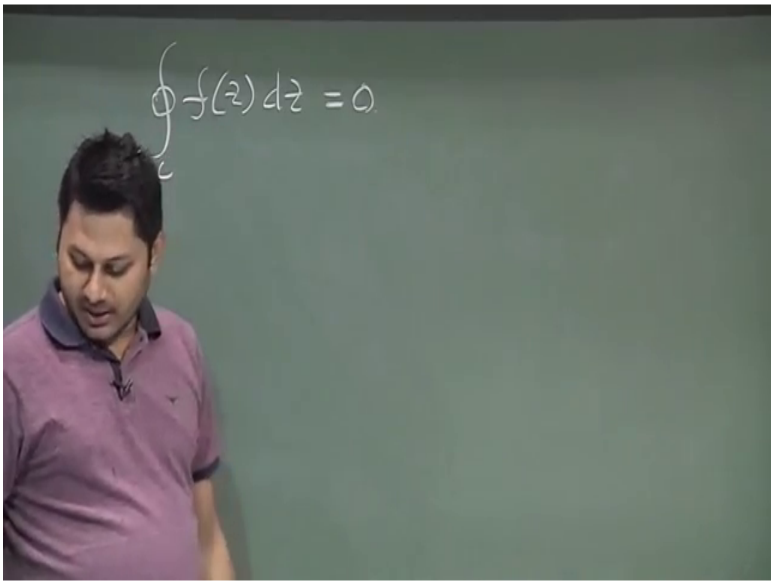
Mathematical Methods in Physics-I

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 44
Cauchy’s Integral Formula

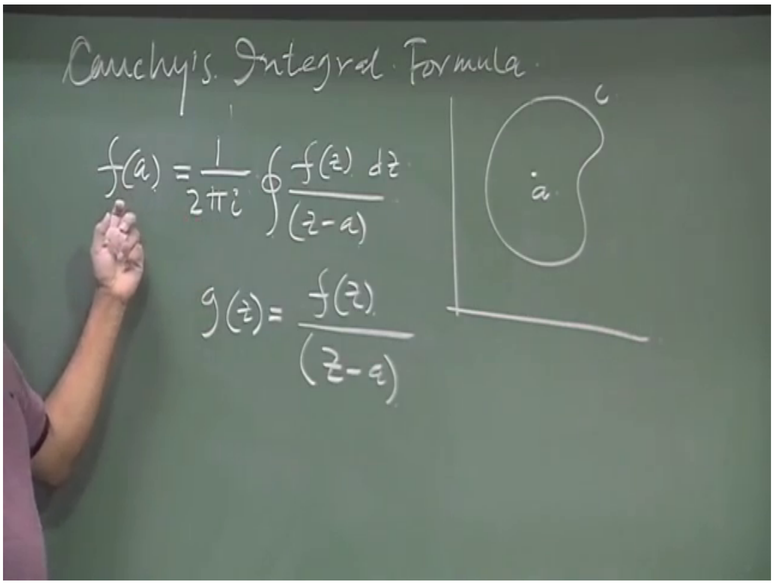
Welcome back student to the next class of Complex Analysis. In the previous class we mention very important concept of the integration so let me remind you once again.

(Refer Slide Time: 00:29)



That this is my function and this will be 0 when f z is analytic over that region, and may not be 0 when the f z is not analytic. So, last day I mention that if f z is not analytic in some region, so how to find out the value of the integration is there any way. So, indeed there is a way it is called Cauchy’s integral formula. So, let me first state that and then I will prove.

