Mathematical Methods in Physics-I
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Lecture - 45
Cauchy's Integral Formula (Contd.)

Welcome back student to the next class of complex analysis. In previous few classes will learn few important things so let me summarize first today that is.

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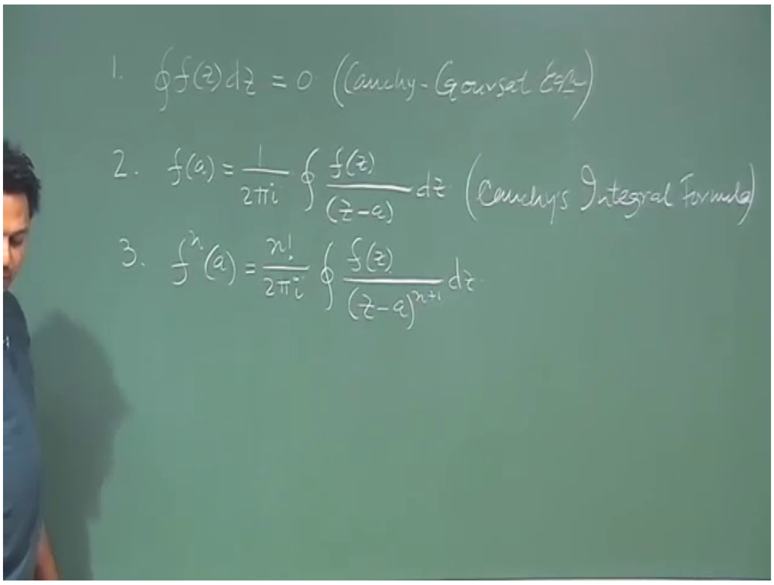

First one is this important expression which we called Cauchy-Goursat Equation or Cauchy-Goursat Equation. We suggest that this expression means that if the function is analytic, then if I want to make a closed integral of the function in the region where this function is analytic we will get the result which is 0. From that we also find 2 very important information and that is f a is this. If the function, f z is analytic, but please mind it f z divided by z minus a this is another function which is not analytic, because z equal to a point the function blows up it call singularity we will come to this point later; what is the meaning of singularity extensively we will study. But here you should understand that at z equal to a point this function is not analytic. But if you do the integration then you will get the result and the result will be f a multiplied by 2 pi i; that is the additional outcome that we have from Cauchy-Goursat equation.

Aand thirdly the extended form of this it is called Cauchy's Integral formula. Now, here I am doing the same thing, but just having the derivative up to n order, then this expression is slightly modified and I have some modified expression like this. This is also the one extension of this formula, just I make a derivative with respect to a and then I will get this kind of form.

So, now in the last class we did few problem using that, so we will go with that we will continue that and we will do more problem so that you can understand how this things are working; for several different kind of contour we will do that today.

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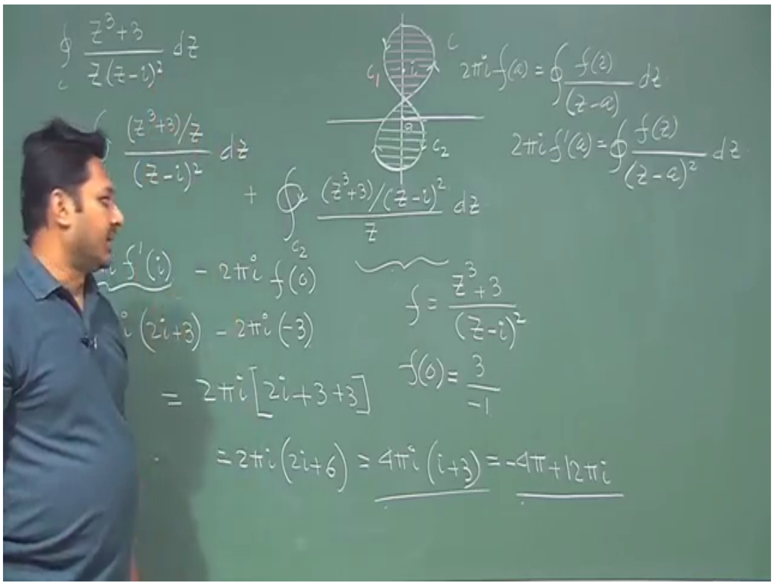

So, let me start with one problem say example 1 or problem 1 whatever. The function, the integration is given like this: z cube plus 3 divided by z my z multiplied by z minus i whole square d z. This is the standard integration that you find in the book by Zill. But interesting thing is that the contour: contour is given like this. This point is 0 and this point is i, and it is going like this. So this is the contour over which I need to calculate this integration; this is my total c.

