Mathematical Methods in Physics-I

Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 40
Complex Line Integration, Contour, Regions

Welcome back to the complex analysis course. So, in the previous few classes we understand interesting aspects of the analytic function.

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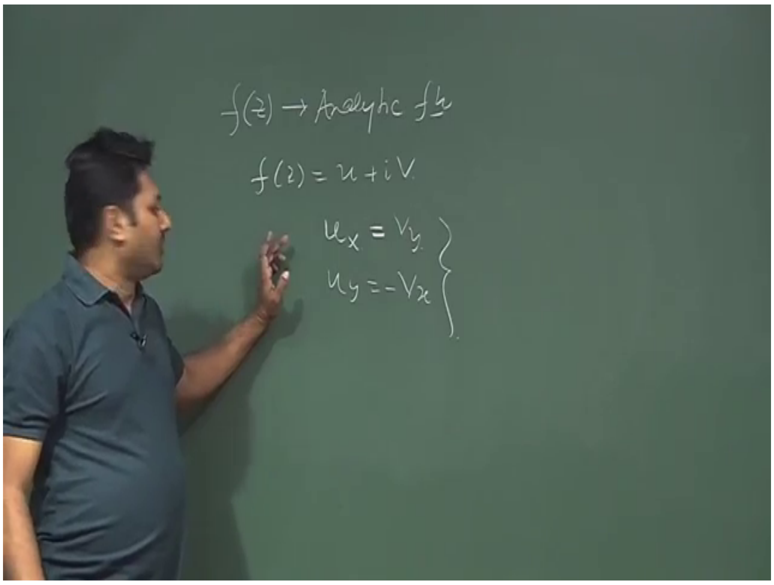

So, fz is an analytic function and fz is u plus i V, if it is analytic then u x some is equal to vy and uy is equal to minus of Vx this is the essential condition. So, that the function is analytic.

Today we will since the analytic function is now, is well defined we will go to the next level in complex and which is important and that is complex integration.

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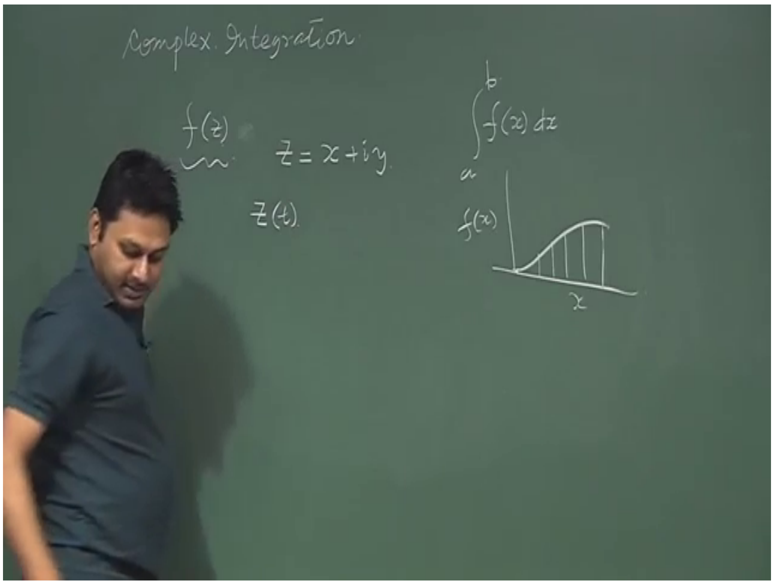

So, normally what happened in normal integration that if we have a function fz, fx in real case. So, x is changing like this and this is my fx. So, I will just calculate the area under this curve because it is changing and every time I will calculate this quantity and as a result whatever I am getting some limit say a to b, I am having some value here.

In a complex case also the things are similar, fz is equal to fz is my complex function and z is x plus iy. Now here x is a parameter single parameter that is changing from one value to another value and as a result I am getting this integration in this form.