(Refer Slide Time: 00:58)



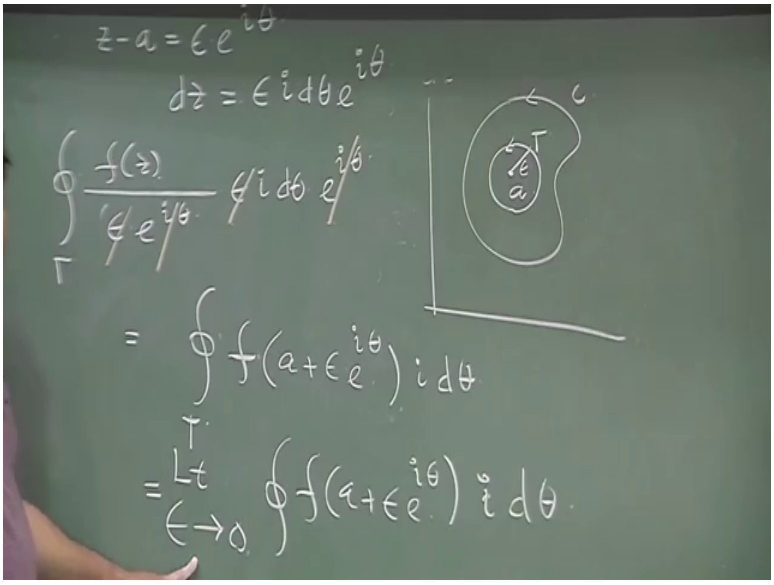
So the thing is Cauchy’s integral formula. Let us go to the statement first then I describe and try to prove how the things going on. So if we have, let me draw it here. This is a regions say c and a point here is a inside the region, then the statements state that let me right it here then I will explain.

If f z, this is the statement; this is the statement of Cauchy’s integral formula. So, after writing the statement let me explain one by one what is the meaning of that. T z is a function which is analytic in this region, right. But please note that this quantity say I write g z, this g z is f z divided by z minus a, which is not analytic in this region please note that.

So, here I have a function g z d x which is not analytic. If I try to find out the integration over c then the value of integration will be just 2 pi i multiplied by f a, where f is an analytic function given here that is the statement. The straight forward statement is if f z is analytic over the region c and a is some point here, if I manage to find out the functional form in this way f z minus z minus a d z then this integration can be evaluated and the value will be 2 pi i multiplied by f a; that is the statement.

So, now let us try to try to prove quickly how we can prove it. This is the statement, let me put this figure here; try to prove it in this.

(Refer Slide Time: 03:47)



So first: again z minus a, I put this integration c and if I closed an circle here with epsilon radius this 2 things are same last day we prove that. If you remember that gamma 1 is equal to gamma 2 we prove this. Some integration over gamma that means a bigger circle is same as the smaller inner circle if I put some smaller inner circle inside this region. So, here this things and this small one is a same.

So, now I put for this small circle which is the value of this which is same. So, I just want to evaluate this quantity. So c, I just wanted to evaluate this quantity z minus a d z. This is the quantity I want to evaluate. Now this is equivalent to; if I say this is my gamma equivalent to gamma closed integral f z divided by z minus a d z, fine. Now, I change this c to gamma because I have already shown that these things are these things are same. So, for this integration now I am putting my variables like this. So, z can be represented as we did it earlier 1 problem that z minus a is this quantity.

