Mathematical Methods in Physics-I

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Lecture – 43

Application of Cauchy - Goursat Theorem

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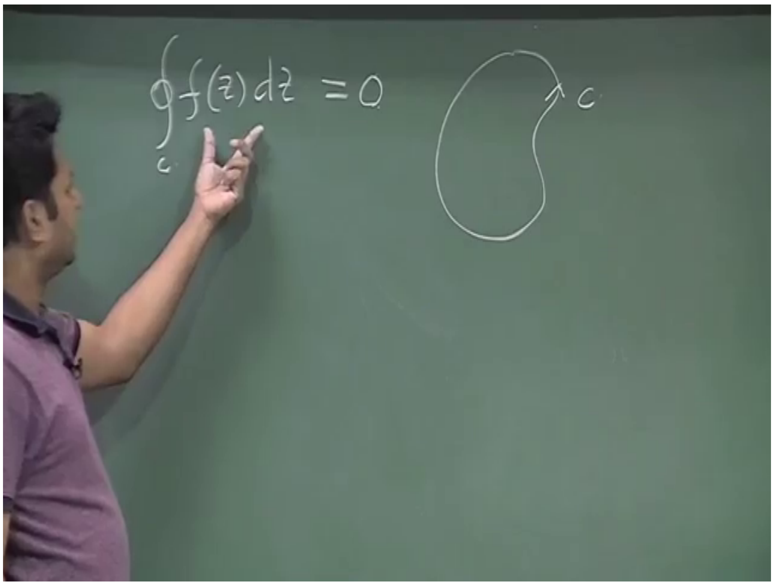

So, welcome back student next class of complex analysis. In the last class, we discussed something about Cauchy-Goursat equation or Cauchy-Goursat theorem, we suggest that if we have a region bounded by some curve c and if the function f z is analytic in this region then the total integration of this value will be 0. So, let us try to find out few things what is the meaning of the function is analytic in some region.

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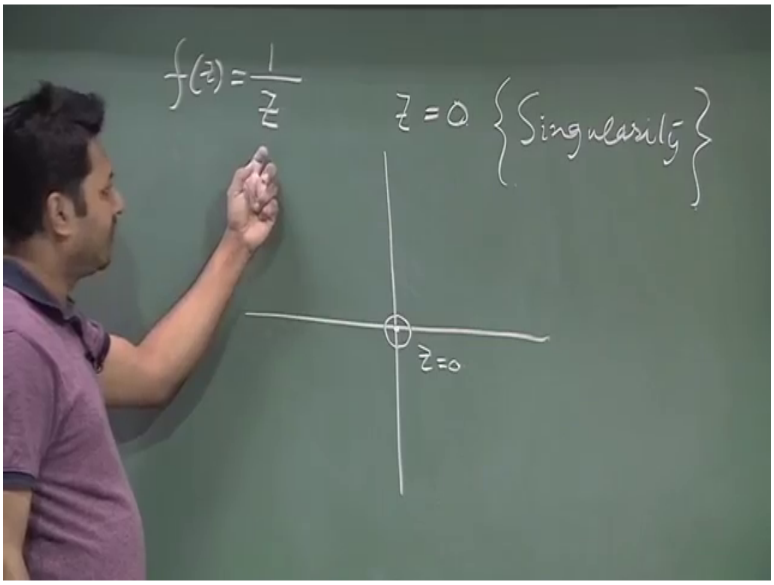

So, let me give you first straightforward example. So, f z is 1 divided by z this is a function. Now, you can readily see that the function at z equal to 0 point, the function is blowing up. So, z equal to 0 point is called a singularity. We use this term earlier singularity, but we will going to learn about these things more in the future class, but you should now realise what is the meaning of singularity; that means, the function is failed to analytic at that particular point that means, z equal to 0. So, if this is my region, and if this is my point z equal to 0 except this point, the function is analytic everywhere, so that means, there is a region there may be some region where the function is failed to analytic. I can give you more examples, so that things will be cleared to you.

