Mathematical Methods in Physics-I
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Lecture - 42
Cauchy - Goursat Theorem

So, welcome back student to our next class of complex analysis.

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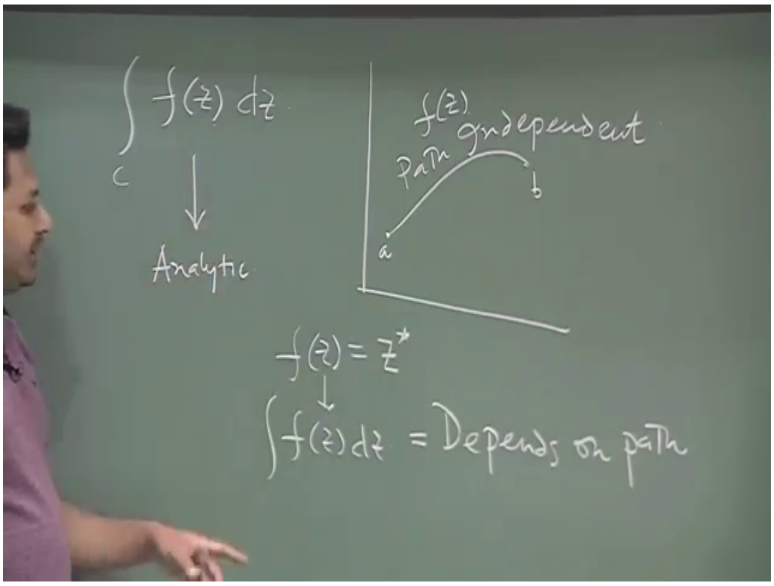

In the previous class we learn a very important thing and we find that if a function complex function f z is given and if I want to integrate it over some curve. So, this is the plane and I want to integrate it over some curve a to b, then this integration the value of this integration depends on only the initial and final value. When this function f z is analytic if a function is analytic then this integration which is over this line is depend only on the initial and final value. Will show few examples and show that that if this function is not analytic if you remember f z is equal to z star this was this function is not an analytic function. So, when you integrate this we find that it depends on path integration f z dz when f z is this z star is depends on path. But if it is analytic it is path independent. So, this integration if f z is analytic then it is path independent this is an very interesting concept since it is a path independent we should not bother about which path I am going that is one issue another important issue that I like to mention today which is equally important and very useful.

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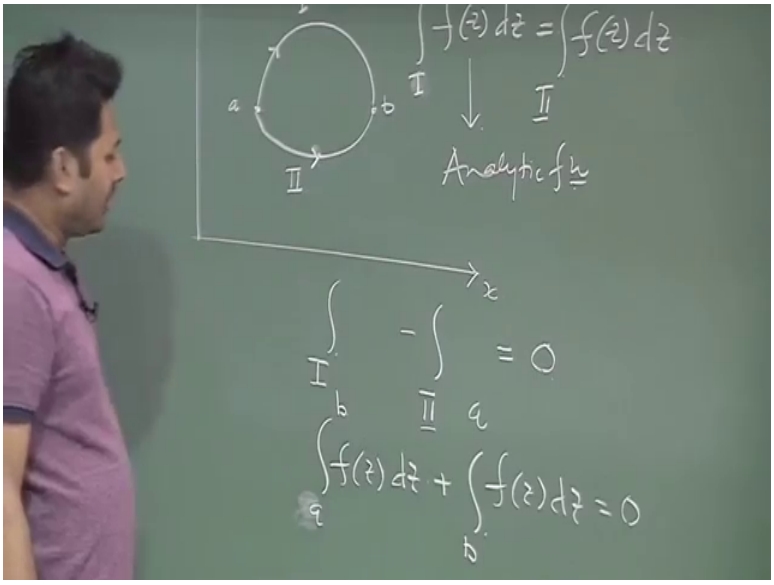

So, this is suppose this is my coordinate system and I am going from this path to this path a to b and then return back from here to here, let me first do in this way this is path 1 and this is path 2. So, first I go a to b through path 1 and then I go a to b this point b through path 2. So, integration of over path 1 dz is equal to this because it is a path independent integration when it is possible when f z my f z is analytic. So, here f z is an analytic function.

