Mathematical Methods in Physics-I
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Lecture – 41
Complex Line Integration (Contd.)

So, welcome back student to our complex analysis course where we are in the process of learning line integration. In the last class we show some real function and how to do the line integration so line integration.

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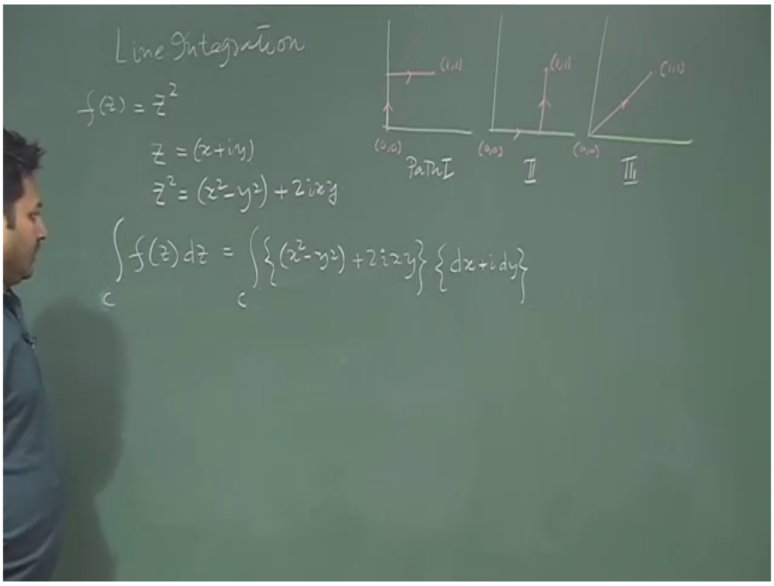

So, today we will start with function z square this is the function that is given. So, this is some sort of example I am doing here I need to evaluate this particular function z square for different 3 different path the problem is there are 3 different path is given. So, one path is I need to go this point 1,1; one path is from 0 to 0 I need to go one path is this second path this is 0 to 0 this is 1, 1; second path is this to this and third path is this. So, there are 3 different paths, that is given to you that I am going from this point to this point and this point to this point this is 1 line or path this point to this point and this point to this point another path and directly from this to this. So, I need to go 0,0; to 1,1; 0 0 to 1,1; and 0, 0 to 1,1; both the cases I am going to the same point to same point, but the approach is different.

So, let us do the calculation this is path one; this is 2 and this is 3. So, z is x plus i y and z square is x square minus y square plus 2 i x y this is the 2 thing I have. So now for path one; so the function f z d z is simply be c this quantity x square minus y square plus 2 i x y multiplied by delta z which is d x plus i d y this is the complete form of the integration I need to evaluate these things for these 3 path 1 2 and 3 this is path one.

So, now let us start with path one; so path one if I calculate here, so let me erase this path and calculate here. So, the integration 1 integration 1 means I am doing the path one. So, x square minus y square plus 2 x y here I am going to from this path to this portion to this portion. So, when I am going to this portion to this portion there is no change in x. So, let me define x y and here this is x, this is y, this is y, this is x, this is x, this is y. So, I am going from here to here, so there is no change in x so only change in y. So x is d x is 0 and also x is 0; because this point x is fixed and the fixed value is 0.

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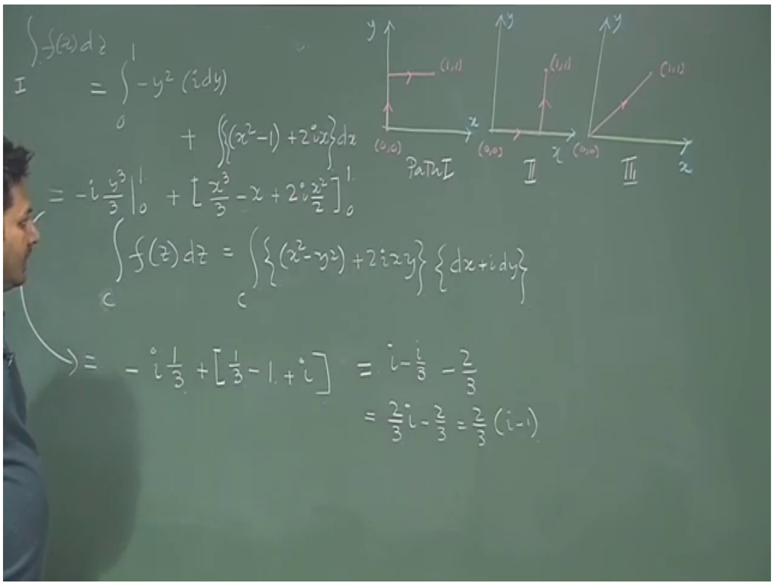