Now, in this contour one important thing you need to note that in this portion it is one direction and in this portion it is another direction. In this portion the direction is positive in the sense my region which is this one; this one is my region where I try to integrate the stuff the region like this. This is my region here. And in the next case the region is different. Why it is different? Because according to my convention if I go in this path, this is the direction along which I am going; if I go to this path the left hand side if I go right this my left hand side will be the region then I considered this direction as my positive direction.

On the other hand in this case if you find I am going this direction and the region is my right hand side. So that means, my right hand side since the region, so the integration here whatever I will get should be with the negative sign. That is the main important thing for this kind of contour you should be careful about that. Another thing is the 2 singularities that is here: one i, because in this particular function if you look this function z equal to zero point is a singularity where the function can blow up, z equal to i is another singularity where the function can blow up and fail to be analytic. So, we need to calculate these things by keeping in mind that in this region i is here and in this region another singularity 0 is here; so 2 singularities are there, but 2 different regions.

So, what I will do here that I will divide these things to 2 part. So first part is say this portion that is the rate portion I called my c 1 contour and another portion here I called c 2 contour; one is c 1 and another is c 2. Now I try to calculate the total contour which is the summation of c 1 and c 2. See if I write it will be c 1 with this sign because this is my positive sign; let me write it a bigger way so that you can see properly this. When I write this in c 1 my i is here c t. So, z cube plus 3 divided by z, I should write the function in this way so that the singularity portion in the denominator and whatever the function I have here it should be a analytic function at that region, which is z square plus 3 by z is analytic in this region red region.

Also I need to add the other part, where the direction of the integration is in different direction c 2. And here I should write it as z cube plus 3 divided by z minus i whole square whole divided by z d z. I am writing the same function, but here I am writing this nominator in such a way that the function whatever the function I get is analytic in the region c 1, here I write this function z square plus 3 divided by z minus 1 whole square this particular function in such a way that this is analytic in this region where z equal to 0 is my singularity.

So, now things are done, because now if I write here the form that my f a 2 pi i multiplied by f a is my solution the beauty of this expression is you never need to do the exact or. This is the 2 formula that I am going to use. So, here if I put this thing here then my solution will be 2 pi i, if I called this function f z then f prime at i this is my first result I need to calculate this one however; minus because I am now going to change this symbol. So, this is in the different this is in the clockwise I may need to make it anticlockwise so that is why it should be negative sign. Then 2 pi i in C 2 z equal to 0, so simplify if I say this is my function so this function at 0 point.

Please mind that this function and this function are 2 different function, but I am noting the same way f z you can do that in g z whatever, but just I need to put the value of this things. So, that is why I am not going to put any different function form. So, if whatever the function here is written here make a derivative of these things and put i here make a derivative for this case no derivative is required because it is just z so just put 0 and you will get the result ok.

So, what is my f here for this case f z is z cube plus three divided by z f prime z is how much derivative of this so 3 z square multiplied by z minus this quantity z cube plus 3 and derivative of the new minuter which is 1 by z square which is 3 z cube minus z cube so it is 2 z cube minus three divided by z square something like this. Now if I put f prime I just need to replace z to i and then I will get the result. So, f prime i here is 2 i cube minus 3 divided by i square this quantity is how much i cube is minus i.

So, minus of 2 of i minus 3 divided by minus 1. So, eventually I have 2 i plus 3 this quantity is 2 i plus 3 so let me put it here. So, this value I have 2 pi i f prime i I calculate just I calculate it is something like this.

Now the next part, next part is 2 pi i f 0 means I need to put the value 0 of this function here, what is my for this case what is my f, my f is something like z cube plus 3 my f is z cube plus 3 divided by z minus i square this is my f for this particular function this particular form.