Now, from this we can write a very important and interesting thing that if I write I whatever the argument is here, here minus this that should be equal to 0; that means, integration of this I take this integration to this side and that should be equal to 0. Now, minus of these things is nothing, but I am going b to a so; that means, I f z dz plus path 2 in opposite direction. So, if I put this limit. So, it is a to b and now my limit will be b to a f z dz this is equal to 0; that means, eventually I am coming back to my original position a. So, I start with here I go here and return back to this; that means, I am completing a closed loop.

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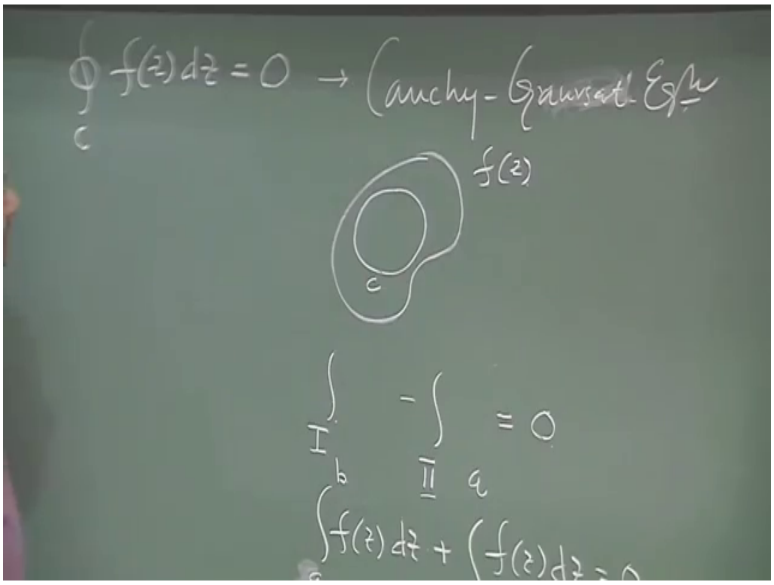

When I am completing a closed loop what I find that my integration is 0 so; that means, if I write here the total form the complete form that closed integral f z dz is equal to 0 that is a very important outcome that we find and last day we mention about this equation. This equation is called Cauchy-Goursat equation.

This particular integral form if f z is analytic over some region say this is the region where f z is analytic let me describe what is the meaning of this equation this is a region where f z is analytic and if I have a closed loop here c and if I want to integrate this integration f dz over this closed loop. So, I will get the value 0 this is because this function f z whatever the function f z I am talking about is an analytic function. Now we will prove that in more meticulous way that how to show that. So, let us start with this.

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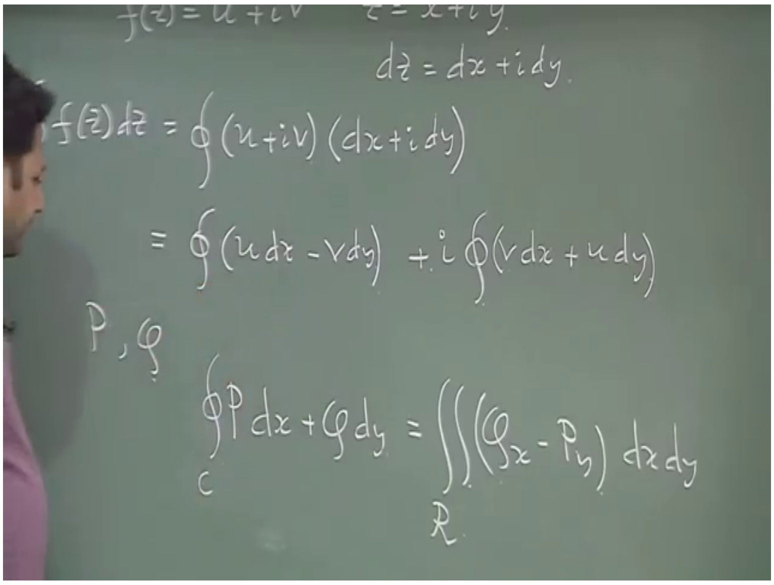

So, f z is u plus I v and z is x plus I y as usual. So, dz is d x plus I d y now if I want to find out the integration f z dz it will be something like closed integration u plus i v this I can rearrange this slightly u d x minus v d y this path plus v d x plus u d y. I separate out these 2 integration like this I just separate out.

