Mathematical Methods in Physics-I
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Lecture - 49
Taylor Series

Welcome back student to the next class of Complex Analysis.

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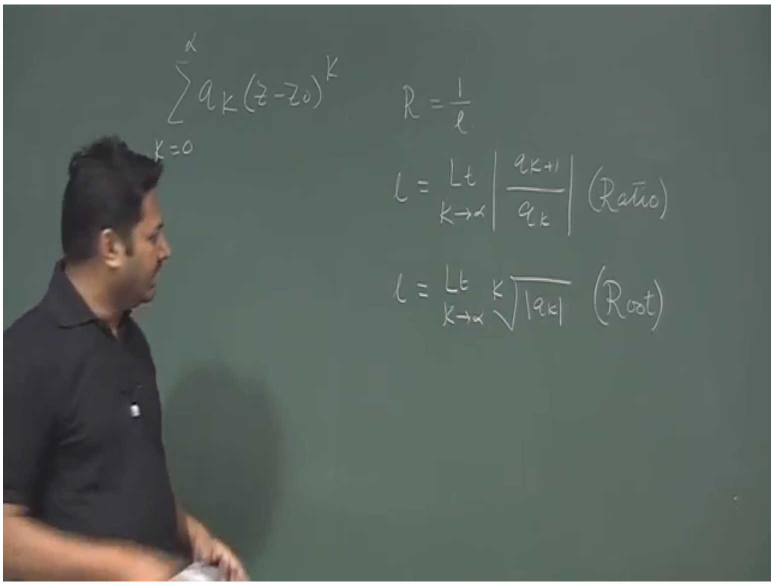

In the last class, we started something which is called power series. A series can be represented in this way which we called power series and then, find the radius of convergence of this particular series. Radius of convergence is defined as 1 by l r is equal to where l is limit, k tends to infinity a k plus 1 by a k. So, we are normally using this ratio test, but today I will be going to show just doing the next thing before going to the next topic, I just show one thing that l can also find out with root using the root test and sometimes if the coefficient of a is such that it is difficult to find out the ratio test, then we may go to this root test also.

So, let me show you one example where maybe it will be easier to find out the root rather doing, so just one example.

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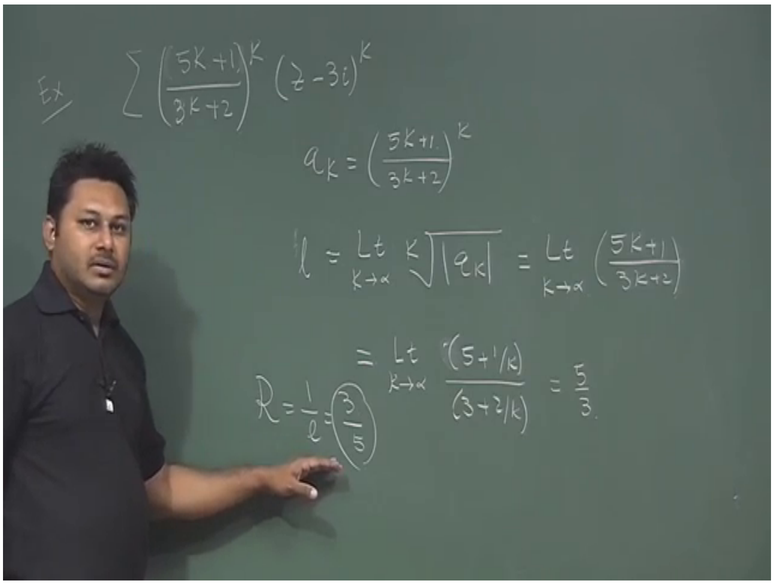

So, as a continuation of the previous lecture, I have something 5 k plus 1 divided by 2 k say 3 k plus 2 whole to the power of k and then, multiplied by z minus 3 i k. So, this is the problem that is given to us, where I have something like this. So, now if you check what is your a k, it check forward that it will be 5 k plus 1 divided by 3 k plus 2 whole to the power k, but what is your a k plus 1. It will be 5 k plus 1 plus 1 divided by 3 k plus 1 plus 2 whole to the power of k plus 1.

