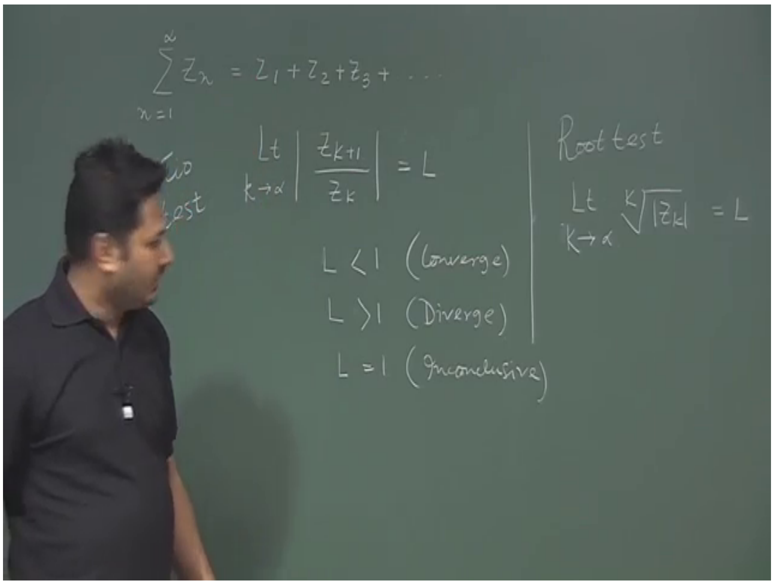
Mathematical Methods in Physics-I

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 48
Circle and Radius of Convergence

So, welcome back student to the next class of complex analysis in the last class, we started very interesting thing which is series and sequence and we mention that.

(Refer Slide Time: 00:31)

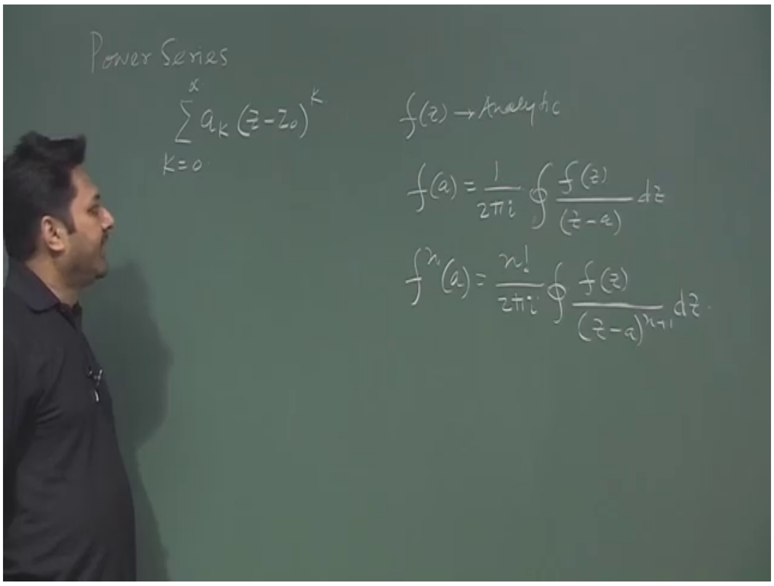


If we have a series; something like that n is equal to say 1 to infinity which is Z 1 plus Z 2 plus Z 3 few complex number, then there is a way to test whether these things, this partial sum are going to some converging limit or not and we mention that if say limit K tends to infinity z a plus 1 z k if this quantity is L then this partial sum will going to converge if L is less than 1 it is converge.

If L is equal to L is greater than 1, then it will diverge and if is L equal to 1, then there is no conclusion in conclusion there are another way this is called the root test now this is called the ratio test a similar kind of test is also available as I mentioned; it is a root test we called this root test sometime also; we required that where we need to find out this and the condition over K is same if K is less than 1, then it is convergent if K is greater than 1; it is divergent and if L is equal to 1, then we cannot say anything about the convergence of this particular series that is written here.