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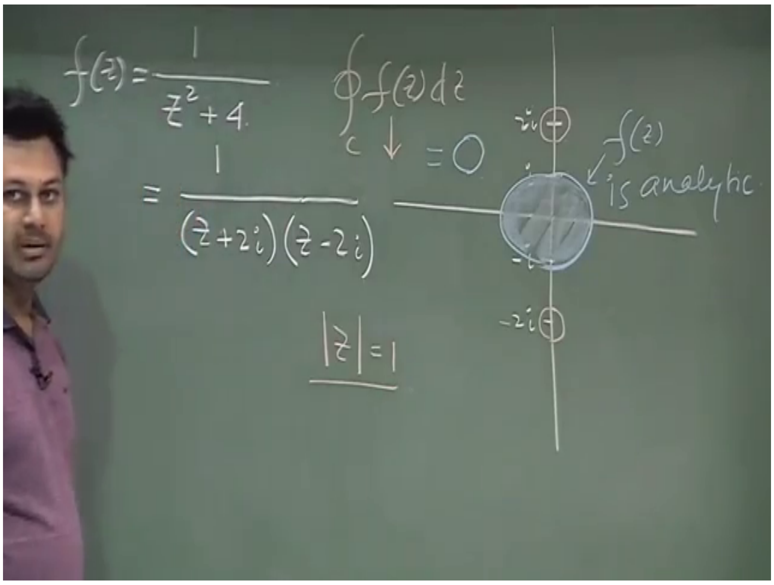

Let me give you one another example 1 divided by z square plus 4, 1 divided by if I simplify that plus 2 i z minus 2 i this is my function right. So, if z equal to 2 i then you find the function is blowing up z equal to minus 2 i again the function is blowing up. So, if I draw here, this is i, this is 2 i, this is minus i, this is minus 2 i. So, i will have two points here, say here and here, this is minus where the function have that problem. The function is failed to analytic at that particular point except that it behave like an analytic function. There are few regions of few points where the function is failed to analytic and that is why this integration when I was doing the integration these things we need to mention that f z is analytic over some region and the region is maintained or mentioned by this closed curve c.

So, if c is given in such a way that this function is analytic then we can use our formula. For example, here if the z curve for this particular problem one function z square plus z is given like this region that means, I am having a region like this mod of z is 1 that means, I am having a region here. In this region, this region, this entire region, this given function f z is analytic; for this region the function f z is analytic. So, f z is analytic, which f z I am talking about 1 by z square plus 4. Why it is analytic, because my singular point where the function is blowing up it is outside the region. So, in this region, it is analytic.

Now, if I want to integrate these things over here. If I want to integrate this function these things over here in these region, which is given by z equal to minus one I will have straight way I will have the value 0 that is the thing I wanted to mention. After having the knowledge of this region where the function is analytic, let us do few calculation.

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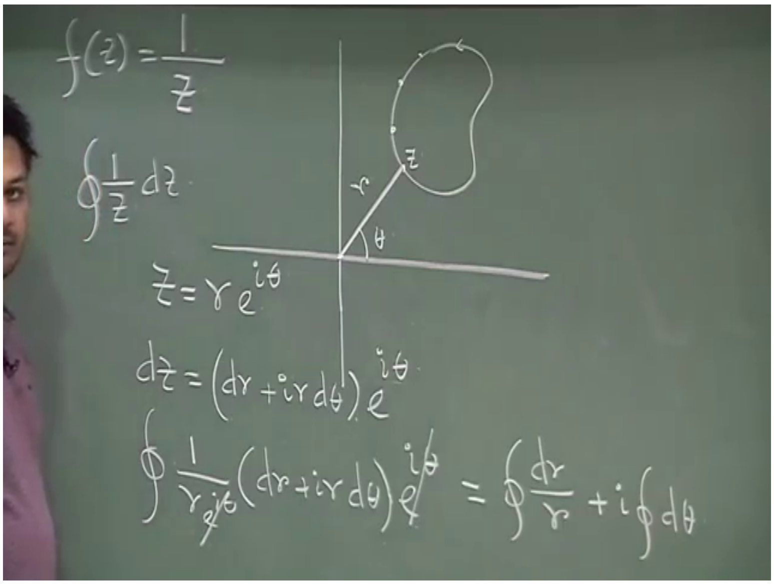