Now, the next thing is that now we going to apply one relationship that we know that if P and x is function of x and y P and Q is a function of x and y then we know the relationship that this is over some curve P d x plus Q d y is equal to post curve is equal to this quantity over the region Q x minus P y d x d y we know this theorem very well.

So, P and Q if they are function of x and y and if this quantity is continuous and if I integrate over some region C then this quantity this is a closed curve and this closed curve is encircling some regions say R. So, that is why line integral become the line integral become the surface integral we use several time in the vector analysis if you remember then this is the case.

Now, if I try to apply here then let us see how I get this. So, now, let me erase this, this part because black board is slightly small here so.

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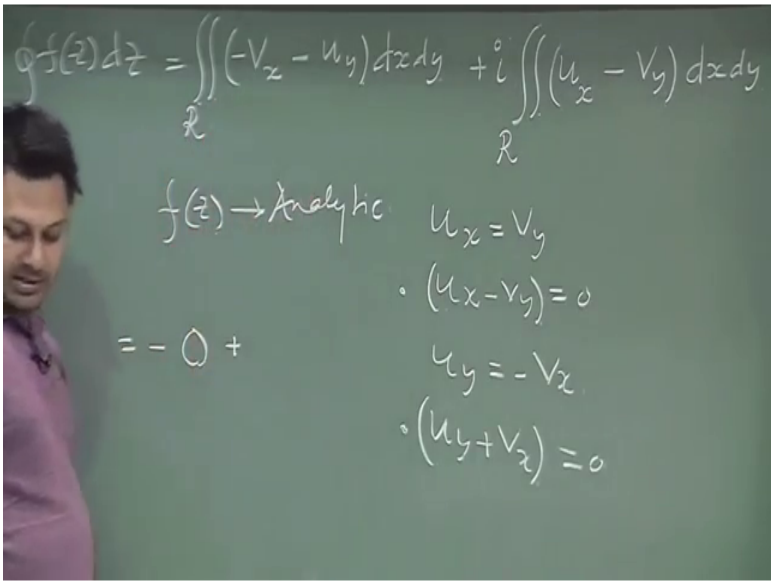

So, this quantity this quantity and this quantity I will replace through this theorem. So, here I should write it u d x minus v d y so; that means, here u is P and v is Q minus v is Q rather so that means, I should write it here integration over some region the region is enclosed by this circle minus of v x minus u y d x d y this is the first part. And second part plus I integration over this region u y this u which is here it should be u x minus this v will b v y d x d y. I ended up with this expression when I try to find out the total integration.

So, now I know that my function is analytic. So, now, you can understand if my function is analytic. So, f z is analytic that eventually means that I have this relation u x is equal to v y or u x minus v y is equal to 0 this is one expression I have and u y is equal to minus of v x. So, u y plus v x this quantity is also 0 this is true only for the analytic function. If this is true then we can say the function is analytic and now I have 2 expression if I put this expression here then you can really find that this is 0, so here if I write u x plus v x plus u y v x plus u y which is 0. So, it is 0 plus if I take minus common from that then it will be v x plus u y and this quantity u x minus v y u x minus v y is this quantity u x minus v y which is also 0.

So, if I put this here then the real and imaginary part both are 0 individually so; that means, this integration has to be 0.

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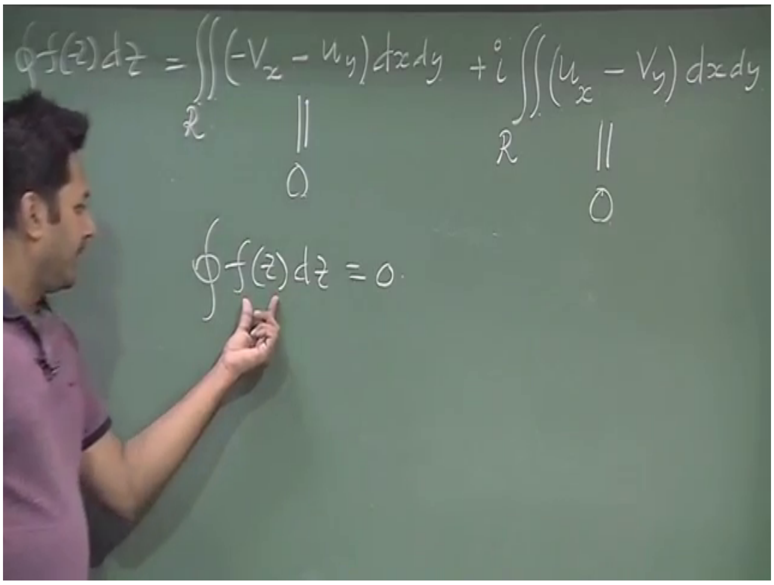