This was the last days things; we will going to exploit this more before that I need to give you some important thing which is called the power series normally a power series is represented.

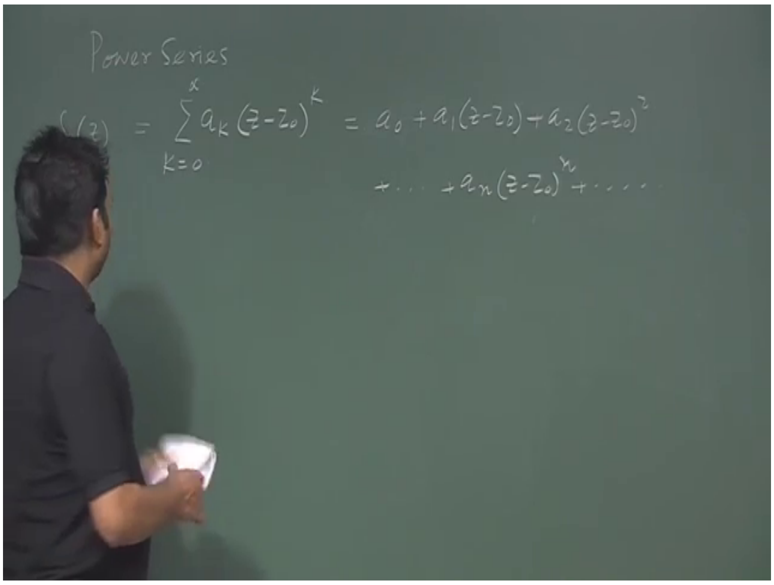
(Refer Slide Time: 03:11)



In this way, this is the representation; representation of a power series now if a function f z is analytic in some region then from goes from Cauchy's integral formula; we find that if a; that means, the value of the function some point a; can we evaluated as this not only that I can also find the value of f n z f n a; that means, n th order derivative mind it when I am writing this expression fz is an analytic function.

Now, I try to find out the n th order derivative of that particular function f z at that point a and it was something like this; that means, n th order derivative is possible for this analytic function and keeping that in mind we can say that a function if it is analytic we can somehow put this in some serious which is in this form; now what is the beauty of this form let me know explicitly write this.

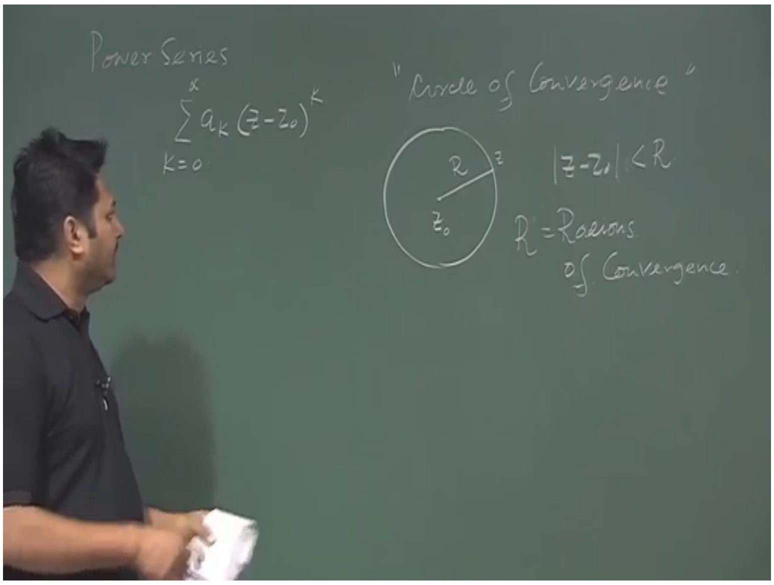
(Refer Slide Time: 05:22)



So, you can have the n th order derivative of this things which is possible. So, first term is say a 0 second term is a 1 z minus z 0 third term is a 2 z minus z 0 square and so on.