So, let us start with some trivial calculation. So, let us start the function that I wrote few minutes ago 1 by z. So, now I want to integrate this over this region, say this is my region i over I want to integrate. So, you can readily understand that over this region the function is in this region at least the function is analytic. So, this is the value of r, this is the value of theta, this is my z point. So, if I do the integration, I need to calculate 1 by z for this closed path dz, which quantity I need to calculate. Now, z is r e to the power of i theta this quantity, dz is how much, here please mind that every time z is changing, this is the point where z is changing or in this way whatever r and theta both are changing, it is not constant both are changing. So, that means, it is d r plus i d theta i r d theta e to the power of i theta if z is r e to the power i theta then d z has to be d r plus i r d theta. For this case d r e to the power i theta will be d r. So, I take the e to the power i theta both side common. So, it is outside and I will get this straight forward.

Now, I will put it here, so in the inside the integration. So, my closed integration 1 by z means 1 by r e to the power i theta and this quantity d r plus i r d theta whole multiplied by e to the power of i theta right. Now, this e to the power i theta and this is e to the power i theta will cancel out. And I will have to integration one is this related to r, and another is related to theta. So, it will be something like d r by r and plus i d theta, simply I have this. Now, these things I can write in general because I am going to let me erase this because I need to use this board here.

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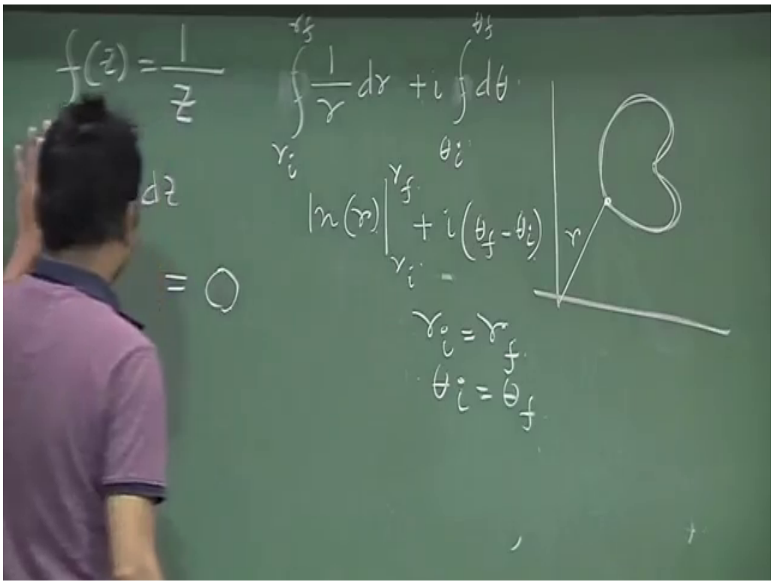

So, integration of one by r d r plus i integration of d theta that I am going to calculate. So, now you can understand. So, let me do one thing here let me draw all the figure once again. So, this is my r. So, if I write in state of having the closed if I write then it will be r initial to r final instead of having this things if I write theta initial to theta final - this. Now, you can say you can find that this is how much l n of r r 1 r f r i r f plus i theta f minus theta i this is in general if I go from some initial point to some final point. Now, the thing is that I am going to this path and then returning back to the same point here. So, my r i is eventually r f and theta i is eventually theta f when my r i is eventually r f then this again quantity this quantity both will going to vanish. So, we will have 0 which is expected because we know that it has to be 0, because the function one by z is analytic in this region and I am making the close integral over that. So, essentially this is 0 when my region is giving like that.