So, this is 0, this quantity is 0; that means, the closed integration f z dz is 0 when f z is an analytic function if it is an analytic function then only we can say this is 0. The example we have already shown the last days class.

Another important thing I like to mention in this point that we are talking about the region and all these things.

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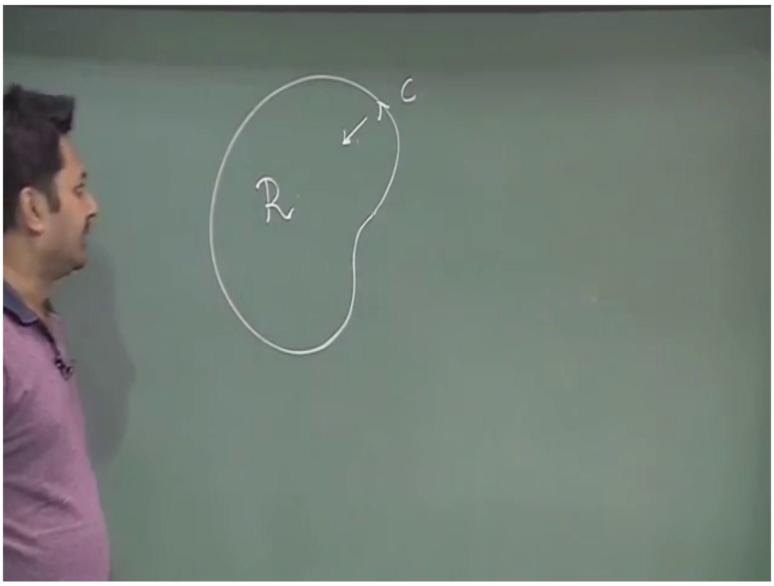

So, if a curve is like this bounded some region this is the region and this is the curve I am talking about closed curve means it is bounding some region. So, normally what happened that it depends on which direction I am going, so this direction is normally taken as the positive direction because the left hand side is the region. Normally if the direction of this curve is such a way that it encircling the region and this region is in the left hand side when I move on to this curve then this is considered to be as a positive direction we need to use these things later, but this concept or this notation on this convention we should follow.

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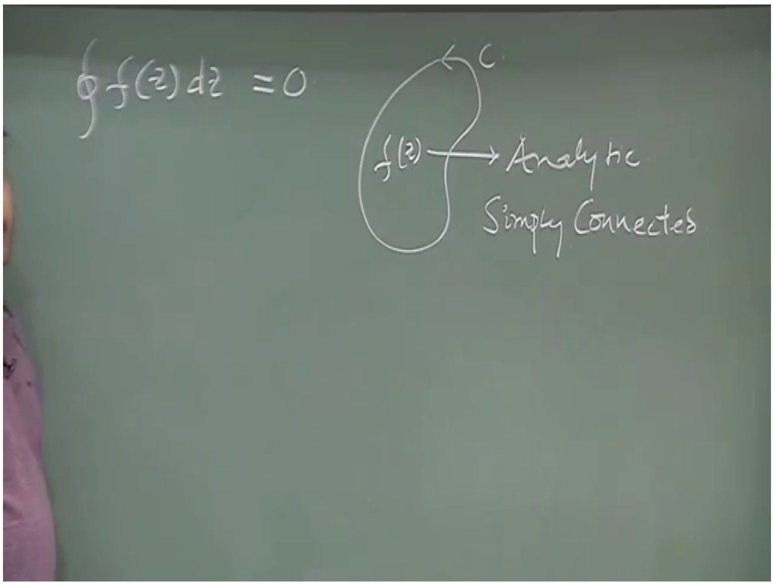

So, what we get so far? We get let me write once again this expression that f z dz is equal to 0 when C is a curve enclosing some region R and in this region the function is analytic. So, f z is analytic in this region f z is analytic in this region. Now this is for this curve is a simply connected region this curve is a, this is simply connected curve simply connected now we try to find out what is the form or how this Cauchy-Goursat equation is valid for a multiply connected region because multiply connected region is something which is different from the simply connected region.