So, it is a polynomial with z minus z 0. So, if a function f z is represented in the series, then we can say this is an analytic function and having n th order derivative because if you make a derivative of this function you will have this quantity the right hand side you can have the derivative of that.

(Refer Slide Time: 06:28)

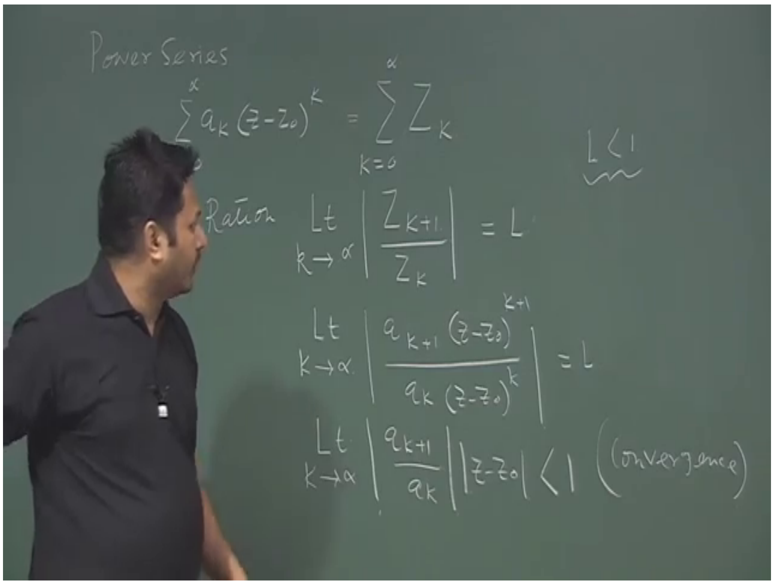


So, that is why the power series is important, but also one thing is important here for this power series, you cannot expand this series to according to your choice. So, there is something called convert circle of convergence circle of convergence; that means, you can expand the function in terms of power series if this is the point z 0 and here, I have z and if my radius R. So, in this region you can expand maybe it is possible to expand this particular series.

So, that the series is convergence. So, the in order to have the convergence of the series, we should have something called the circle of con convergence and the point is that you cannot take any z and expand in this series, there should be some restriction and this restriction is coming through this R. So, mod of z minus z 0 is less than R this is your limit for which you can able to expand a function in power series if this condition is not valid, then there is a possibility that even though you are expanding something it may not converge. So, converging the series is an convergent convergence of that series is an important criteria.

So, in order to converge this series you need to need to also find out what is the circle of convergence and this is R is called the radius of convergence R is called the radius of convergence this R; however, we can find out with this ratio test and whatever the test we have mentioned just before. So, let me do that. So, how to find out the circle of convergence or this radius of convergence for that particular series to be convergent?

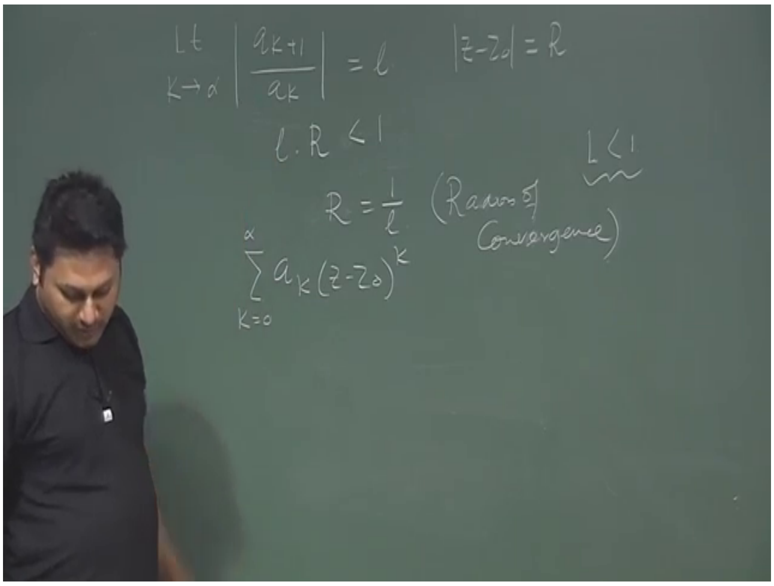
(Refer Slide Time: 08:47)



So, let me right this series say big z in k. So, I am just write this thing as big z K where K is 0 to infinity as usual, then my ratio test suggest limit K tends to infinity big z K plus 1 divided by this things.