So the first path, so f z d z simply comes out to be this quantity this quantity what I put this is 0, this is minus y square, minus y square changing x is 0; along this path so this quantity shouldnt be there d x is also 0; because there is no change. So, only thing I have is I d y, the limit from here to here is 0 to 1. Next, I will go from here to here, so I just divide this entire paths c into two paths c1 and c2 that is all. So, entire path c if I called this is the path c then I divide this in to 2 path.

The next path what happened, in the next path my y is fixed and this fixed value is 1, but d y is 0 and it is going to this direction, so that means my x is changing y is not changing, but I am having the value 1, so I directly put this 1, here it is plus 2 i x is changing, but y is not changing so y is 1 and d y is 0; because there is no change in y here. So, I should put d x here

So now I need to evaluate these things. So what should be the value here, it seems to be minus of i y cube divided by 3 0 to 1, this is the first value first integral value plus this quantity. If I integrate it will be x cube by 3 minus x and then I have plus 2 i x square by 2 the limit is 0 to 1 fine.

So, let us go back here I need this so I am not going to erase that, so this quantity if I put the limit it will be minus of i 1 by 3; this quantity if I put plus 1 by 3 minus 1 plus i, it seems to be something like this. So, let us check what value I am getting so before that I need to check once again everything is correct or not. In path one, I am going to this way this is my the value of z square is this x plus y whole square. So, x square minus y square and 2 x y and d z is d x plus i d y it is correct. Now I am going to path 1 in path 1 if I go from here to here 0, 0 to just 0, 1; so my x is not changing x is 0. So minus y square and then this is 0; minus y square d y i d y; so this is so this path is fine.

So, let me first find out what is the value then I will check things are correct or not. So, it seems to be i minus i by 3 minus 2 by 3; so it is 2 by 3 i minus 2 by 3. So, it is 2 by 3 i minus 1 it sums to is comes to be something like 2 by 3 minus i with path one.

So, let me write down the value I will do that for other path. So, let me write down the value here, so for this path the value of integration I write it here it is 2 by 3 i minus 1 for path one; I will do the same thing for path two and so on. So, let me erase these things and put it somewhere here. So, that would be easier.

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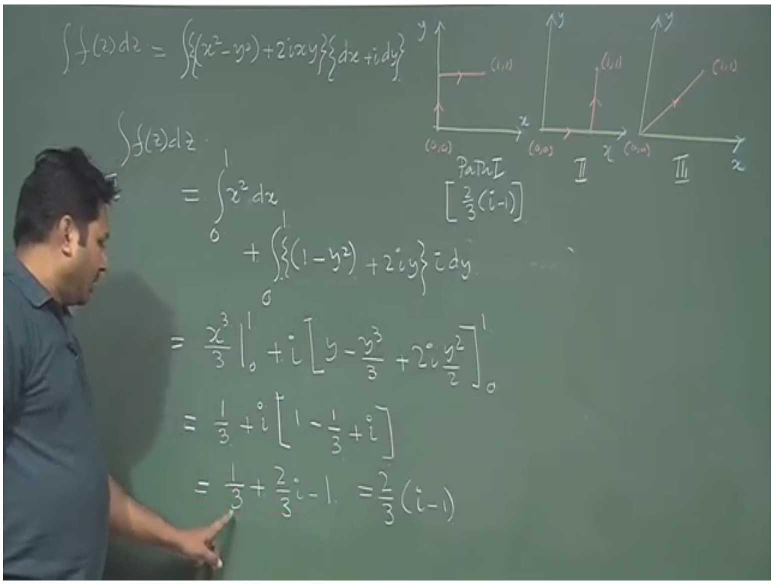

So, my f z d z integration is comes out to be x square minus y square plus 2 i x y d x plus i d y this is the integration total integration I need to evaluate. So, path one I have already evaluated it is 2 by 3 this.