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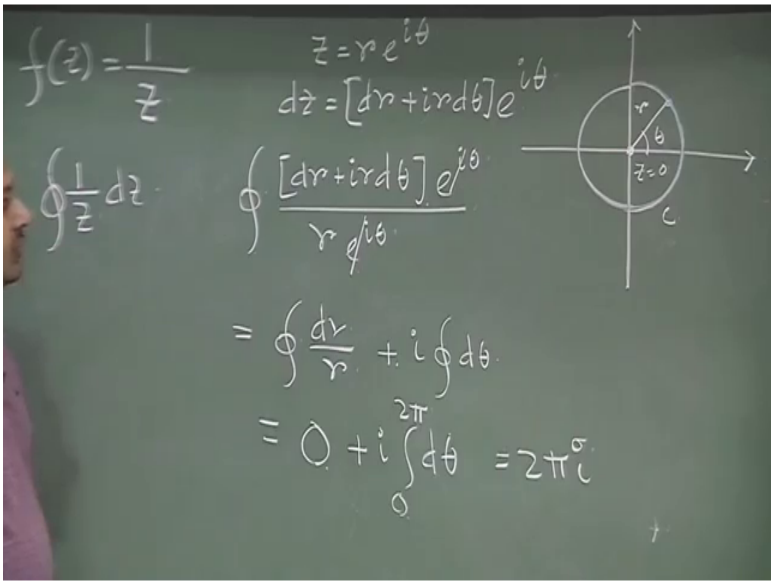

I will do the same problem for another region. So, let us find what we are getting this is interesting. So, let me draw the region that side, so that I need not erase this every time. So, this is the region now. I am enclosing this z equal to 0 point. So, this is my c; I define my c in such a way that it is encircling the z equal to 0 point. Now, the function f z is not analytic in this region, f z please mind it is not analytic in this region. So, integration of this quantity obviously will not be should not be equal to 0. So, let us try to find out how the things. So, here again this is say my r and this is my theta. So, again z is equal to r e to the power i theta exactly the same procedure I will going to follow. So, d z is equal to d r plus i r d theta e to the power of i theta - this, then the integration is d r plus i r d theta e to the power of i theta divided by z which is r e to the power i theta. So, this quantity will going to cancel out and d r by r as usual when the previous case whatever the things I got is ok and then I find i integration of d theta.

Now, here is the interesting thing, exactly the calculation is exactly the same that we have done in the previous problem, but there something extra we have. The extra thing is that the theta is now rotating like this, enclosing the point z equal to 0 that means, initial theta and final theta may not be the same one. Here initial r are final r will be same, so I will have 0 for this, but here it will be d theta will be 0 to 2 pi, because it is rotating 2 pi. I start here and it rotates like this and come to this point, so that means, my range of theta is 0 to 2 pi. From here, I return back to that point which is 2 theta, so d theta. So, eventually I will have 2 pi I this is a very important relation that 1 by z d z if I try to find out the integration in this region, the value should be 2 pi i; the value should be 2 pi i. So, let us now do the same problem in different for other case. So, let me check which other problems I have for you.

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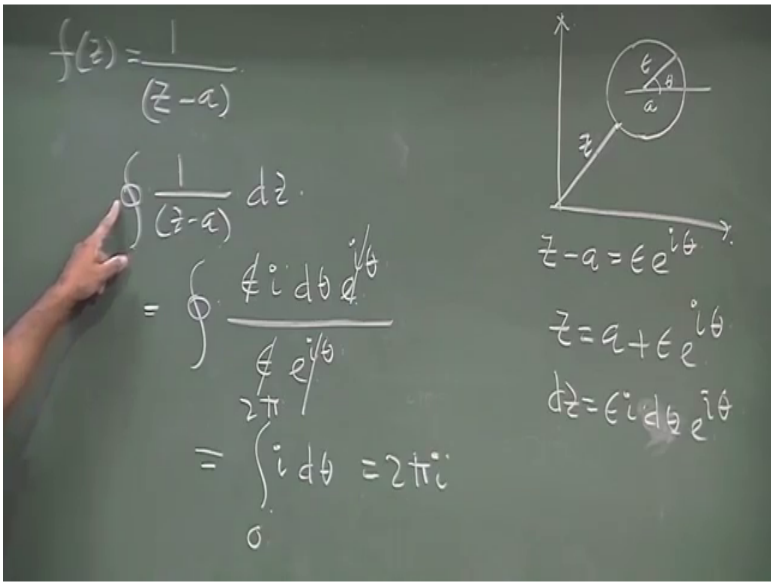