So, now when you try to find out what is your a k divided by a k plus 1 divided by a k, then there is some problem because this will not cut. The cancellation will be not that easy and then, you need to take the limit. It will not be that simple, but if you take the root of that, then things will be simpler. So, if I want to take mount to take the value with this test k tends to infinity a k, then mod of this to the power k simply comes out to be limit k tends to infinity, this quantity 5 k plus 1 divided by 3 k plus 2.

So, now things have done. So, you have this quantity. So, now if I put the limit, then limit k tends to infinity. This will be if I take k common out, then it will be 5 plus 1 by k divided by 3 plus 2 by k and if I put k tends to infinity, then this term will be going to vanish and finally, we will have something 5 by 3 l become 5 by 3 r which is 1 by l will be 3 by 5. So, here 3 by 5 is my radius of convergence and the previous way we do in the previous case, the previous problems here also we can find.

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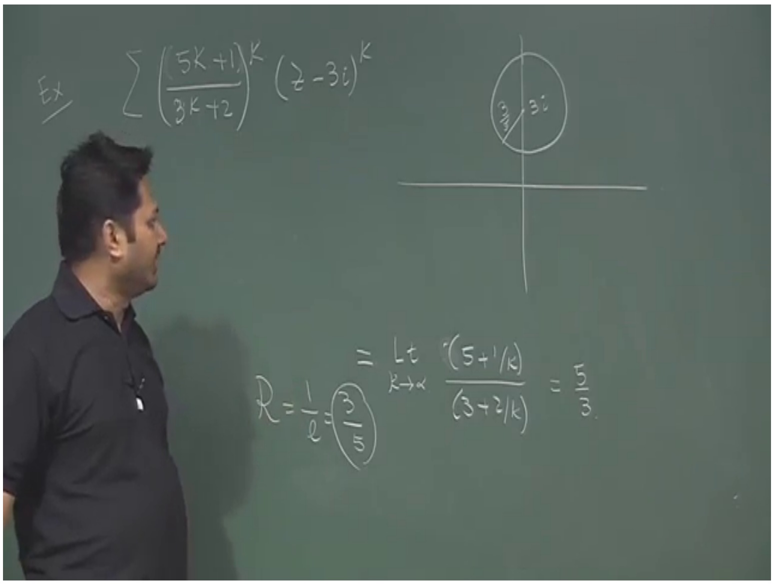

So, 3 i is somewhere here. If I make a circle around 3 i whose radius is 3 by 5 say something like this, then these are the region of z for which it will be going to converge. So, this is one example where you can use the root test rather ratio test just to show that in some problems, you may need to apply the root rest also. It is not necessarily always you will be going to apply the ratio test. Ratio test is convenient the way problems are given normal. We start with the ratio test, but it is not necessarily that always you use the ratio test. To test is another way to do that. So, next in our syllabus, we will have something interesting which is called the Taylor series of complex function.

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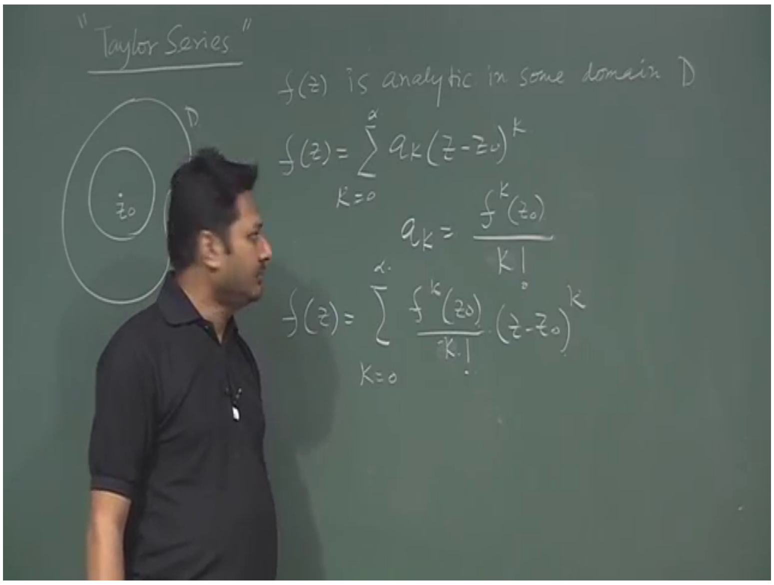