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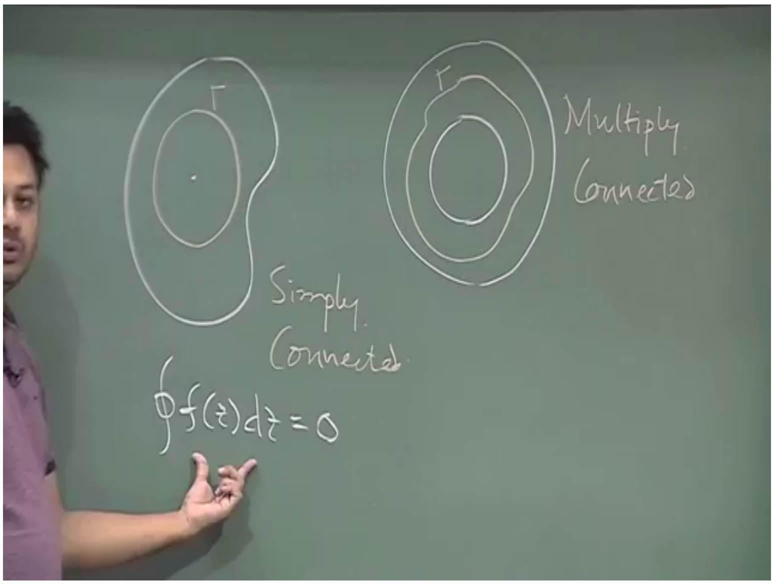

So, let us go back to our old class where we mention what is the meaning of multiply connected region this is the region which is multiply connected this is the annular region why because if I take any closed curve here, so let me remind you once again. If I write if I draw a closed curve this is say gamma is a closed curve over this region this region is a annular region then what happened that if I want to shrink this (Refer Time: 17:05) curve to make a point I want to shrink that. So, that this goes to a particular point then what happened that I cannot do that without going outside to the region. So, I need to go outside the region then only I can do that. If this is the case then this kind of region is called multiply connected region this is multiply connected.

And simply connected I have already shown. So, if this is a region and if this is my curve some curve closed curve say gamma if I want to shrink that it can readily come to this particular point without escaping the region. So, it will be inside the region and it will come to this point it collapse to this point this is a simply connected region simply connected. Now, for this simply connected I know this is true when f z is analytic in this region, but if f z is analytic in this region is this true or how to show that it is also true for this multiply connected region that we need to know. So, now, let me draw a region somewhere here say.

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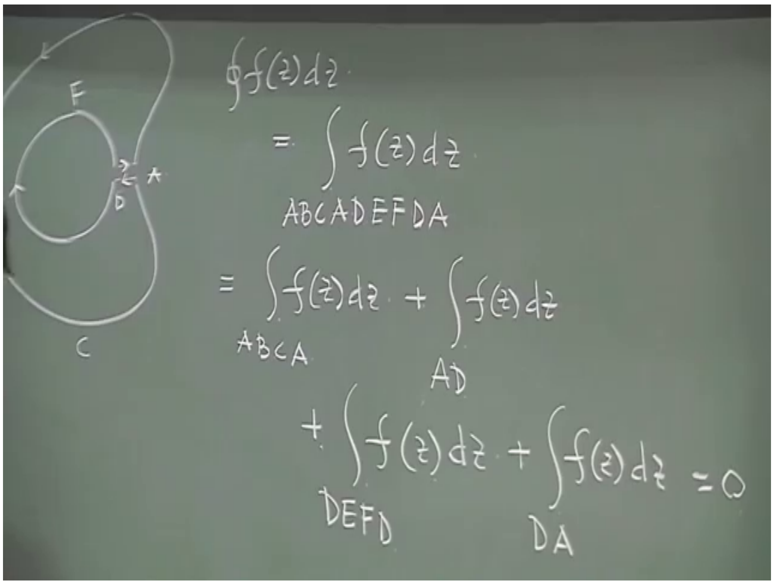

This is the region I am drawing this is the region I am drawing this point A B I am going to this B this is A C and I am going to this region because when I go outside this curve my region is in the left side when I go inside my region should be this side. So, the direction of this curve should be something like this. So, here it is D E F, this is the region I am talking about this is the multiply connected region and now I need to show that what is the form of a Cauchy-Goursat equation for this kind of multiply region systems.