Say it is L this things is L. So, now, if I put these things is L, then I can have limit K tends to infinity this quantity is nothing, but a K plus 1 z minus z 0 K plus 1 divided by this a K z minus z 0 K which I say it is L still now; I do not know whether it is less than 1 or greater than 1. Now I need to put after that I need to put the condition for the convergence the condition is L has to be less than 1 that is my condition. So, after that I can say limit K tends to infinity this quantity a K plus 1 by a K multiplied by this quantity z minus z 0 is less than 1 for convergence; now I put the condition for my convergence. So, this is the condition of convergence this quantity just before I mention is something or R. So, let us erase this.

(Refer Slide Time: 11:33)



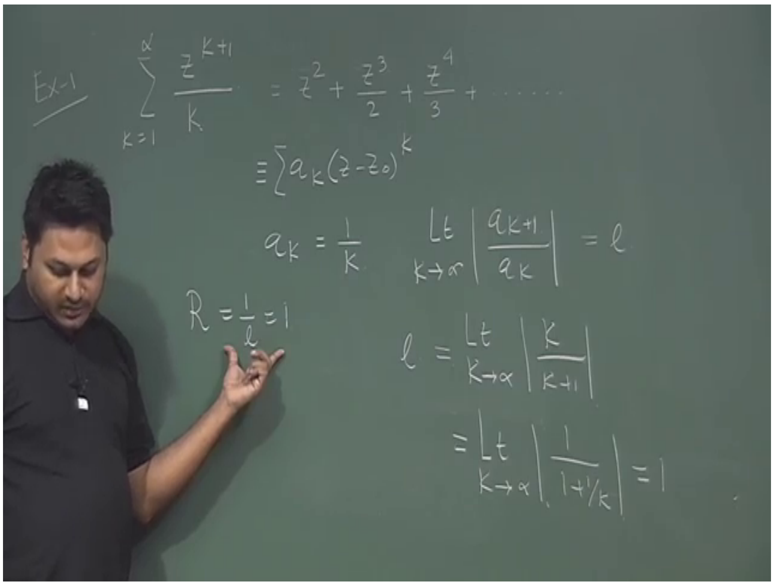
So, say limit K tends to infinity I calculate the quantity a K plus 1 divided by a K; this quantity I say small L this is small l. So, from that and also z minus z 0 is equal R since this is the region where try to find out my expression the expansion is convergence under the condition when z minus z 0 is R and this R is the radius of convergence. So, from that I can find that L multiplied by R is equal is less than 1 or I can say the radius is the limit of radius of convergence is something the limit of radius of convergence is L R equal to 1 by L.

This is the limit because when is equal to when as for as it is less than 1. So, it is convergence. So, when it is equal to 1 which is that limit for which the things are convergence so; that means, the radius of convergence is essentially 1 by L when 1 by L l is defined at this. So, if I have a power series in my hand. So, my first task is to find out what is the radius of what is the circle of convergence. So, the circle of convergence is defined by this radius R and I can find out this radius R in terms of L which is nothing, but this limit that is the thing. So, first a series is giving to me.

So, let me summarize once again this is the series is given to me and I am asked whether this series say 0 to infinity is convergence or not in order to convergent this series in order to converge. So, what is the region where I can expand the series? So, that the it will be a convergent series. So, the region in order to find the region I need to find out this limit when I find this limit I will find l. So, R is eventually 1 by L which gives me the radius of convergence. So, radius of convergence radius of convergence.