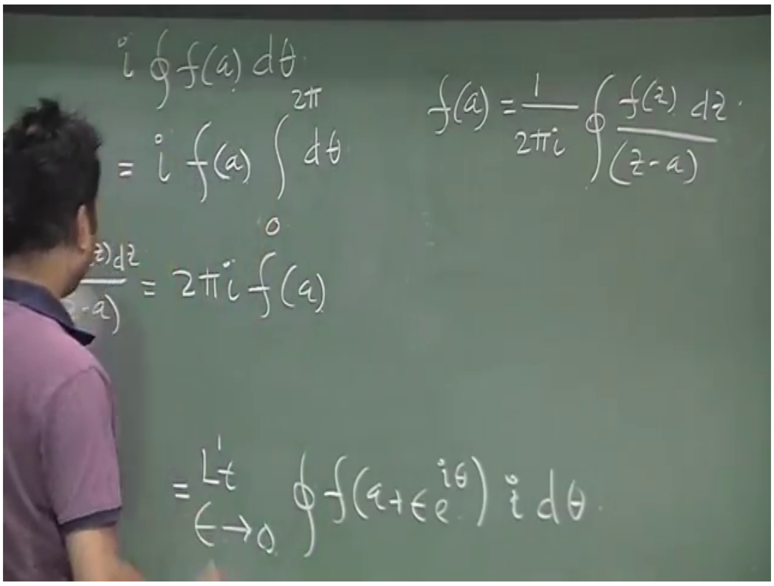
So, what is my d z? D z is epsilon is a constant, so I should not change epsilon i take a constant thing d z epsilon i d theta e to the power of i theta, this. Now I just try to find out this integration that is all. This integration over gamma is f z divided by z minus a . So now I change these things. So, z minus a is epsilon e to the power of i theta multiplied by d z; d z is epsilon i d theta into e to the power of i theta. This e to the power i theta and e to the power i theta will cancel out, epsilon is also cancel out. These things are not be there.

What finally we have here? F z: f z now since I put the value of z, so f z is now a plus epsilon e to the power of i theta this quantity, then i d theta. Now after having this thing we will put some kind of trick here. The trick is: I choose this smaller circle gamma with some radius epsilon, you may choose another with a smaller epsilon, somebody choose another with a smaller epsilon and so on. So that means, I can shrink this epsilon according to my likings and it suggests that every time I will get the same result, because it never mention that it depends on the radius of the epsilon. So, I never mentioned or the calculation never mentioned that epsilon should be some value for which this is. So, if I want to shrink this epsilon and if epsilon tends to 0 then still the result is valid.

Under this condition, because I need to evaluate these things and epsilon is sitting here so what I will do that I will put the limit epsilon tends to 0 over this. And essentially by doing that I can eliminate epsilon, but my this thing suggest that it will not going to change the entire structure. So, if I go from here to here the value of the integration will remain same. So I just put the epsilon; to remove the epsilon I just put this limit it comes like this. So, this is the expression I have in my hand.

After that what I will do? I will just put the limit and find out what is the value.

(Refer Slide Time: 09:16)



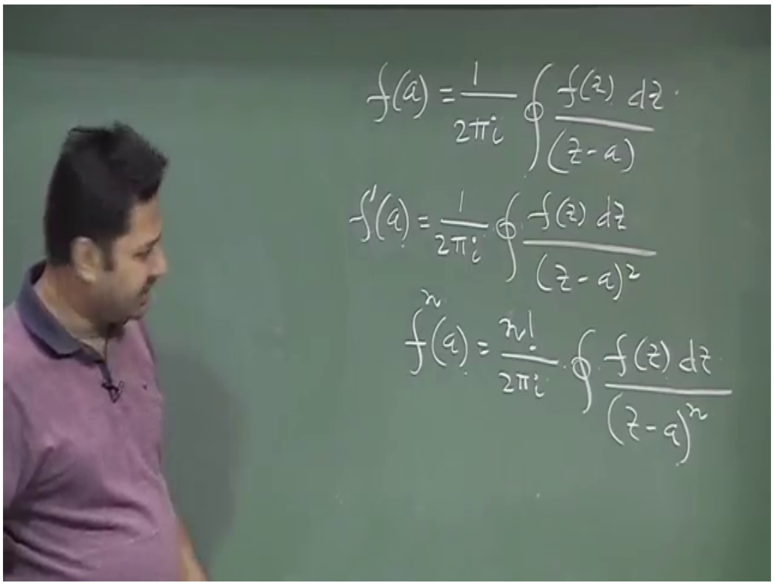
If I put the limit here then eventually I have closed integration f of a i will go outside d theta. Now again this closed integration which is equal to i f of a, which is going outside because f a is not a function of theta after putting the limit epsilon tends to 0, this quantity is vanished, this one is vanished. When this one is vanished we will have only f a in our hand, so we will take this f a outside this integration. And then integration limit will be just 0 to 2 pi d theta. And this quantity is nothing but 2 pi i f a.