Now, what about the path two so integration 2 which is path 2, So in path two; that means, this path what happened initially the change over x is there, but there is no change in y so that means, y is 0 and x is changing. So integration x square minus y is 0. So, I should not bother about y; this quantity is also been not be there d y is not there so it will be just d x this is the first path, the first segment of line is given like this and the rest path is plus the limit is 0 to 1, because x is changing from 0 to 1 here.

What about the next path next path is integration y is changing so limit 0 to y. The value of x is now 1 so 1 minus y square plus 2 i x is 1 y is changing. So, y d y is not changing d x is not changing d y is changing so it is i d y. If I evaluate the first path it is x cube divided by 3, 0 to 1; the first integration. The next path term by term, so I take I take it outside so it is y minus, y cube by 3; integration this quantity is plus 2 i y square by 2 with the limit 0 to 1. The first term comes out to be 1 by 3 plus i y is 1 minus 1 by 3 plus this 2 is cancel out so plus i.

This quantity if I evaluate it is 1 by 3 plus 2 by 3 i minus 1 which is equal to 2 by 3 i minus 1, 1 by 3 minus 1 gives you minus 2 by 3, minus 2 by 3 so on this quantity.

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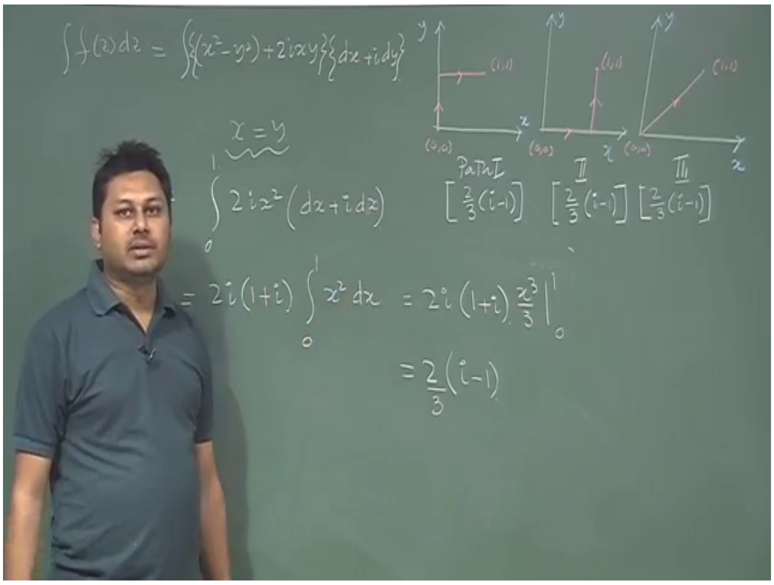

So, path two also I am having the same value 2 by 3 i minus 1. I evaluate the integration over path 1; I am getting this value I am evaluating the integration for this path; I am getting this value the last path is that so let me do that also.

So, in the last path if you see carefully then you will find that it is coming from 0 to it is going from 0 to 0 to 1,1; point and the relationship between x and y is simply x is equal to y. So, there is a line over which x is equal to y is satisfied I am going to that line only where m is equal to 1 rather y equal to m x plus c here c 0. So, y equal to m x m is equal to 1 because it is going 0, 0 to 1, 1, so slope is 45 degree. So, that the relationship between x and y is x equal to y, so it just I replace these things and then things will be easier. When I replace that, the first term will going to vanish because x is equal to y this term will not be there. So the integration is simplified as 2 i x square because y is now x d x and these things so it will be d x plus i d x x is changing from 0 to 1, it will be something like this.