So, before doing that, so let us try to find out these things. So, now I want to continue with another function, this is equally important. f z is now z minus a, I want to evaluate this quantity, I want to evaluate this quantity. So, this quantity let me write the region here, which region I want to find, here says the point is a, and this is the circle I have, and I want to find out the integration here. Now, if you see the point a is inside the circle, that means this is my z point which say this is epsilon and this is my z point. You can see that this a point is inside the region that means this function is not analytic there, so obviously, this function should have some value there in that point. So, let us try to find out.

So, now I can write z is equal to say z minus a is equal to epsilon e to the power of i theta if this is the radius is epsilon then z minus a I can write in this form because it is z is changing z is changing over this things. And it is nothing but this point plus this one epsilon i theta. So, the general z point is defined like this, I can write it. When this rotation because of this rotation a will be, so let me explain you if you have a. So, this is in general the location of z. When epsilon is a 0 z and a are same point, but it is rotating over some radius r radius epsilon. So, when epsilon theta is changing, it will change the signs; sometime it will be a plus epsilon some will be a minus epsilon some will be some values. So, some point will be here say a plus epsilon, some point is here which is a minus epsilon, and another values are also possible depending on the value of theta anyway.

So, what is my d z here d z is epsilon I am saying this is a circle. So, epsilon is fixed epsilon i theta e to the power epsilon i d theta e to the power of i theta, this is my d z. I will put it here. So, total integration of f z is this quantity z minus a d z which is 1 by z d z is epsilon i d theta e to the power of i theta divided by z minus a, z minus a is epsilon e to the power i theta. So, epsilon e to the power of i theta e to the power i theta e to the power of i theta cancel out; epsilon, epsilon cancel out. So, we will have this theta range is calculated against this point. So, my theta is here.

So, this theta is rotating at that particular point, this is the some sort of shifted origin and it is rotating. So, the initial and final value of theta is not the same, but it is 0 to 2 pi i d theta, so it will be 2 pi i again. So, this quantity if f z is this, if I take the total integration, close integration over that point where it is enclosing the point the singular point a, I will have a value 2 pi i. And again it is true because function of z is not analytics, so it should not be 0 at that point, but it is 2 pi i. So, now, I will again change this thing slightly and try to find out.

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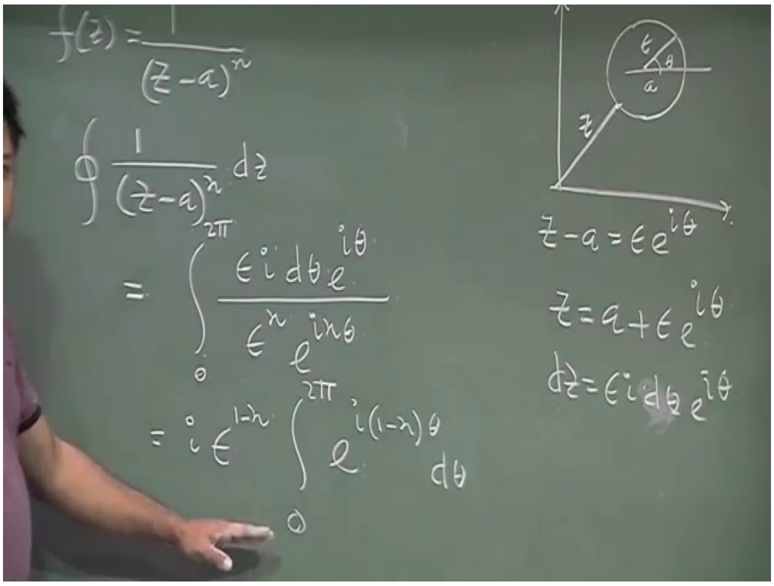