Now, I need to put f 0 here; that means, I just put z equal to 0 and then I am done. So, what is my f 0 here if I put z equal to 0 it is just 3 z equal to 0 and this, this is minus 1 i square is 1 so i square is minus 1 this minus sign will be compensated by this square. So, we will have one plus sign and I should have something like minus 3. So, here I am getting something minus 3 so my first part is this and my second part is this. So, if I now write 2 pi i if I take common then i will have 2 i plus 3 and plus 3 another 3.

So, it should be something like 2 pi i multiplied by 2 i plus 6 or you can simplify more so it is just 12 pi not 12 pi 4, if I take 2 common so it will be just 4 pi i i plus 3 something like that I mean you can further simplify a multiplying these things and then you will get something different. But the result is like this, minus 4 pi if you do that then it will be minus of 4 pi plus 12 pi i this is my final result in the in the a plus i b form.

So, I put it this result is correct also this result is correct these result is correct all the results are correct, but it is convenient and it is always expected that whatever the result you find you should put it as a plus i b form. Because complex number is in general represented by a plus i b this result is correct, but you can simplify to make it as a plus i b as I did here. So, the result is something like 4 pi plus 12 pi i for this particular contour, where the function is given like that.

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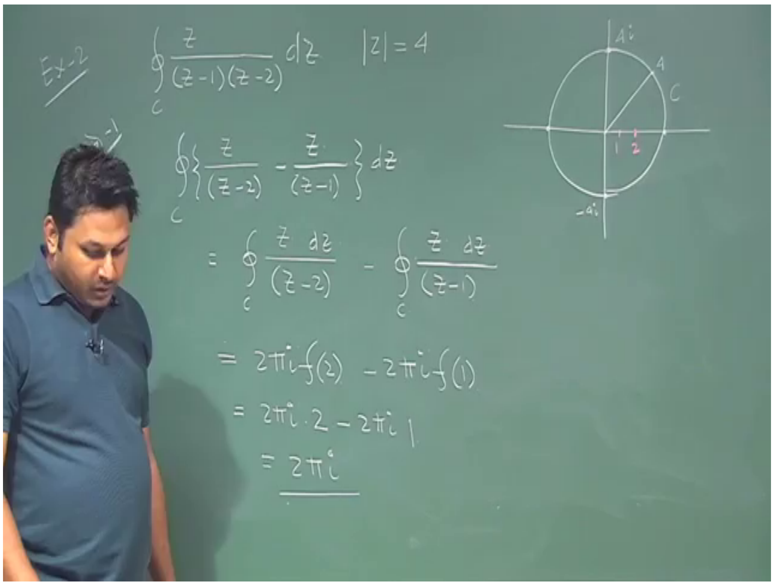

So, let me erase this and try to calculate few other problems which is so example 2 says example 2, so the next problem I have is something like this z divided by z minus 1 z minus 2 dz this is the integration I need to do where the contour is given like this. So, contour this contour c is given z is equal to 4 mod of z is equal to 4. So, first we need to find out how the contour look like mod of z equal to 4 is nothing but a circle here this is the circle of radius 4 so this is 4, so this point is 4, this point is 4, this point is how much 4 i and this point is minus 4 i this is the region that is given to you and if you note in this particular function you will find that there are 2 singularities that is inside this not 1 only

1, 2 singularities z equal to one where the function can blow up so I can say this is one of my singularities z equal to two is another where I have the singularity.

So, these two points are sitting inside this circle whatever the circle is there. So, one point is here and another point is here this is 1 and this is 2, these 2 points. Now if these 2 points are there then I need to calculate this. So, there are 2 way to calculate first say method 1. In method one what we will do that I will just simplify these things a bit this is my total contour c c and I simplify these things as z divided by z minus 2 minus z divided by z minus 1 this.

I just factorize these things, so I just if you factorize this things it will come this equation is come like this after factorization I can separate out into2 part, so one part is z divided by z minus 2 d z minus integration of z divided by z minus 1 d z instead of having 1 integration now I am having 2 integration separately, this minus this for this contour. When I am separating out 2 individual integration, then I can readily calculate because for this case it is easy the function. Now I am going to use the Cauchy’s Integral formula here straight away, where z equal to 2 point is here I should not bother about the one because in this particular function there is no singularity at 1.