Now, it is 2 i multiplied by 1 plus i and then I integrate this 0 to 1; x square d x if I take d x common there is it will be 1 plus i so 1 plus i and 2 i I will take outside the integration is over x square 0 to 1 so it will be something like this. So it will be 2 i 1 plus i and x square 0 to 1; it will be x cube by 3 0 to 1. If I put that it is 2 I now multiplied these 2 inside so it will be i minus 1 and if I put the limit it will be 1 by 3 so it will be 2 divided by 3 this quantity. Again, I am having the same value so my final of this path I will also have 2 by 3 i minus 1, I will have these value 2 by 3 i minus 1.

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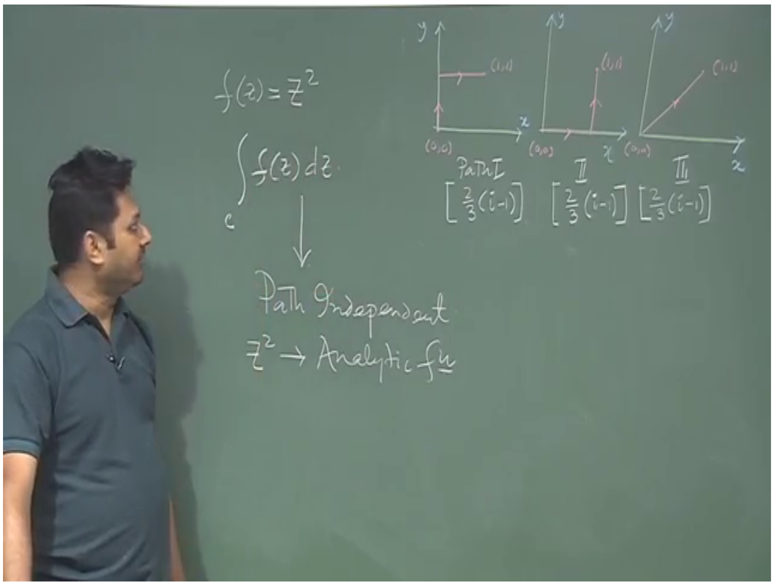

So, here one interesting thing very interesting thing after doing all this calculation lengthy calculation not very much lengthy, but very straight forward calculation, I find that when the function is given like this when I integrate this quantity the result, whatever the result I have is path independent that means, whatever the path I choose at least for this particular form of the function there is no such change if all the values are merging to is same value and it only depends on the final and initial value.

So, it that is why it is path independent whatever the path you like you can go, but the end result is same. Now why is that? The reason is that here, I will again show one example that will be more easier to understand the reason is z square here is an analytic function. So, z square since this is an analytic function this analytic function integration of an analytic function over a path if not, depends on the path at all so it depends only on the initial and final value. So, since z square is an analytic function we have this property, but if I took a different function then things may not be the same.

So, let us try to do that also, I will take now let me try with this things to another function this is path 1 I am doing the same problem for another.

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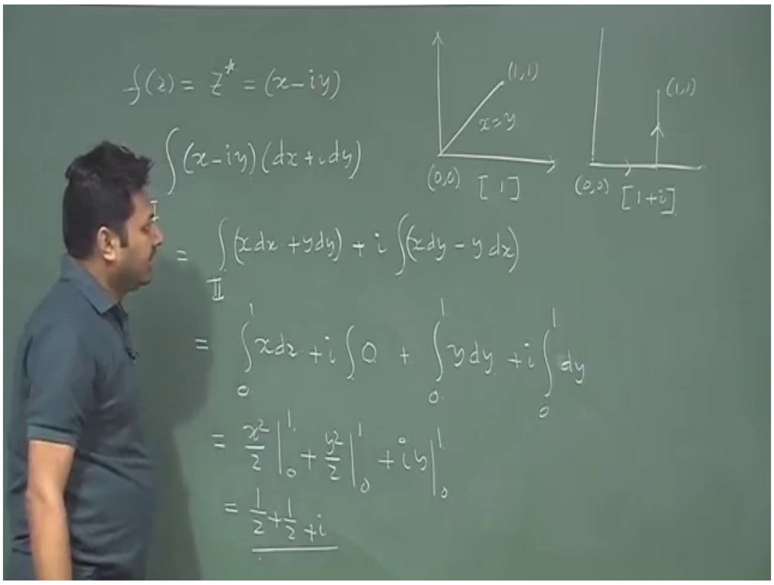