On the other hand here z can be changed like the variable x is changing, but there are 2 variables are associated with that. So, 2 variable should change independently and as a result z will going to change and because of that fz is going to change and then we will have something in the integration form.

So, we can do this in the parametric way for example, t is called the parameter x and y both are changing with the parameter t.

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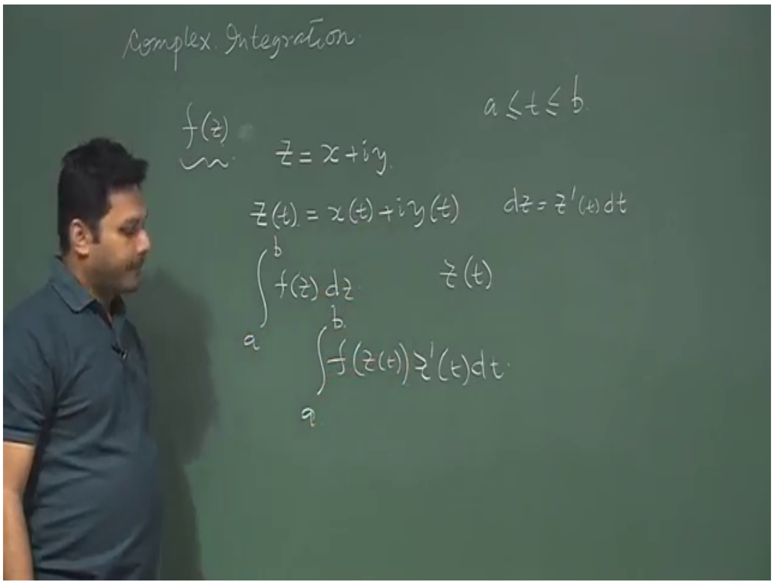

So, t is some parameters say it is this value, then if I write this quantity; that means, I am having the integration with t because t is my parameter here t is changing, here z is function of t, x and y is also function of t. So, delta z, since z is the function of t. So, delta z can be represented as z prime t delta t. So, z prime t delta t if the z as a explicit form of t is known then I can find out this quantity. So, this quantity is essentially then fz function of t, if x and y is given as a function of t explicitly then I can write this and replace this, then my integration be only depend on t, x and x if given as a explicit function of t then I can do this way.

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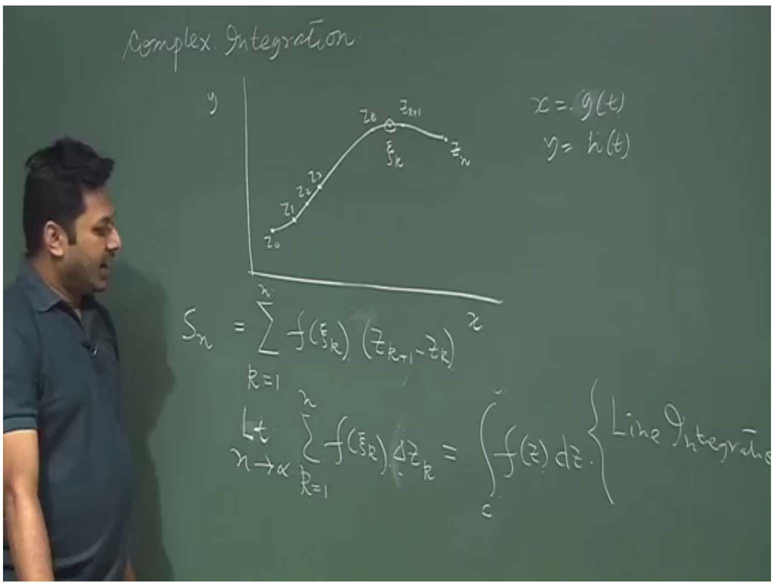

Anyway another aspects of this integration is in the complex plane for example, for every time the segment wise the way we just do for the real case. So, the z is something changing like this and there are small segments say z0 to zn there are n different segments and it is changing like this, for this is x and y the for different x and y value I will have different z value. So, it is now x and y change independently if it depends on some parameter t. So, x t will be x say x is a function of is a function of t and y is a function of t, explicitly then there is should be relationship between x and x and based on that we can draw this curve that is the thing, but even though they are changing independently. So, they are giving some value z 1, z 2, z 3 they are small segments of value for the small increment of x and y together and I will have 1 individual value like this which is forming a curve here say z k here z k plus 1 and so on z n.