I erase this part. In the left hand side what we had? The left hand side I had this quantity, because I started with this quantity. So, f z divided by z minus a integration d z is equal to 2 pi i f a. So, this is nothing but the statement of Cauchy's integral formula that I wrote the very beginning. So f a, is how much? 1 by 2 pi i integration f z z minus a d z.

So, we prove this Cauchy's integral formula and this is a very very powerful formula, because you can have now everything in your hands. So, if I have this integration in my hand I can readily find out what is the value of this integration it is just 2 pi i multiplied by f a. If a region is giving to you were f z is analytic.

So, after having this expression we now try to find more from this expression. And we do it in this way; if I make a derivative of this expression with respect to a.

(Refer Slide Time: 11:42)

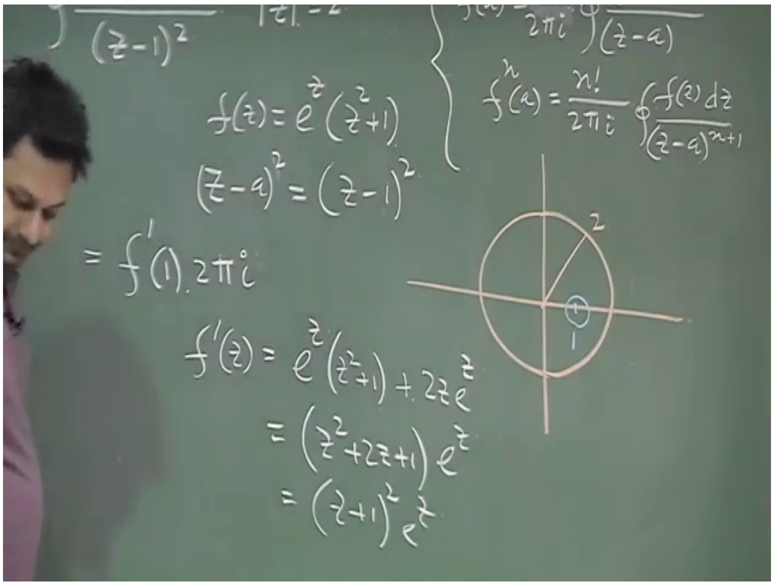


F 1 a, so one value I will have 2 pi i integration of f z d z, this quantity will be z minus a square. One negative sign will come because of this integration, but one negative sign is sitting here just before a, so this negative and negative will be positive to we have this quantity.

Now, if I go on integrating then I will ended up if I have the nth order integration f n a it will gives me simply factorial n divided by 2 pie I integration f z d z divided by z minus a whole to the power n. After doing the integration over these I will have this. So, 2 very important expression is now in our hand. And this expression is 1 is f a and another is f n a. If my functional form is something like this where we have some kind of power sitting here then I will use this expression with this, this is factorial n so it looks like h let me change it. And if I have the function like this, I will use this.

So, after having the expression in our hand; let me just write the expression somewhere here and then start doing the problems. So, let me write this expression here for my convenience.

(Refer Slide Time: 13:35)



So, f of a is 2 pi i total integration f z z minus a d z: this is one formula that I have. And another is: f n a is equal to 2 pi i should be 1 by 2 pi i, 1 by 2 pi i factorial n divided by 2 pi i total integration of f z z minus a whole to the power n plus 1; I mean I do not know what I wrote here it should be n plus 1 because every time when you do the integration. If it is n, so this integration when it was 1, so this integration was here we have 2 here. So, when it is n it should be n plus 1, I do not know I erase that in the last expression what I wrote I forget, but it should be n plus 1, anyway.