So, the radius of convergence can be figure out. So, if the problem is given like this and if you are ask to find out the; what is the radius of convergence I need to just calculate this. So, now, let us go to some examples and then maybe things will be clear.

(Refer Slide Time: 14:53)

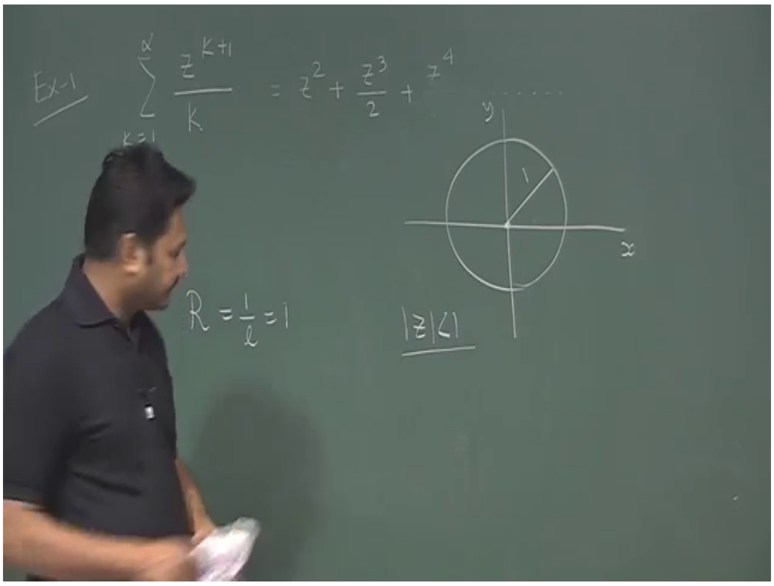


So, let us see what example I have today for you. So, example 1 and the series that is given to us is z K z to the power K plus 1 divided by k. So, this an series first I can put it as a as first series also. So, 1 to infinity this series is given to us. So, now, if I expand this series my first term will be say z square my second term will be z cube by 2 my third term will be z to the power 4 by 3 and so on and so on.

So, what is the; what is the radius of convergence of this things; I mean what should be the value or on the restriction over z for which this should be a converging series you already know the result, but let me do that. So, if I write it in terms of this series. So, this is nothing, but a power series where z 0 is 0 and a K is 1 by K this quantity. So, if I find limit K tends to infinity a K plus 1 by a K which I called L, then this comes out to be L comes out to be limit K tends to infinity a K is 1 by K a K plus 1 is just this things.

So, K by K plus 1 if I take K common from both the sides and put the limit; K tends to infinity this K will cancel out. So, 1 divided by 1 by 1 plus K which is 1 so; that means, the L value is 1 R radius of convergence is 1 by L which is 1. So, the radius of converges convergence is 1. So, what is the meaning of that I can expand this function.

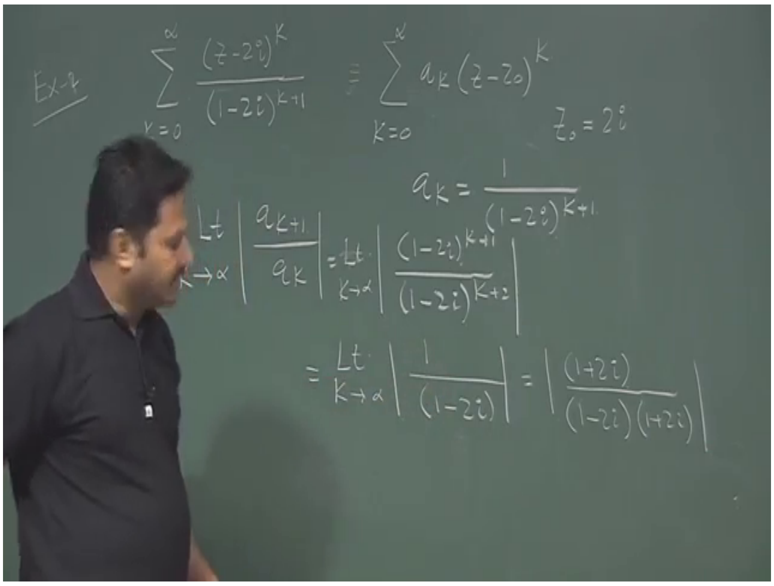
(Refer Slide Time: 17:52)