Let me just state quickly. So, what is the meaning of this Taylor series all of you are aware of. So, if a function f z is analytic in some domain d, so that means if I have some domain here d, it is analytic, then f z can be expanded as power series around some point z 0 like this. No problem with that so far. So, we have some point z 0 here. Around this point I can expand this function, f z is analytic in the domain d. I have a z 0 point. I can expand these things around that point, where a k now a k is the coefficient which is the characteristic coefficient. So, this characteristic coefficient will be this. So, the characteristic coefficient can be represented in this way.

So, in total I can write that the function f z, it can be represented like this. This is the statement of the Fourier series whatever the Fourier series we have. So, a function which is analytic in some domain d, I can expand that particular function around some point z 0 and if I expand this function around the point z 0, then I have some a k. This characteristic coefficient in my hand and this characteristic coefficient will be something like this, where the function derived k time a k is the k time derivative at that particular point divided by factorial here. It should be factorial k. I should not use n here factorial k and this, ok.

So, quickly try to understand how these things are happening.

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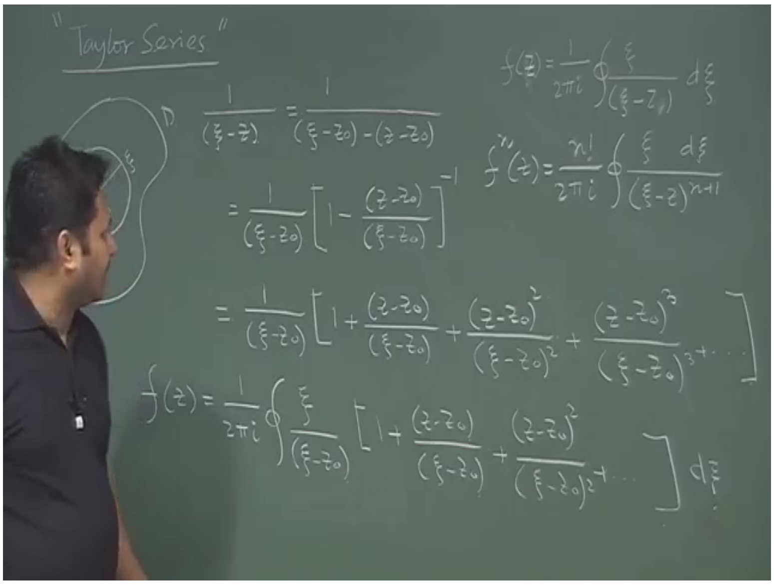

So, if I have some point say z here and this radius is i. So, inside that i I am expanding. So, z is the point inside this, this region then I have. So, let me first write Cauchy's integral formula because I am going to use that the Cauchy's integral formula suggest this is 1 by 2 pi i xi minus this, where I write xi which is variable here. It should write a z, I write the Cauchy's integral formula different way. I write f z in terms of xi here. Normally if you remember that it was something like f a was 1 by 2 pi i integration of z divided by z minus a d z. This index I just changed to xi and this a i changed to z.

So, z is a point here. Try to find out the value of the function and xi is the region having a circle say and I just change that. So, the thing is not going to change. That thing is same. So, f n z is 1 by 2 pi i with factorial n here sitting xi minus z whole to the power n plus 1 d xi. These two expressions I have form these Cauchy’s integral formulas. This is very important expression I mentioned earlier. So, these two I have in my hand.