For example, if I change this quantity to z minus a whole to the power n then what should be the value of this integration; n is an integer, so it is interesting because now I am increasing the power. So, I am try to calculate this quantity z minus a this d z, which is d z will be same as before, there is no change, d z will be same. So, it will be integration epsilon i d theta e to the power of i theta divided by z minus a whole to the power n this is the place where things are changing. So, it will be e to the epsilon to the power n e to the power of i n theta I have this with the limit 0 to 2 pi, this closed integral and this things are identical now, because epsilon is not going to change. So, closed integral here is eventually means that my integration is 0 to 2 pi. Now, it is i epsilon 1 minus n integration 0 to 2 pi e to the power of i 1 minus n theta d theta, I am having this expression now in my hand. So, we are almost there.

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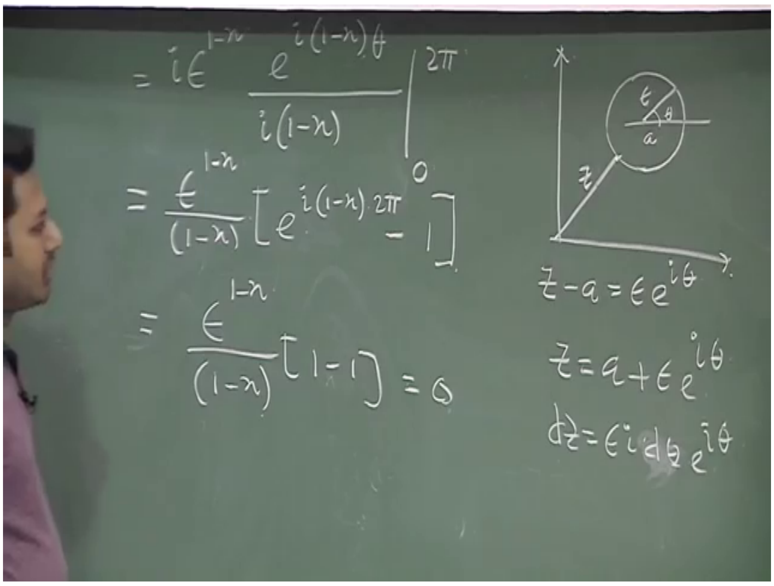

So, this quantity is i epsilon 1 minus n and this quantity is e to the power of i 1 minus n e to the power of i 1 minus n theta divided by i 1 minus n with the limit 0 to 2 pi n is an integer. Now, if this is the case you can find that epsilon 1 minus n divided by 1 minus n e to the power of i 1 minus n into 2 pi minus 1. e to the power 1 minus n 2 pi is essentially 1 because it is cos 1 minus n 2 pi multiplied by i sin 1 minus n 2 pi, so it is 1; sin 1 minus n 2 pi is 0; cos 1 minus n 2 pi is 1, so it is 1. So, I will have epsilon 1 minus n 1 minus n 1 minus 1, which is 0 this is also important outcome that in this region when I calculate. So, let me summarise these things quickly.

