, 2 linearly independent vectors can be possible in 2 dimension. So, this is also an important outcome that step by step we are going. So, first equation we have already find first conclusion, that in 1 dimensional maximum linearly independent vector should be one in 2 dimension.

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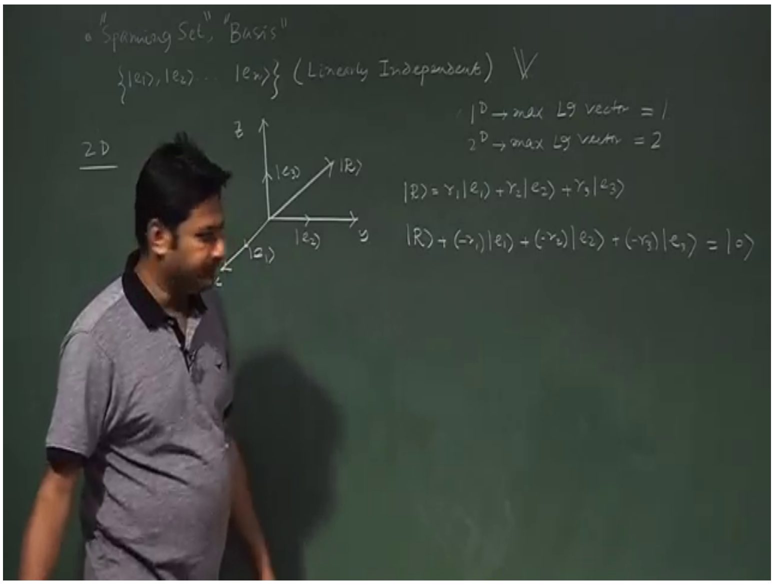

I can extend this comment that maximum linearly independent vector one can have is 2 what about 3 d. Now it is now you can understand what actually I am trying to do in 3D if I write this as my x coordinate y coordinate z coordinate then x y z is a 3 independent direction.

So, these vectors unit vector along this if I write it is as e 1 unit vector along this if I write it e 2 and unit vector along this if I write e 3 are the 3 independent direction vector which are not related to each other they are linearly independent. So, any fourth vector say r this is my fourth vector r can always be represented in terms of e 1 e 2 and e 3. So, I can write it readily; that means, this equation e 1 e 2 e 3 are all in the vector space. So, this equation one can modify and write in this form, just to show the way it is defined for linearly dependent independent vectors 0. So, now, I have this is 1, this is minus r one this is r 2 and r 3. So, I have a value if I say this is c 1 c 2 c 3 and so on. So, for non c 1 c 2 c 3 is not equal to 0 I am having a solution.

So; that means, if I use 4 vectors then there will be a linearly dependency among them. So, maximum linearly independent vector in 3D is how much there will be 3. So, again I have this for 3 D, I have maximum linearly independent vector is 3. Now is the time that we can generalize the thing.

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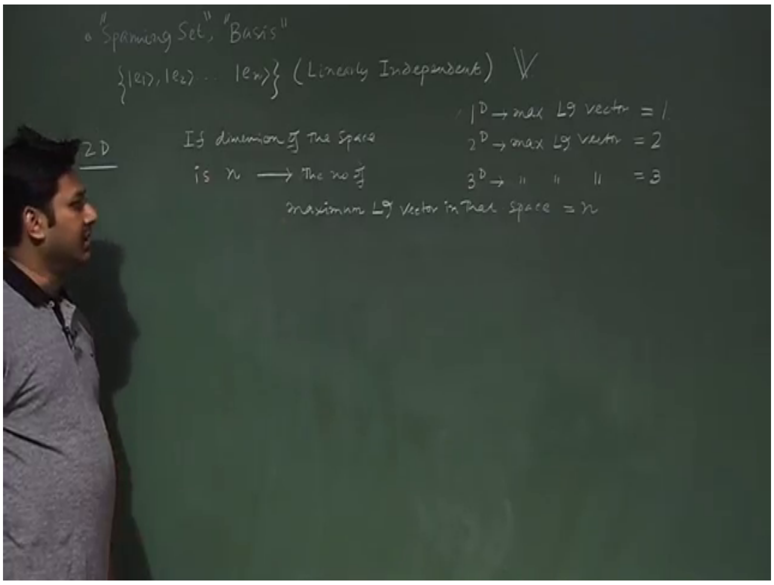

So, for 1D I have 2 linearly independent vector for 2 d, I have maximum 2 linearly independent vector for 3D I have maximum 3 independent vector. So, there is a intimate relationship between the number of maximum linearly independent vector that one can find in a space and the dimension of this space. So, if, if dimension of the space is n, then we can readily conclude that the number of maximum linearly independent vector in that space will be equal to n.