So, I should not bother about the other singularities, the other singularities will be taken care of this integration. So, I separate out that is the reason I separate out where the other part this part and this part there is no connection between this. So, 2 pi i the result will be whatever the function I will put function at 2 minus 2 pi i whatever the function I put this function try to evaluate the function at z equal to 1 point because it is 1 and it is 2. So, now if I do that then what so 2 pi i here function is both the cases same the function f z is z for both the cases.

So, I need to put z equal to 2 just so it is z equal to 2 minus 2 pi i here function is z so at 1 what value I will have at 1 I will have the value 1. So, I will have 4 pi minus 2 pi i so the result is 2 pi i so what I did that there are integration the integration I just divided into 2 part. There are 2 singularities when I divide into 2 part then I calculate one part and another part separately, without bothering about the other when I do this separately then I will get this result in my hand. So, now I can do that specific integration into another way also and that is these 2 point I can have here singularities. So, I can separate it out these 2 points and write the function in this way.

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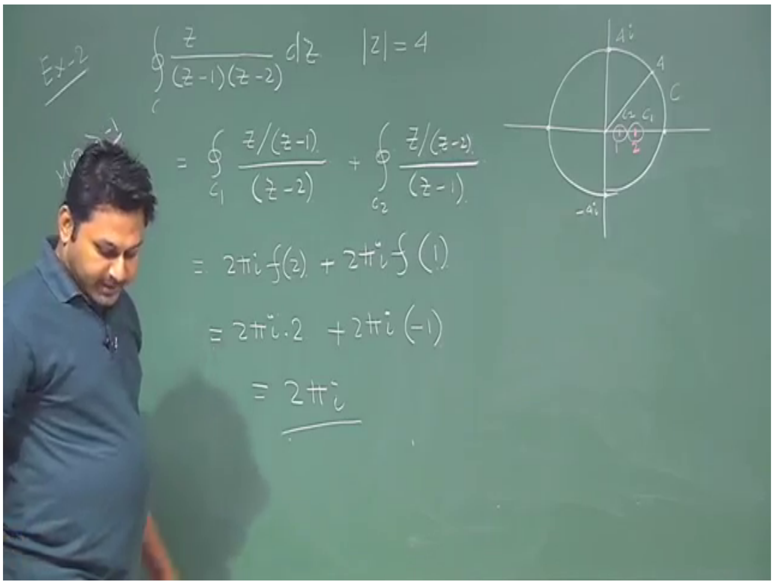

So, this over c is same as this and this that we know from our previous knowledge, so I can have these two separate c 1, c 2, 2 curve I can separate and this result will be for c 1, 2 is my singularity. So, I write this function like this in the second case my singularity at c 2 for c 2 circle is 1. So, I write this as z minus 2 so for this case my singularity is at 2 and for this case my singularity is at 1 and my function in the previous case is my function was z, but here this function is z by z minus 1 here z by z minus 2.

So, if I now calculate that let me find out what value I am getting so 2 pi i function evaluated at 2 point plus 2 pi i function evaluated at 1 point. So, it is 2 pi i this function f 2 means I just put z equal to 2. So, it is z equal to 2 so 2 and 2 minus 1 so, this entire quantity at z equal to 2 is 2 plus 2 pi i 1 so z equal to 1, if I put z equal to 1 I will have minus 1 here this function at z equal to 1 the value of this function at z equal to 1 is minus 1, this is one this is 1, 1 minus 2, 1 minus 1 so this will be minus 1.

So, again the entire result is 2 pi i which is same as the previous 1. So, this is method 2 you can do this in this way also you can have this kind of example more examples.