So, let us try to exploit these two expressions and check whether I can have these things or not. So, if I expand for example, xi z I write this thing in terms of z z 0 because around the z 0, I am trying to expand my function. Remember this is my function f z. Right hand side is something and I am trying to expand this function around z 0. So, somehow z 0 point should come here. So, let me write this as xi minus z 0 minus z minus z 0. So, minus z 0 and plus z 0 I add and I rewrite this same expression in terms of z 0, so that the z 0 terms will cancel out. So, just xi and z is sitting here. Now, what I will do that I will take xi minus z 0 common out of here and I will have an expression z minus z 0 divided by xi minus z 0 whole to the power minus 1.

Again I rewrite the same expression in a different way. I take this common outside, then it will be 1 minus z minus z 0 divided by xi by z 0 and this xi by z 0 1 minus z by z 0 divided by xi by z 0. I can take it, write it in this way with to the power minus 1. Now, if we do, if we now expand this, this quantity, so it will be something like 1 plus xi minus z minus z 0 xi z 0 cube and so on. Also one thing we should note that z minus z 0 is this and xi minus z 0 is this. So, z is inside. So, z minus z 0 divided by xi minus xi 0 if I take the mod out of that, it should be less than 1 because this quantity is less than this quantity.

So, it will be just less than 1, then I have the ability to expand this. In this way this power series now 1 by z is expanded in this way and I can put this here. So, my f z is something 1 by 2 pi i closed integral xi by xi minus z 0 multiplied by 1 plus z minus z 0 divided by xi minus xi 0 plus z minus z 0 square divided by xi minus square and so on with d xi. I just replace this 1 by xi this quantity here and as a result, I have an expansion of f z in terms of this. This is the thing, ok.

So, let me erase this part may like shown to require this also, then my f z.

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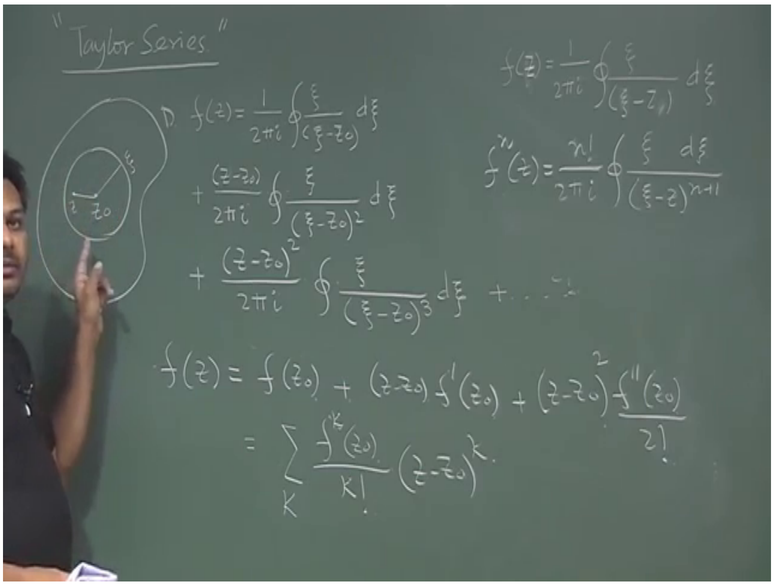

First term if I calculate, it is 1 by 2 pi i, then integration of xi by xi minus z 0 d xi. This is my first term. What is my second term? It is plus 1 by 2 pi i integration of please note that z minus z 0 z minus z 0 square. All things are not a function of xi. So, I can take it outside. So, here it will be z minus z 0 that is going to outside, then xi divided by xi minus z 0 square. This is my second term. What about the third term? Third term will be z minus z 0 square divided by 2 pi i total integration of xi divided by xi minus z 0 cube d xi and so on.