So, these two is in my hand. Now if it is in my hand then I can readily calculate few integration. Let us do some problem straight away. So, let me check which problem I have in my hands. So, one problem is here. Example 1: how to calculate this integration? The problem is given as this function e to the power z z square plus 1 whole divided by z minus 1 whole square. This integration you need to evolve where the contour, since it is a close integration the contour is giving to you and this contour is z is equal to 2. This is important. Every time the contour is very important over which contour I want to find out.

Now, after having this expression you can rearrange these things in terms of this; that is the first thing you need to do. So, if I do that then I readily find here my f z is e to the power of z z square plus 1. This is my f z. And what is my z minus a here z minus a whole to the power n, if you remember that this is square and we have something here so that means this square is nothing but z minus 1 square. So a is here 1.

Now the thing is that, first we need to check whether these functions, the entire function is analytic over this region or not. So, this region first we need to drop. Let me draw it here: mod of z is equal to 2 means which region they are talking about, they are talking about a region this, this is z equal to 0 and this point is 2, the radius is 2; this is radius of 2.

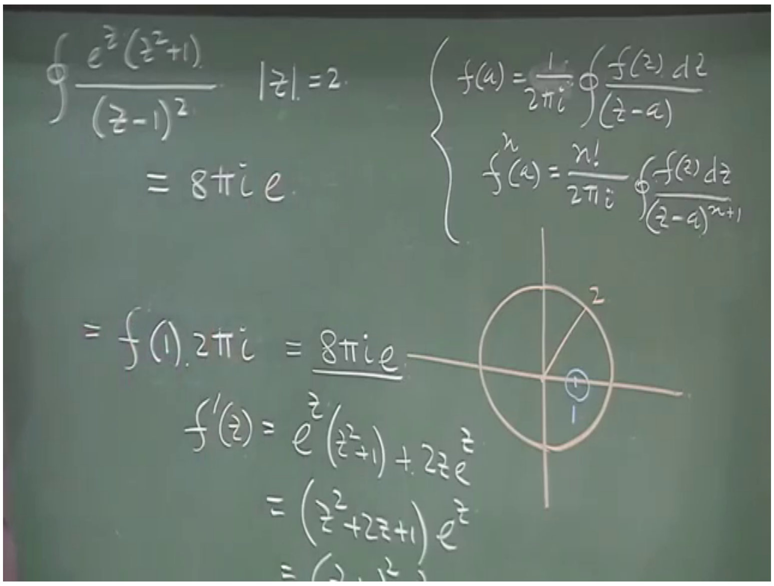
Now, where is my a? Z minus 1 square, so a is how much; 1. So, if a is 1 this point is sitting over here, this is the point 1 which is inside the region that is given. That means, this entire function has a singularity at z equal to 1 so that means this function is not an analytic function. So, close integral over this will not be 0 that is the first thing. Second thing is that once I have the singular point here I will just go into apply the formula that I have.

The formula is straight forward once you have. So, the formula of this is straight forward. It will be f prime 1 into 2 pi i. I am using this with n equal to 1, because if I put n equal to 1 then this equation comes out to be this. So, this things I need to evaluate. In order to evaluate what I need to find out; the function I need to evaluate the derivative of this function first derivative of this function at 1 I need to evolve; that is I need to figure out, that is the thing.

So, what is the value? So, function z is already I read in that (Refer Time: 18:35) e to the power z. So, function of these things is how much? E to the power z z square plus 1 plus 2 z e to the power of z. If I take e to the power z common, then z square plus 2 z plus 1 e to the power z or seems to be z plus 1 whole square e to the power of z. Now if I put f prime 1, so this is my f prime z so f prime 1 is I just need to put it here the value 1; if I put 1 then this value.

So, let me erase this. So, this value comes out to be f prime of 1 is simply comes out to be this is 1 so it is 4 e to the power of 1: 4 e. So, now my total value is this.