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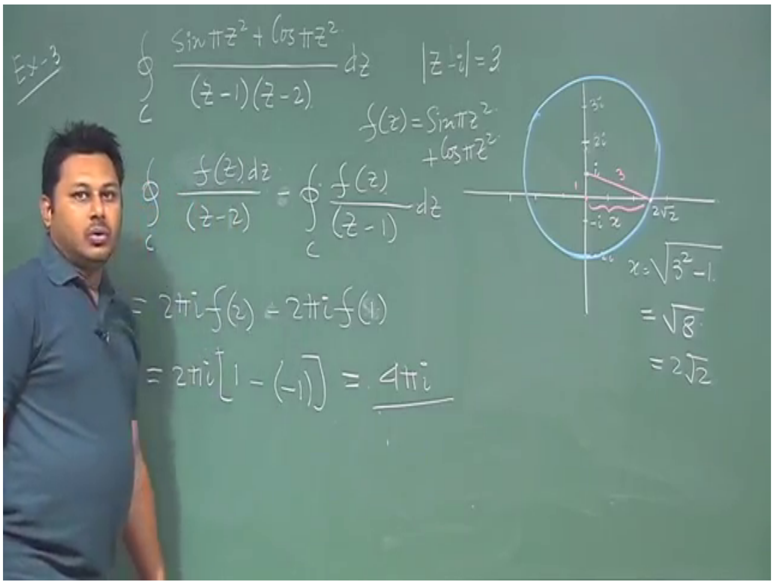

So, let me give you another one where example 3, so the function that is given as something like sin pi z square plus cos pi z square this is a similar kind of exactly similar kind problem just to phi z this is the function that is giving to us.

Now, also we need to mention about the contour which contour I am calculating so c should be defined and c is defined as z minus i is equal to 3, this contour is interesting because this is slightly different kind not very different kind of contour this is just another sort of contour where the circle this is the circle and circle is where. So, this point is say I, first we need to find out whether any singularities are here how many singularities are there in that contour whatever the contour I am going to draw that is the first thing for every problem you need to find out. So, this is my centre so, this is i, 2 i, 3 i, so let me put this i here this is my i.

So, this is minus i this is minus 2 i, this is 2 i, this is 3 i, this is 1, this is 2, this is 3, this is 1, this is 2, this is 3, this is my coordinate I need to draw the co coordinate properly. So, that now I am going to draw these things. So, z minus 1 means this and I will have a circle with radius 3 with radius 3. So 1, 2, 3 so 1 2 so here the circle will be something like this 1, 2, 3. So, it will be going something like this ok this is the circle, where this radius so slightly upper part slightly on up so it should be something like this actually this is my circle.

Now the important thing is that the value of 1 and two whether it is inside the circle or outside the circle we need to find out. So, this value is mind it 3 right because this is the radius this circle is of radius 3. So, this value is 3, what about this value this value is 1. So, you know what is this value because, your points 1 and 2 are placed here somewhere. we need to you need to ensure that this circle the point 1 and 2 are inside this circle. So, it is what is the value here so it will be root over of 3 square plus 1 which is root over of 10, it is not it is called minus 1.

So, it will be root over of this quantity if it is x. So, x is three square minus one. So, it is root over of 8, it is say 2 root 2, so 2 root 2; that means, this quantity up to the here is 2 root 2. So obviously, minus 1 and minus 2, should be inside this that is the thing we just figure out. Now the rest part is straight forward, so I will again use that previous method. So, I will just write this as a function, so function of z divided by z minus 1, z minus 2 plus integration d x d z function of z divided by z minus 1 d z.

Just factorize these 2 where my function of z is sin pi z square plus cos pi z square, now I put the value. So, 2 pi i function of z at 2 plus 2 pi i function of z function of z at evaluated at point 1 so this is 2 pi I, if I take common what will be the my first 2. So, if I put z equal to 2 it will be sin 4 pi plus cos 4 pi sin 4 pi is here I should have a minus sin, so sin 4 pi is 0, cos 4 pi is 1, this is minus minus then I put 1. So, one is sin pi again 0 and cos pi cos pi is minus 1 so I will have minus 1. So, my end result seems to be 4 pi i.

So this is the standard few problems I derive in the last class as well as in this class. So, that we can familiar with this kind of problem and just use the cos is integral formula to evaluate the integration whatever is given to you. First you need to find out how many singularities are there for the given. So, when the problem is given to you the thing that is also given to you is the contour over which or the region on which you have to derive your have to integrate. So that means, this region is important: first you find out the region and then find out how many singular points are there in the region. For this case here two singular points are there in the region. For example, if it is 2, then this 2 was outside then you will get completely different results. So, you need to find out that in the function how many singular points are there and then you just apply the straight forward Cauchy’s Integral formula and you get the result that is all.

With this note let me conclude the class here. In the next class we will start something new which is called series and sequence of complex numbers, which is very very important to go to the next part of this course. So, with that so let me conclude and see you in the next class.

Thanks for your attention.