Now, in between z k and k plus 1 I assume one value say 5k and this 5k, for this 5k I will have the functional value k, I just construct the formation of the integration. So, integration is just I am just taking this 5k every time multiplied by z k plus 1 minus z k with the summation k, 1 to n and I call this summation as s n. So, this is the form I am going to use. So, I am just changing these value to these value then in between this value to this value I have one value say psi k and this psi k if the function is giving to me I put this function psi k then multiply we will have something and then every time for every point I can have the same thing and if I sum over k then I will have something s n. Now if this points are too close to each other then I can write this things as, when the limit n tends to infinity; that means, I will have the segment infinite number of segments. So, this distance is infinitely small.

Then this quantity I write it now delta z k, now this quantity converted to a integration with the limits say some value initial value to final value say a to b like this. So, this is over some line because I am taking the integration like this. So, instead of putting initial to final I should write it is over some line c like this. If the line is different; obviously, this values going to change and I will have something different this is called the line integration by the way, I will have a line in complex plane and I try to integrate my function over this line and when I try to integrate that I will have this kind of form.

After having this now let me give you some idea about this kind of integration, but before that I need to cover one important thing also.

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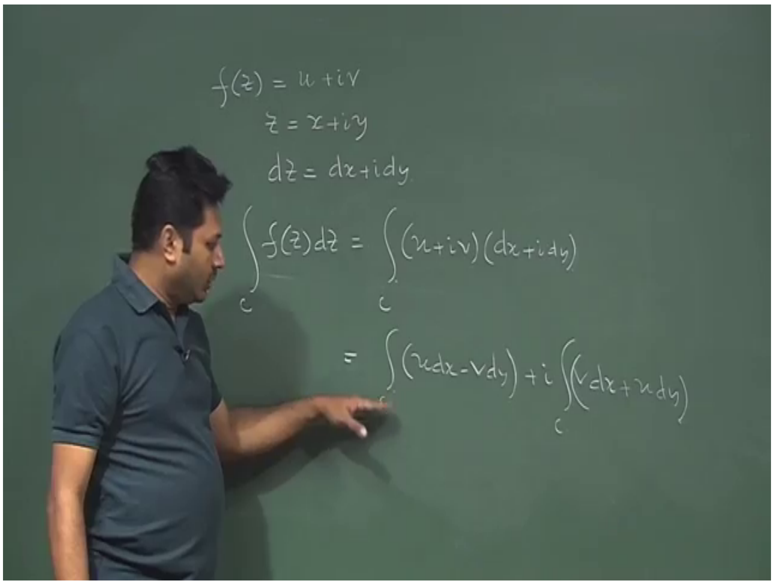

So, fz in another form I mean how this line integral, I mean how I do this calculation fz is u plus i v and z is x plus i y positively state forward. So, delta z is d x plus i d y. So, integration, whatever the integration is given to me in the form fz d z it will be this line integration fz is u plus i v and it is d x plus i d y I will have u d x minus v d y plus i integration of v d x plus u d y. If the functional is given to me I know what is my u what is my v then I need to put that and then based on the if the curve is given to me I know how these d x d y will going to change and then I will evaluate the entire integration using this, this is the procedure or the recipe to calculating the integration apart from any other any other I mean theoretical detail.