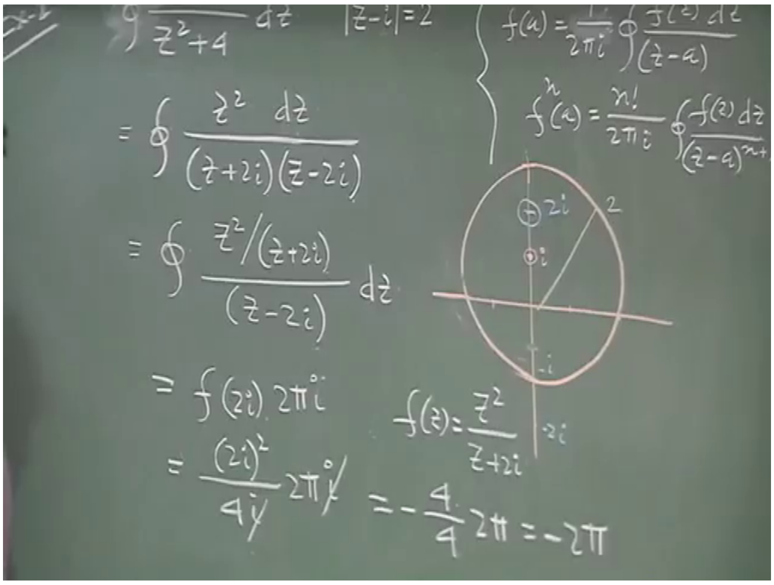
(Refer Slide Time: 19:33)



So, the integration is 8 pi i multiplied by e; that is the value of the integration that I wanted to find. So, total solution. So this value, this equation when I am evaluating it at z equal to 2 this region that means this region the value comes out to be 8 pi i e. So, this is the first problem.

So, now after that we will go to few other problems, let me do that. I believe you now have the idea how to evaluate this kind of integration which is very straight forward problem 2; using just using this two Cauchy's integral formula either you need to use this or this.

(Refer Slide Time: 21:00)



Example 2: the integration is given as close integration z square divided by z square plus 4 d z; that is the integration you need to evaluate, this is the integration we need to evaluate. But before evaluating I need to know what is the region that is given. The region is given that z minus I is equal to 2, over this region I need to find out my integration.

So, before doing anything first I need to just draw this region; what region they are talking about and then I need to find out where the singularity is there and accordingly I calculate by using that formula; so first the region. So, I let me draw the region here this side. Z minus I is 2; that means my coordinate is somewhere here this is the point i. And I will going to encircle this point with a circle of radius I need to this is my centre, I need to encircle these things with radius 2.

So, I this is 1 unit, this is 2 unit, this is 1 unit, this is 2 unit, this is 1 unit, this is 2 unit, this is 1 unit, 2 unit. So, my circle is this something like this; it is looking very bad looking circle so let me; something like this.

In this kind of problem you need to know very precisely where you try to calculate, what is the region and where you try to calculate this integration. So now, I know that my I point is here and this is the region where this radius is 2. Now I simplify this expression: z square divided by z plus 2 i z minus 2 i. This is a familiar function, because in the previous class I have already mentioned this function z plus when I try to find out the singularity at two different points.