So, in n dimensional space, I can have at most maximum n number of linearly independent vector. I shouldn’t have more than that because if I have more than that vector then what happen that that vector can be represented in terms of other linearly independent vector. So, that we have already shown for 1D 2D and 3D cases. So, now, after having this important conclusion, now let us go back to another important topic which I mentioned in the last class, which is inner product because there is a relationship between these things e 1 e 2 e 3 lastly I mentioned also that.

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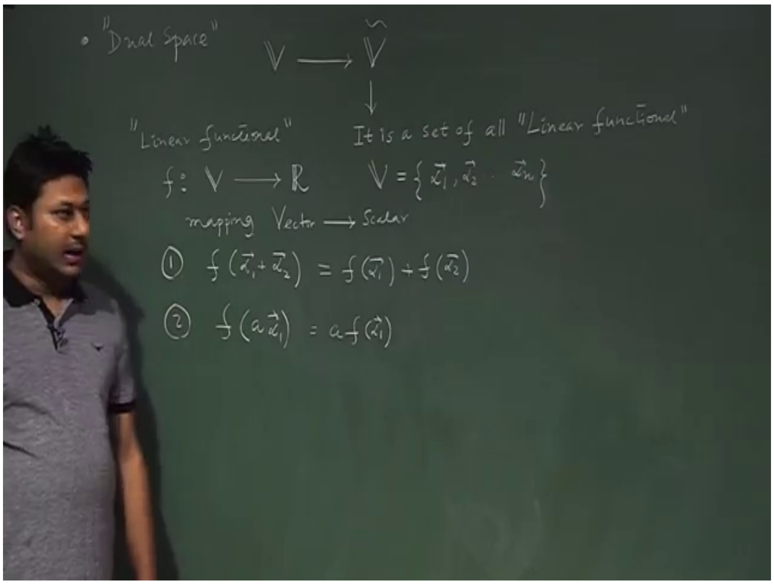

If e 1 e 2 e n are the vectors, then there is a relationship that they are orthogonal. We called it orthogonal right now I am not writing anything because I am not defining anything right now, but you should know that they are orthogonal which is important property the orthogonality property, only we can define or only we can understand once we learn the inner product thing. So, inner product is already you know for normal vector, but this is not the fundamental for fundamental understanding we need to know few things more. So, let us do that.

So, very important concept conceptually you should understand is what is happening which is called dual basis or dual space for the time. So, dual space what is dual space. So, if we have a vector space v then associative to vector space, I can construct or have another space I write it as v tilde which I called dual space. Now what is the specialty of dual space what it will do and what is what contain I mean what is what kind of things are there in the dual space.

So, dual space is containing it is a set of all linear functional. It is a set of all linear functional. Now I defined in order to define dual space, now I define another new term linear functional so; that means, in order to understand what is v tilde essentially I need to understand what is the linear functional because this is set of all linear functional. So, what linear functional will do let us try to first find them again we will go back to and understand what is the dual space. So, linear functional is essentially a rule the rule suggests that, if I apply if say it is a rule let me finish this and then I will understand make you understand linear functional is essentially a rule which we can apply on the elements in the vector space v which are vectors and as a result I am having something which is not a vector rather scalar.

So, this is a mapping which gives vector to a scalar. So; that means, I am operating something over a big vector which this rule, and as a result I am having something which is a real quantity or complex quantity for the timing let us consider this is a real quantity that is scalar. You know that this scalar field can be constructed by the real quantity real elements or real numbers as well as the complex numbers. So, here where what the mapping is suggesting it is suggesting that I will apply on the vector and I am having a scalar this is the most fundamental thing. Next it has 2 properties. If I apply say inside the v the elements are something like alpha 1 alpha 2 alpha n.

These are the these are the suppose these are the elements in the vector space which are vector. Now if I apply this function over to element alpha 1 and alpha 2 like this. It will be given as this is one property; that means, I will going to add this vectors thing and then I apply the rule over that whatever the thing it will give; that means, it will give this f rule which is applied over a vector will give something which is a scalar quantity. So, I will take one vector from v another vector from v I will add them, and apply the rule over that and I will get a real number or a scalar I will do the opposite thing not opposite. I will do the different thing I will take the vector alpha operate that over the rule, I will take another vector alpha to operate that over the rule I will take I will have now, 2 real quantities and add them this equation this things what whatever I am getting and these things are same things.