In this way I can expand this function in this way for all the zs with 0 point is 0. So, this is 0 and this is 1 unit radius and this is my z plane x and y. So, all the z less than 1 whatever the z, I put inside the circle should be less than 1 and for that less than 1, I can expand this series in this way.

For which the radius of convergence is this. So, if I put this if I take z outside, then whatever the value, I have will not be convergence here whatever a z here should give you only the convergence; that means, it will go to some value which converts and radius of convergence is 1 let us go to the next example this was very trivial example.

(Refer Slide Time: 19:15)



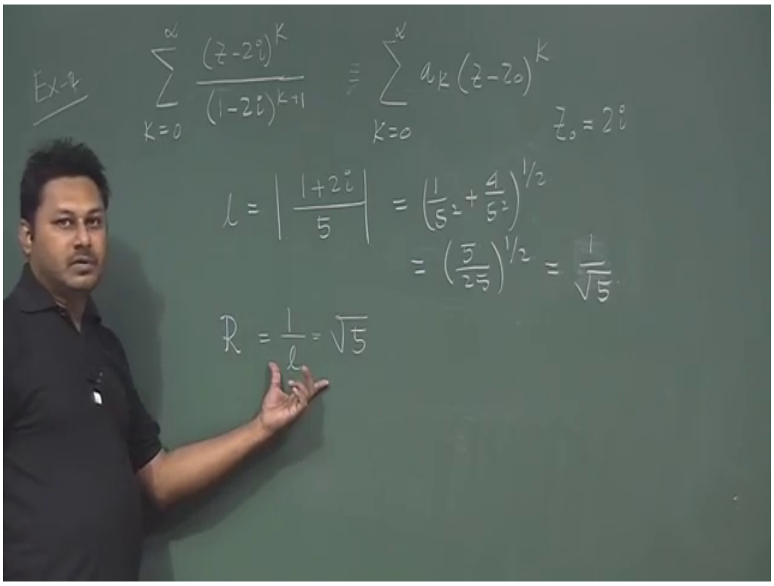
So, the next example is example 2 is suggest summation K 0 to infinity and the function is given as z minus 2 i whole to the power K divided by 1 minus 2 i K plus 1.

So, again we have a series it is also equivalent to some power series is nothing, but a power series where z 0 if I compare this to this we have 2 i and what about a K; a K if I compare; it should be 1 by 1 minus 2 i; this quantity; right. So, a series is given to me and we need to find out the radius of convergence for this series. So, my goal is to find out the radius of convergence and this series can be written in this particular form a K z minus z 0 to the power K which is nothing, but a power series where z 0 is that and a K is this quantity.

So, now the rest part is quite mechanical I need to again find out what is the value of this quantity because this is my small l; the small l is this quantity once I find small l; the R will be just 1 by l; small l, then I can find out the radius of convergence. So, if K tends to infinity this I put. So, a K plus 1 will be 1 minus 2 i a plus 2 because a K is this plus 1. So, a K plus 1 will be 1 by this and a K will be 1 by I will have this the limit is as usual K tends to infinity. So, this term will be simply 1 divided by 1 minus 2 i mod of this quantity mod of this quantity.

So, the next part is to make it a plus i b form. So, limit K tends to infinity. So, the limit K tends to infinite is meaningless here because the K is already gone. So, let me erase this and then need to find out what is the mod of things. So, 1 minus plus 2 i divided by 1 minus 2 i 1 plus 2 i this quantity. So, that quantity we need to just find out that will be our l.