Now, what we will do here that first I will make this region multiply connected region to simply connected region how we will do that. So, let me first. So, what is the concept of doing this? So, I want to replicate the same thing. So, suppose this is the curve I am just replicating this 1 here the same thing what I will do that I will erase this part and make a I will cut this and make a path like that. Please note carefully how the path is making this path I make a small cut here.

So, that I can have a path here now this entire thing is simply connected because if I want to have some region this region can shrink and to collapse to a point, so there is no multiply connected path is here only thing was this I make a passage here. So, that this portion is now empty and now I have a entire complete curve here which is simply connected and this is true the Cauchy-Goursat equation is correct for simply connected or it is already valid for simply connected region. So, I can make use of this concept here also and then try to find out how things going on. So that means, I am making a cut here in this figure if I do that then I make a passage here like this.

Now if I write the Cauchy-Goursat equation for this case the closed integral f z dz is equal to what value A B C, A B C I am going to this path then again I return back to A then go to D then go to E then F then again D and then A please mind it this portion is infinitely narrow. So, this D point and this point are almost same that is why I write it here D. So, it will come back to D and then go to A point this is a very very narrow path just I cut missing a line over single line, so that this become a simply connected region.

So, this is the integration that I have in my hand the closed path is now represented by A B A B C A D E F D A, this is the thing. Now I separate out because this closed path is made of many simple single line. So, I can separate it out this single lines. So, I can write that this is A B C A this is one path of this integration f z dz this path plus integration A B C D A then A D next path only this strip I am talking about this small strip, f z dz plus D E F, D E F D I will returned back to my d point again. And finally, I return back to my A point through the opposite direction it is in this direction I am returning back to the opposite direction please note it. So, plus integration of D A f z dz this quantity is 0 this quantity is 0. So, once I have this entire thing and separate out these things then I find one interesting thing that one integration is here having A D and one integration here having opposite direction D A this, this one, this one and this one the second integration and the fourth integration.

So, this second and fourth integration let me erase this are opposite to each other so; that means, integration of A D f z dz is equal to minus of integration D A f z dz its a negative one because it is for this integration it is coming A to D and another integration it is coming going D to A the opposite direction. So, when it is in opposite direction. So, I can write this as this; that means the summation over A D integration plus D A integration is 0; that means, in this total integration this part and this part cancel out this part and this part will going to cancel out. So, let me erase this part and erase this part and finally, I have 2 expression in my hand which suggest one is A B C D A and another is D E F D.

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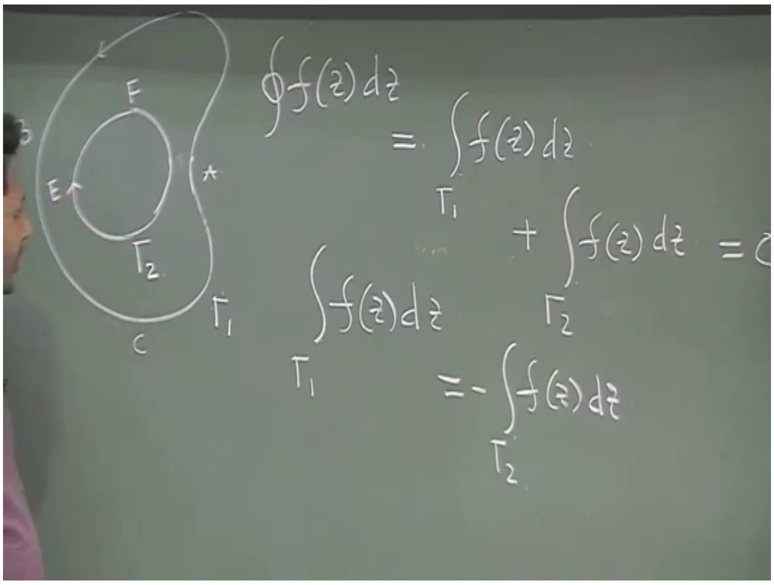

Now, there is no the joining is not required because I have already derived these 2 circles, that means, this is total f z dz is for this circle if I say this circle is gamma 1 and this circle is circle means this closed region gamma to assume this is a circle. So, this closed region I say gamma 1 this closes region I say gamma 2. So, this quantity is gamma 1 f z dz plus gamma 2 f z dz that proves that this closed integral is basically true for multiply connected region also this is the multiple connected region and for this region I can say this is the case which is equal to 0.