So, let us do that f z is f z I will let me change the path. So, I am not going to do the entire 3 path, but two different path. So 0,0; 1,1; I am going to this path 0 ,0; 1,1; I am going to this path just to where, I am going to directly from here to here and another I am going to this just to check this is only for checking. So, my function is z equal to z star f z is equal to z star. So, we know that z star is not an analytic function. So, that is why if I try to find out the path integration it is likely that the 2 values may not be the same. So, we do it we try to do it quickly so that means, it is x minus i y and the integration over path 1 is x minus I y d x plus i d y this is the integration I have.

Let me simplify this integration path 1 so it is x d x plus y d y plus I integration of x d y minus y d x this I separate out and then I put so the first case the first path is like the previous problem the relationship between x and y is just x equal to y. So here x equal to y is the relation, when I put this relation here then I will readily find the integration if I put x equal to y so 2 of x d x because y will be x d x will be d y and then if I put it will be something like this and my limit is 0 to 1.

However, when I put x equal to y here and d x equal to d y, here this quantity seems to be vanished because this minus this seems to be 0; so I will have 0. So, here what value I am getting 2 x square by 2; 0 to 1, which seems to be 1 ok 1 value I am getting which seems to be 1. So I will have the integration the value of these things is 1.

What about the other path for path 2, what we are getting for path two in the first path from here to here I dont have any change in y I just only change in x so and y is 0; so this quantity going to vanish. So I will have x d x plus i this quantity d y is 0 and y is 0. So, I will have something like 0 here, x is a changing from 0 to 0, but d y is something which is not changing so that is why I put d y is this is 0; here y is 0 so I need to put it and then, so the limit is this is the first path the limit is 0 to 1.

what about the second path the second path I mean second portion of this path this is second portion of this path. So, what happened so x is now fixed to 1 value so d x is now 0 y is changing. So, integration of y d y with the limit 0 to 1 plus i x is 1 and d y is changing. So, I should have this quantity x is 1 and d y is changing d y, but here d x is not changing so this quantity is not be there so this is the entire thing I have.

So, here I have something x square by 2; 0 to 1 this is 0; plus y square by 2 0 to 1; plus something here which is interesting plus i y 0 to 1. Let me check once again, the first path y 0, 0; 1, 1; there is no change in no change in d y so d y is 0 and only the x is changing so that is why I have these things here d y is 0 and y is also 0 so this quantity is also should not have any contribution here.

In the next part of these things, this portion have x d x so x is not changing so this will not be there. So, only we have y d y these things and for this part x is not changing it is a fixed value that is 1 and d y is changing so I should have this contribution, but here d x is not d x is 0; because x is not changing so this contribution not be there. So, I will have integration of i d y that I have.

So, after doing all these things I find that this is half plus half and this quantity is plus i, so as a result I will have something 1 plus i so that means, if I go from here to here and if I go from here to here both the cases I will have the integration, but this integration have quite different values. The reason behind that as I mentioned this is a very simple checking that whether the function is analytic or not.

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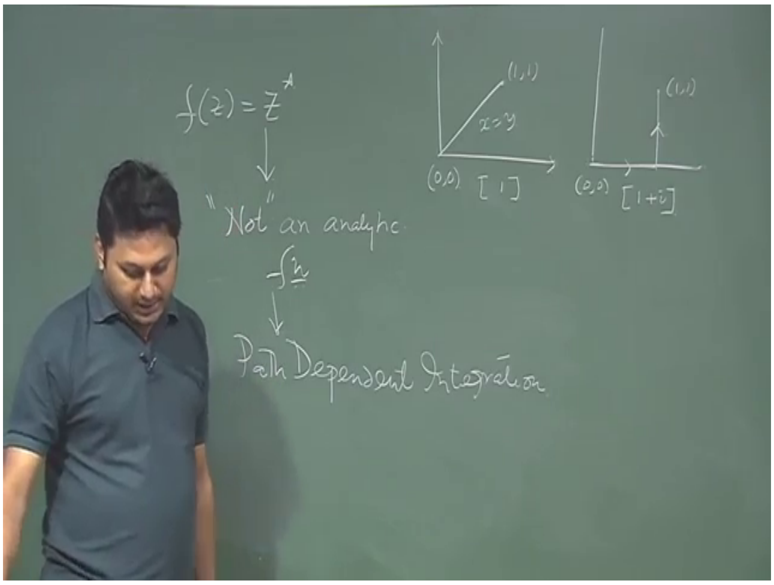