So, this is rule number one rule number 2 is if a is some scalar if I apply this scalar multiplied by this thing over that it is same as this. So, now, I can understand what is linear functional, and this linear functional suggest that if I apply these 2 things over that I will have 2 different quantities. If I apply this over that I will also have a quantity, but a will be going outside. So, these 2 with these 2 properties I can form the linear functional. So, so far whatever I am showing is nothing, but the definitions, but with these definitions say will not going to understand anything. So, let me give you the specific examples of these things, then you will be understanding that exactly what is going on in linear functional what is linear functional. So, let me start with 2 examples and then gradually I believe you will understand.

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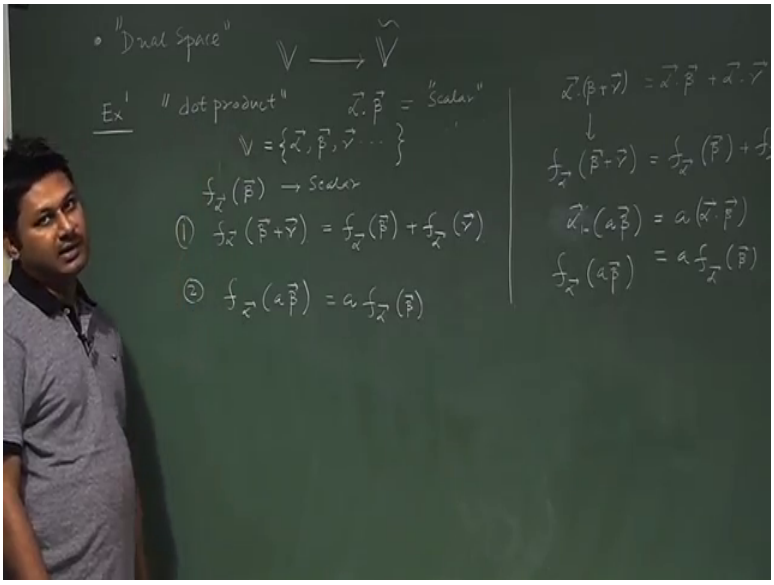

So, example, one example one example is dot product this is my, rule this is my rule dot product is my rule. How it is working you know that dot product if I have one alpha vector if I have one beta vector then when I am making the dot product; that means, I am applying a rule over that dot product is a rule, that I am applying over that then what I am getting is scalar quantity remember 2 vectors are associated with that I am applying a rule which I call dot product and as a result I am finding a scalar. So, I will write this things in our normal notation. So, for example, in a vector space I have the vectors like alpha beta gamma and so on. I am applying the rule dot product and when I am making the dot product from the left side I need to note this alpha.

So, I am noting that this is alpha which I am making which I am operating over a element say beta so; that means, this is my operation I am operating this. So, this will be a scalar quantity as I show here now my rule one suggests that, if I apply this over beta plus gamma applying this beta plus gamma. So, in the right hand side what should I write according to my rule it has to be this plus this whatever the rule I defined just before. So, now, let us consider with this in our usual notation things maybe you will be facing some kind of kind of uneasiness by looking this this kind of things, but let me do that. So, alpha dot beta plus gamma, how I write this I can write this alpha dot beta plus alpha dot gamma, no problem in that if I do then if I write in my notation it will be like this alpha is operating over that. So, I have beta plus gamma and this rule I can simply write in this way what about the next rule.

So, first rule is what about the next rule? The next rule you will now understand by yourself that if I do that if I make a constant, if I make a scalar and then operate it is nothing, but scalar this. So, now, if I write in the same way alpha in our normal notation say I will making alpha over a scalar beta with a dot product, then it will be simply a alpha dot p I can write it a alpha dot p. Always I write in this notation this and this one can write this this one can write. So, 2 of the properties are fulfilling with the fundamental rule that, if I operating these over that I will having a scalar it essentially means that dot product is following all the rules for linear functionals. So, dot product is a linear functional.

So, with that today I will going to conclude that class. In the next class we also start with this because dual space is a very important concept, we need to know in the next class we will start from this and we will start from a new example it is not that dot product is a only example. There are many examples which can follow the linear rule of linear functional. So, next class we will start another with taking another kind of rule which also follows the linear functional with that let me conclude that. So, see you in the next class.

Thank you for your attention.