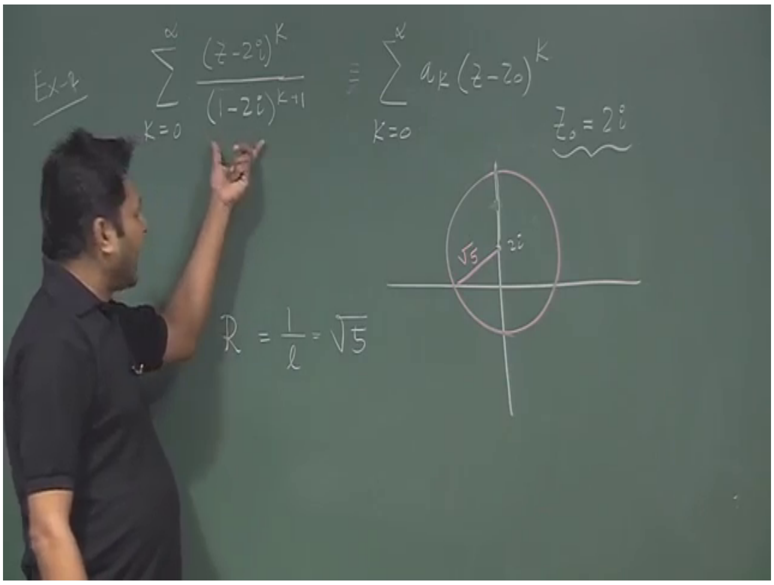
(Refer Slide Time: 23:16)



So, let me erase this part which we have completed. So, my L; this is l. So, L is mod of 1 plus 2 i divided by a plus b into a minus b it is a square minus b square. So, a square is 1 b square is 4. So, it will 5 and mod of this quantity is 1 by 5 square plus 4 by 5 square whole to the power of half or 5 by 25 whole to the power of half or 1 by 5 or 1 by root over of 5.

So, my L value is turns out to be 1 by root over 5 so; that means, the radius of convergence R the radius of convergence R will be 1 by l; that means, root over of 5. So, this is my radius of convergence. Now let us look what is the region the region is important you need to find out where is the re what is the region.

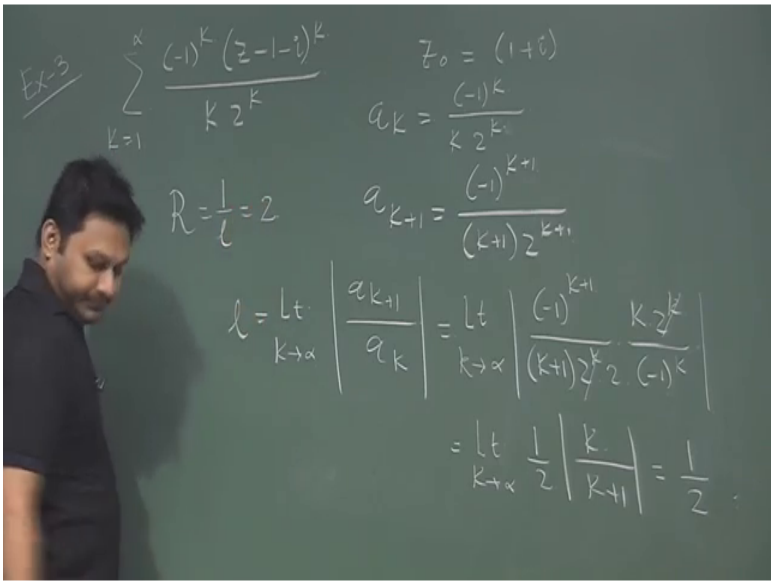
(Refer Slide Time: 24:49)



Where this power series is valid and it is converging what is the region for which is converging. So, this is z 0 is 2 i. So, this is say z 0 point; let me put it here this is say z 0 point which is 2 i and with 2 i; I need to make a circle I need to make a circle here with radius root over of 5. So, this is my circle of convergence and this is my radius of convergence which is root over 5 that I figure out what about the z is here if you all this z inside this circle weight circle.