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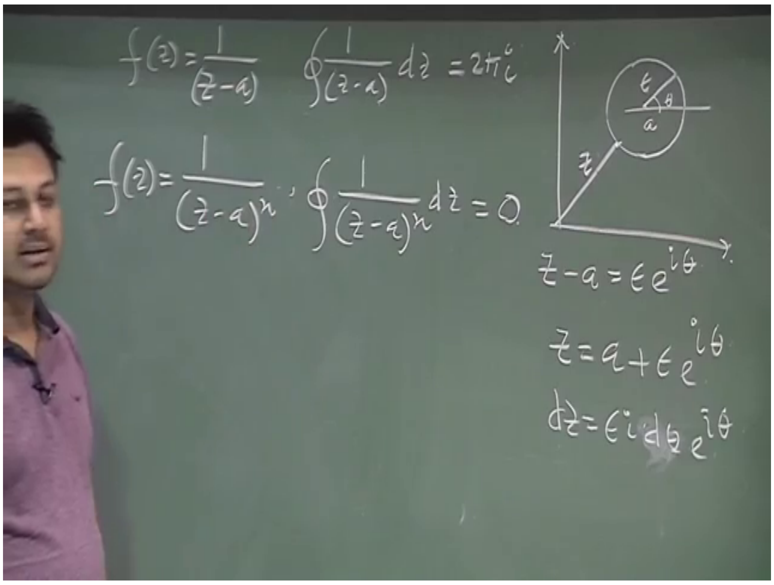

So, when my function f z was 1 divided by z minus a, when I calculate this quantity, I have 2 pi i as a result. When my function is z minus a whole to the power n and when I calculate this quantity 1 by z minus a whole to the power n d z my result was 0. So, this two are very important outcome regarding these things. So, we calculate that using extensively the calculating everything we find that it is this value is 2 pi, and this value is 0. We will going to use this actually, why I we are doing this because we will going to use this concept in our future classes; and then we will exploit this more to find out the value of the integration.

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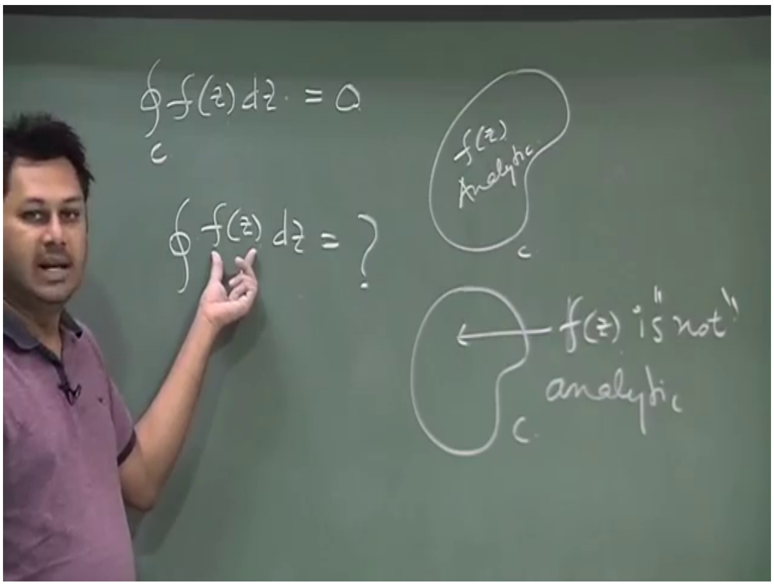

So, before concluding this class, one very important integral concept I like to make here that is please note that so what so far we have. When I have f z d z equal to 0 for the region if the region is giving to me where the f z is analytic then it is ok that f z is analytic I have a region c where f z is analytic, so this value is 0. But if f z is not analytic, I do not know what is the value of a given region. So, f z is not analytic in a region. So, this region is here c where f z is not analytic then I do not know how to calculate this kind of integration. But in the next class, we will start from here and exploiting this concept we try to find out whether is there any way to find out the value of this integration in the region, where f z is not analytic. But I can do some procedure, so that I can readily figure out whether what is the value of this integration.

So, this to find out the integration of this kind where the function giving here is a non analytic function on the region and still I can have a recipe to find out there is a well known formula for that which is called the Cauchy's integral formula. So, in the next class with start with that and find out how the Cauchy's integral formula gives us the values of this integration where f z is not analytic. With this note let us conclude here in the next class we will again start from here and try to figure out how Cauchy's formula enable us to give the values of integration where the function is not analytic.

So, thank you for your attention. So, see you in the next class.