So, here my function f z which was z star was not an analytic function. Since, this is not an analytic function the integration is not depends it depends on path, so it is a path dependent integration. So, for this case we have a path dependent integration and this path dependent integration is coming with the fact that if the function is analytic.

So, today we will learn a very important thing the first thing is that if the given function is analytic and if I want to integrate over some given contour. If the function is analytic, I should not bother about the contour the line rather the initial and final value is important. Because the integration is entirely depend on initial and final value, if I go from one point to another point to any path I like. Let me show that.

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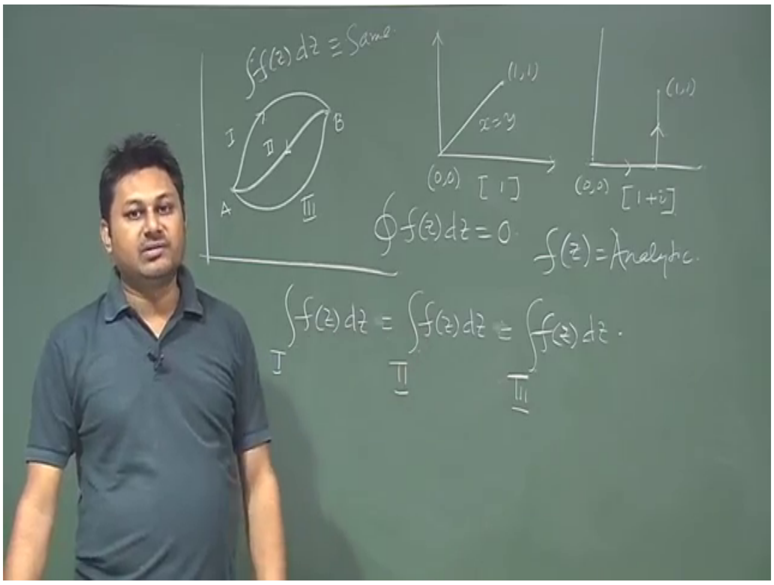

From this point to this point, if I go with these with these or with these, since this point is fixed integration of f z d z will be same for all the path so if it is path one, if it is path two, if it is path three, then integration f z d z is equal to integration f z d z is equal to for all the cases it is same.

And with that we can have a very interesting thing that we will show in the next class that, if I go from this point say point A to point B with this direction and return back to the opposite direction. So, what happened path one is equal to minus of path two and if I add 2 then I have a closed path. So, A to B and B to A, so I will have a closed path and if I integrate that closed path I will always have 0, so that is the very important outcome that is because I am going from this to this and from here to here.

Since, it is a path independent if my initial and final point is same for analytic function; obviously, here f z is analytic, I will always have this relationship it is a very very important relationship and in the next class we will explore more about this relationship and try to find out more interesting thing regarding this relationship which is generally called Cauchy's Cauchy- Goursat Theorem. This is one theorem which is called Cauchy-Goursat Theorem. We will go in the next class we will try to understand in detail and try to prove that in a different way that how these things is working in and all and how this can be useful to calculate some integration, so with this now let us conclude this class.

So, in the next class we will start from here the path integral is a important thing as a most important thing in the complex analysis so we just started the path integral. So, we will continue this concept in the next few classes and try to understand using this the relationship this beautiful relationship the close integral of an analytic function is 0. How we can evaluate different form of integration and how do I know I mean the different integral forms. So, with these let us conclude this class.

Thank you for your attention. So, see you in the next class.