For that this series will be convergence and if the z is outside then this series will not going to converge that is the thing we find. So, quickly we will do another example before this class; let us take what extra example we have. So, quickly we will have on a final example.

(Refer Slide Time: 26:15)



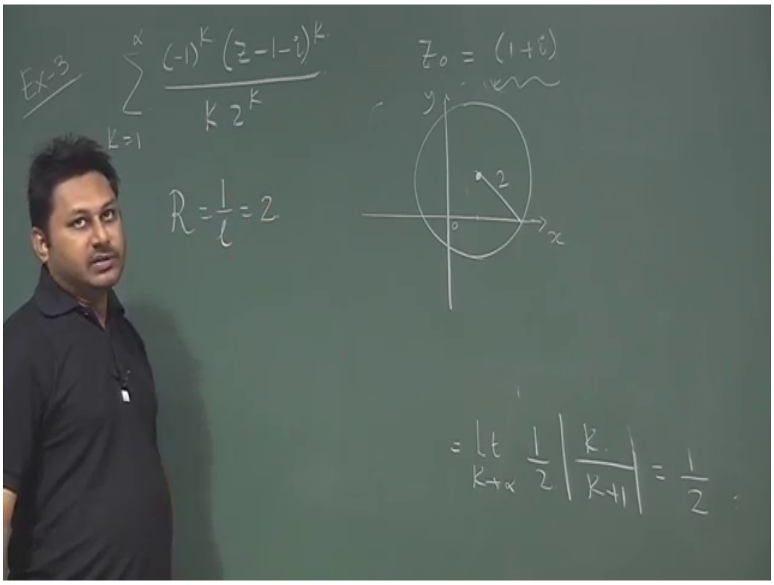
Example 3 say; so, the series that is given to us is K 1 to infinity minus 1 whole to the power K z minus 1 minus I whole to the power K divided by K 2 to the power K; this something like this. So, a K z 0 is nothing, but 1 plus i; this is my z 0 z minus z 0 z 0 is some point here which is this and K a i K is minus 1 to the power K divided by this 2 the power k.

A K plus 1 is minus 1 K plus 1 divided by K plus 1 into 2 the power K plus 1 L is limit a tends to infinity a K plus 1 quite mechanical, but if you practice, then it will be good limit K tends to infinity this quantity, first I need to write. So, minus 1 whole to the power of a plus 1 divided by K plus 1 2 to the power K into 2 and then a K is K 2 to the power K divided by minus 1 into K mod.

So, since we are using mod. So, I should not bother about this sign this sign will be minus 1; 1 K and 1 will cancel out minus 1 will remaining, but this minus sign since m am taking a mod of that we should not bother about that it will go away 2 to the power K 2 to the power K will cancel out and I will have limit K tends to infinity this 2 half will come out and only I have K by K plus 1.

Again if take K common, then it will be 1 plus K and then if put K tends to infinity, it will be just 1. So, eventually will have half; that means, my R which is 1 by L is equal to 2. So, where you should be the radius of convergence and how these things; so, radius of convergence is 2 i find. So, for which set of z values we have the series to be converge.

(Refer Slide Time: 29:30)



So, 1 plus i point is somewhere here this is 1 and this is i; this is the point and if I take a radius circle of radius 2 where this is circle of radius 2.

Then these are the points these are the z points this is 0 x and y all the points inside this circle if I write take this all z then if I expand this then it will be convergence. So, this is the; I mean taste of convergence and find out the radius of circle. So, I believe today's class with this example you are now familiar about the fact that if a function is given to you which is a power series then this power series will be convergence or not and what should be the radius of convergence you can figure out from this treatment. So, with that I should stop my class here.

So, in the next class we will start interesting thing which is the Taylor series and in the Taylor's how a function can be expand in terms of Taylor series we will going to find. So, with that note see you in the next class.

Thanks for your attention.