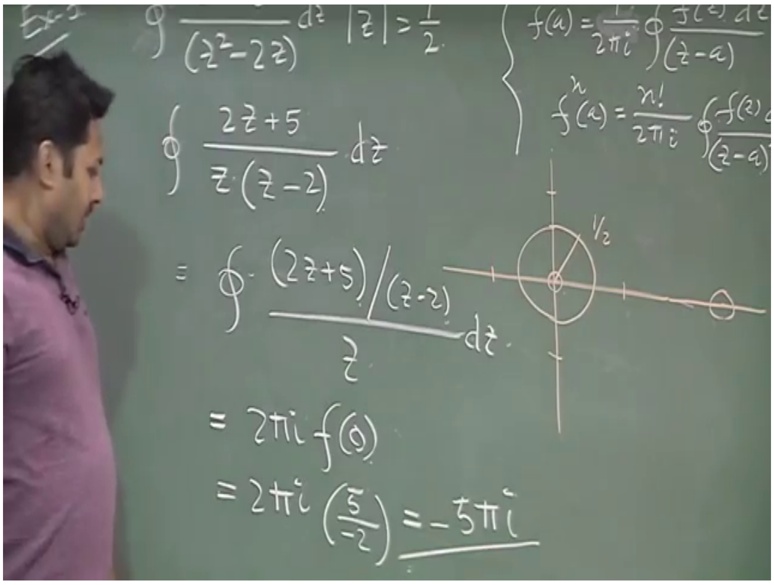
So, now you have two points where the function is having some singularity: one point is 2 i and another point is minus 2 i. You need to find out in this region which point is inside this region. You can readily find that 2 i point this is i, so 2 i point is somewhere here is inside; this 2 i point is inside. What about minus 2 i? Minus 2 i is not inside it is outside. So, this is i, so here this is minus i so somewhere here it is minus 2 i which is outside the region.

That means if I write this function, I need to now rewrite this function slightly modified way. So, z square divided by z plus 2 i whole divided by z minus 2 i d z; then I have the form. So now, this f z is analytic. Here if you remember I mentioned that my f z has to be analytic, total function may not be analytic or should not be analytic because some singularity will be sitting here. Here I am writing the functional form in this way. So, what should be the value here? The value will be function here function of z is how much; this quantity z square divided by z plus 2 i I just replace this function. So, it should be function at 2 i point multiplied by 2 pi i. Functional 2 i points means how much? Z square is 2 i square divided by 2 i plus 2 i it will be 4 i then it will be 2 i 2 pi i

So, here I will have 1 negative sign will come because of this square I square so 4. So, minus of 4 divided by 4, this i i cancel out and 2 pi, so we will have minus of 2 pi. So, the result here is minus of 2 pi. So, this integration if I solve this integration, this integration results the result is minus of to pi.

Before concluding let me give another simple example and then I will going to conclude today's class; so quickly another example; similar kind of example.

(Refer Slide Time: 27:09)



So example 3: the function is given as 2 z plus 5 divided by z square minus 2 z and the region that is given is mod of z is equal to half; mod of z is equal to half. This is the expression. So, mod of z equal to half what region they are talking about quickly we should draw. Mod of z equal to half means this is 1, this is 1, this is i, this is minus i, so half means it is encircling something like this where the radius is half and this is z equal to 0.

Now here how many singularities are there we need to find. So, 2 z plus 5 divided by this is z multiplied by z minus 2. So obviously, 1 singularity is at z equal to 0, so z equal to 0 we have 1 singularity and z equal to 2 we have another singularity, because at this 2 point the function blow up I need to put a d z here.

But, my region suggest that z equal to 2 point is outside. Z equal to 2 point is somewhere here; this is z equal to 2 point, because this is 1 so somewhere here. So, it is well outside the region. If it is well outside the region then my, but z equal to 0 point is inside. So, z equal to 0 point is inside here. So, I can rewrite this function, this integration like this: 2 z plus 5 divided by z minus 2 divided by z d z. What will be my answer? My answer will be 2 pi i multiplied by the function whatever the function I have at 0 point. So, which is 2 pi i this function at 0 means I have 5 by minus 2. So, my result will be minus of 5 pi i. So, this is the result.

With these small few examples like I would like to conclude my class here today. So, today we learned a very very important concept that how by using Cauchy's integral formula you can evaluate the integration like this. At some region you need to find out the points and that region whether this points are giving some kind of singularity or not: for here in this case z equal to 0 point is a singularity and you rearrange the function in such a way you can use this expressions whatever the expressions is given to you, which is the basically the recipe or formula to calculate the integration. And then you do straight away what is the value of that.

So, with this note let me conclude the class here. See you in the next class.

Thanks for your attention.