Now, what I will do? I will just try to find out what is the meaning of this term. This term if I look here it is nothing, but f z. The first term is nothing, but f at z 0. Second term is how much z minus z 0 multiplied by f prime at z 0 because this quantity 1 by 2 pi xi integration of xi minus z 0 square is nothing, but f prime z 0. What about the third term? Third term is z minus z 0 f double prime z 0 divided by factorial 2. I need to put a factorial 2 because this term will come here.

So, if i check from here, factorial 2 is 1 by 2 pi i integration of the rest. So, I am taking this term where n is equal to 1 n is equal to 2. So, that means cube term will be here. So, here we have the cube cube terms. So, here we have a square term by the way and I should have a factorial 3 here, factorial 2 here. So, that means if I go on with this, I will eventually have this quantity f n th order derivative or f k th order derivative z 0 divided by factorial k z minus z 0 whole to the power k which is nothing, but the Taylor series expansion that the first expression that I have written.

So, we can know that how to calculate the Taylor series. At least an idea that how this Taylor series is expanded with the fact that if a region is given to me, I can expand my Taylor series in this way. So, after having a very rough idea, how would the Taylor series we prove. The Taylor series is defined in this way. Let us go back to some problems because at the end of the day you need to do few problems and when you do the problems, you need to be careful enough to to expand before that. So, let me again write just the expression of the Taylor series.

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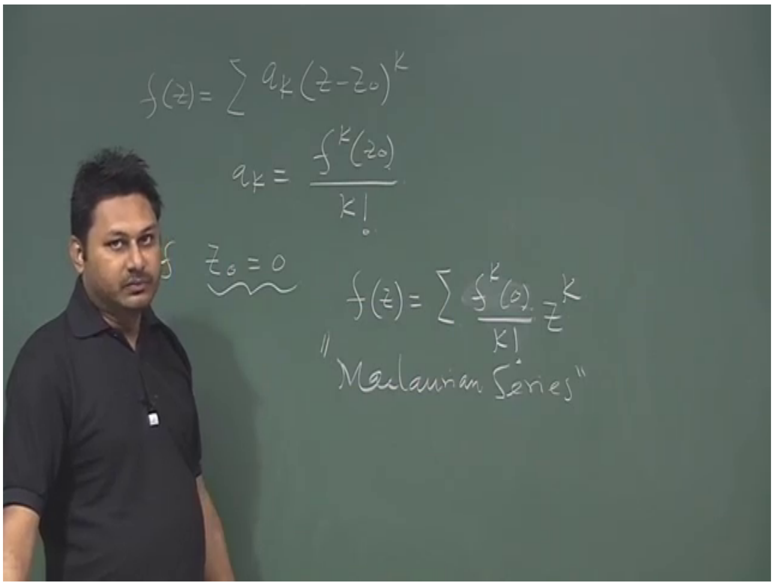

So, Taylor series suggest if the function is analytic in some region g, then it can be represented in the power series in this way, where a k is defined like this factorial k, this is the ex, this is the expression of the Taylor series. Now, if z 0 is a point around which I try to expand my function, if this point z 0 is equal to 0, this function is now represented like a k let me write total. So, f k at 0 factorial k z to the power k; the same function is now represented in this way.

Since I put z 0 is equal to 0 because z 0 is the point along I mean around which I try to expand my function, but I put z 0 point is 0. If I do that, then I have another expression. So, this is called the Maclaurin series. So, that means if the Taylor series is if I expand a function in terms of Taylor series, if this z 0 point is equal 0, then that series is nothing, but a Maclaurin series.

So, Maclaurin series is a special case of Taylor series, where this point z 0 is taken after this Maclaurin series and Taylor series. Now, we are a position to do some problem as I mentioned.