So, we prove that Cauchy-Goursat equation or Cauchy-Goursat theorem is true for multiply connected region also. Now the thing is that if it is the case then I can write this as gamma 1 f z dz is equal to minus of gamma 2 f z dz how the picture will look like here let me erase this. So, initially this was the picture. So, gamma 1 is in this direction gamma 2 is in the direction I am saying gamma 1 integration whatever the integration over gamma 1 and that is equal to the negative of gamma 2; that means, negative of these things. Now, if I reverse this sign if I want to reverse this sign this direction will going to be changed now it is the outer circle or outer region outer closed path is in the anticlockwise and if it is clockwise then this clockwise should be represented as anticlockwise by removing this sign.

So, let me draw that clearly then we will going to understand what I am trying to say.

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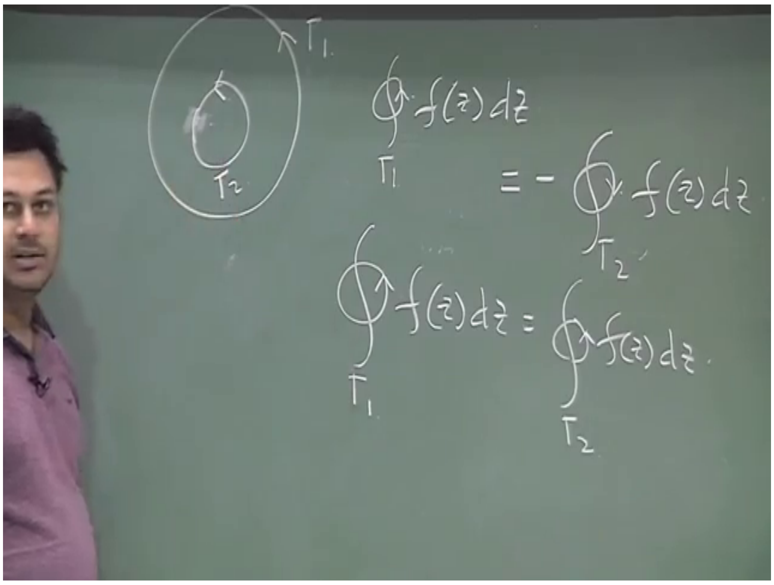

So, this is my one region and this is my another region this is gamma 1 in this direction and this is gamma 2 another region. So, integration gamma 1 f z dz, now I put a sign here for this close integral because this close integral now requires this direction is equal to minus of minus of this gamma 2 f z dz right.

I am writing the same thing in terms of this new sign which shows it is anticlockwise and clockwise respectively. Now if I remove this sign then what happened this sign will going to change. So, I just I wanted to remove this negative sign if I do that then readily I can have so; that means, I will have the same sign in my hand. So, this then I will simply have gamma 1 f z dz is equal to gamma 2 f z d z; that means, if I have a region here what the entire region if my function is analytic. So, whatever the integration I have for this closed region gamma 1 the same value the same value of the integration I should have if I put another circle inside that with gamma 2 notation, the direction is same. So, I will just replicate this things to this things gamma 1 is a bigger region, bigger closed curve gamma 2 is a smaller closed curve. So, the integration value of the integration will be same for both the cases

With this I should conclude my class here. So, today we learnt very important thing that is how the Cauchy-Goursat equation is true from multiply connected region also and you also prove the Cauchy-Goursat equation by using this (Refer Time: 32:19) and all this things. So, with that let me conclude the class here. So, see you in the next class where we learn more about going to use this Cauchy-Goursat equation to find out some integration of some integral form. So, with that, see you in the next class.

Thank you for your attention.