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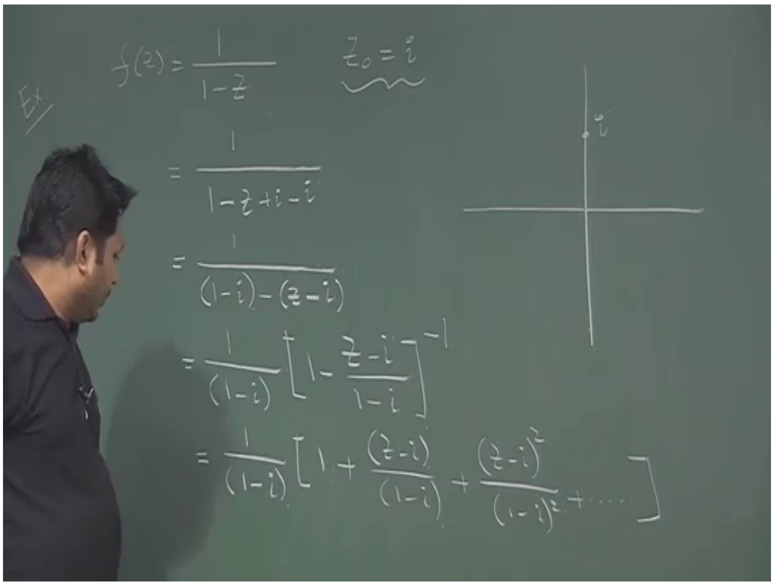

So, let us try to find out some example. So, example is function f z is equal to 1 by 1 minus z. This is a function. Now, the question is asked that please expand these functions around this point z 0 is equal to 1 around this point. I need to expand this function. So, now the process is straight forward. So, there are two ways you can do that. You can write straight way this function in this form z 0 is giving to you.

So, that means this is i. So, a k how do I find a k? You can just make a derivative with the function is given. So, you just make a derivative of this function; first, second, third or fourth and n times. You have a general expression of the derivative and put this value at i n factorial k divided. So, this is the standard way to do the calculation, but apart from that you can do in different way also. The way it prove the Taylor series, I will use the same logic here, the same thing.

So, the point is somewhere here around which I need to expand my function. So, this is point i. So, f z is 1 minus z i will add this i and substract this i like this and then, what I will write 1 minus i minus z minus i. Then, I will do 1 minus i common, then it will be just 1 minus z minus i divided by 1 minus i minus 1, right. So, this expansion, whatever the expansion I will have is simply the Taylor series expansion. So, if I do, then it will be 1 minus i 1 plus z minus i 1 minus i plus z minus i whole square 1 minus i whole square and so on.

So, I already have a series expansion for a given function around the point z equal to 1. So, what is the general form? Let me erase this part. What is the general form? I have the general form is in the summation form is summation z minus i k divided by 1 minus i k plus 1.

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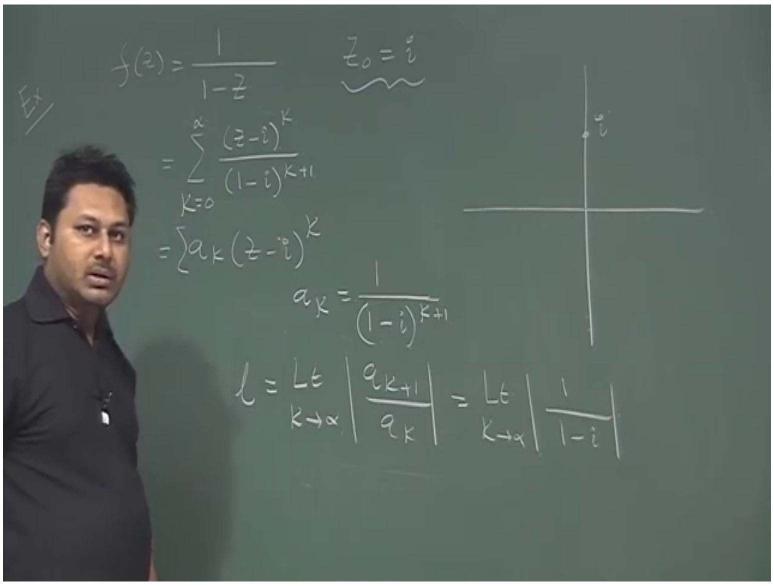

So, this is the expansion. So, k from 0 to infinity, this is the expansion I have. Now, it is equivalent to the power series as I mentioned. That means, I can now find what my a k is. A k is 1 divided by 1 minus i whole to the power of is k plus 1. This is my a k.

So, now once I have a k, next thing is that I can expand this function around point z equal to z 0 equal y. That is true, but I need to know the radius of convergence also because I cannot expand these two entire regions. There is a restricted region along which I can expand this in terms of Taylor series. That is the point. So, once I have a k, I can readily find out what my l is which is limit k tends to infinity mod of a k plus 1 divided by a k which is a k plus 1 by a k means limit k tends to infinity 1 by 1 minus i; this quantity a k plus 1 if I put, it will be k plus 2 and a k will be k plus 1 in the up stair. So, in the denominator we have just 1 by 1 minus i.

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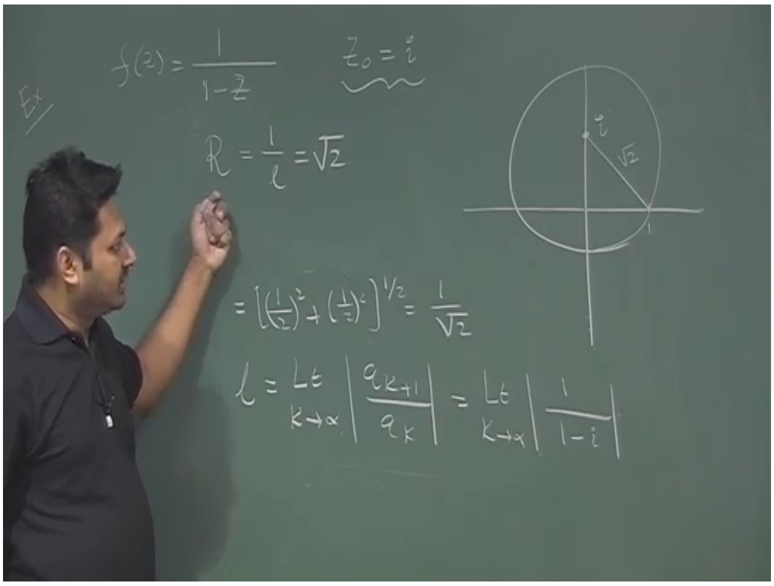

Once you have that, we can have the value of l since it is a mod. So, k is not important here because k is not there at all. So, 1 minus plus i divided by 1 minus i 1 plus i. So, this quantity is how much? 1 plus i and this is divided by a square plus b square. So, it is 2 mod of this quantity is 1 by 2 square plus 1 by 2 square whole to the power half. So, it is just 1 by root 2, right. 1 by 2 square is 1 by 4 plus 1 by 4. So, 1 by 4 is 2 by 4. So, half is 1 by root 2.

So, what is my radius of convergence here? R original function I should write here. R radius of convergence is 1 by l is equal to root over o 2. That means, this is 1, if I have a circle like this where this radius is root over of 2, then in this region this Taylor series is valid because I can expand the Taylor series, but I should keep in mind that my radius of convergence is also there and the radius of convergence is root 2. So, whatever the value of z is here for that z, I can expand the Taylor series the way I expand it.

So, student I would like to stop my class here. In the next class, we will go to few other problems related to the Taylor series. We will do another 2-3 problems and then, go to something interesting which is called the singularity. Before going to the next step, we will learn Taylor series in in the next class. Maybe in the next to next class, we will learn another important, very important series called series, but before going to learn the series, we need to learn more about the singularity. We have a rough idea about that.

So, in the next class we will do few more problems related to the Taylor series expansion and try to learn the meaning of the singularity and different kind of singularity with this note. So, let me conclude the class here.

Thank you for your attention.