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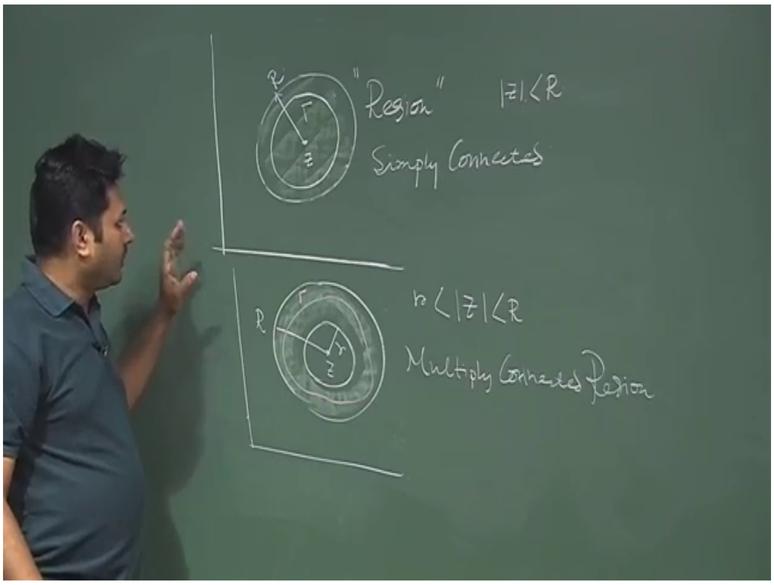

So, this is the way we will do the all the calculation that is why I show.

But now one important thing I need to say about. So, this is my complex this is the plane I am talking about I am saying that this is my line. So, this is the line over which I am doing the calculation or integration whatever.

Now, in some cases we will need to close this line. So, I will just integrate over some close line, like this it is possible because this is a line I go and then returned back. So, it is a close something close, now this form a region. So, this line whatever the line which is closed are encircling some region this is the region, is encircling this is called the region. Now this region is connected by a single line this is called this region which is connected by a single closed loop or something like that is called simply connected. I will come back to these thing in more detail, but since we are dealing with the integration line and region it is better to just introduce how these things are working. So, it is a simply connected region, why it is simply connected? Because if I want to shrink. So, this is the region given I want to calculate the integration and this is say my closed loop inside the region.

If I want to shrink this region, shrink this curve I will have a point here like robot it will shrink and it will come to a particular point without leaving this region, if that is the case then this is a simply connected region, but it is not always the true I will show you another example; that means, here my z is if this is some point z. So, mod of z for example, is less than some value big value r if this is the radius r. So, this is the reason this kind of region where I can have a closed loop and if I shrink this closed loop inside the region and come back to a particular point without leaving the region then this is a simply connected region, but as I mention this is not always true. So, let me draw another region enclosed by 2 close curve like this.

So, now my region is this one, this is the annular region and you are aware of this angular region because in the previous class we show and the say z is I mean here z point is here. So, here if I say this is my z point. So, z is less than r greater than small r. So, if this is my small r if this is my big r. So, this is the region I am talking about mathematically which is given like this.

And now if I draw one like this, this is the closed curve I am talking about say big gamma. So, this closed this region is not a simply connected region because if I have a closed loop inside the region and if I want to shrink that it will not going to shrink a particular point without leaving the region. So, this is called the multiply connected region, we will use that in the future class, but today just I will just show you what is the difference between simply connected and multiple connected region and the idea of that multiply connected region ok.

So, far we have learnt something about the line integral; however, the line integral calculation we did not do, we will do shortly and also know that if the line is closed then I will have a closed kind of loop and there also I can integrate and this loop can be associated with certain region in some region it is multiply connected in some region is simply connected and based on that I mean I define that which kind of integration I am talking about it is a multiply connected region or the simply connected region integration.

So, we will do this things later and then it will be helpful for you this brief very brief introduction will be helpful for you about the region and all this thing. So, now let us directly go to our original, the problem I started with that is the line integration. So, let us take one example.

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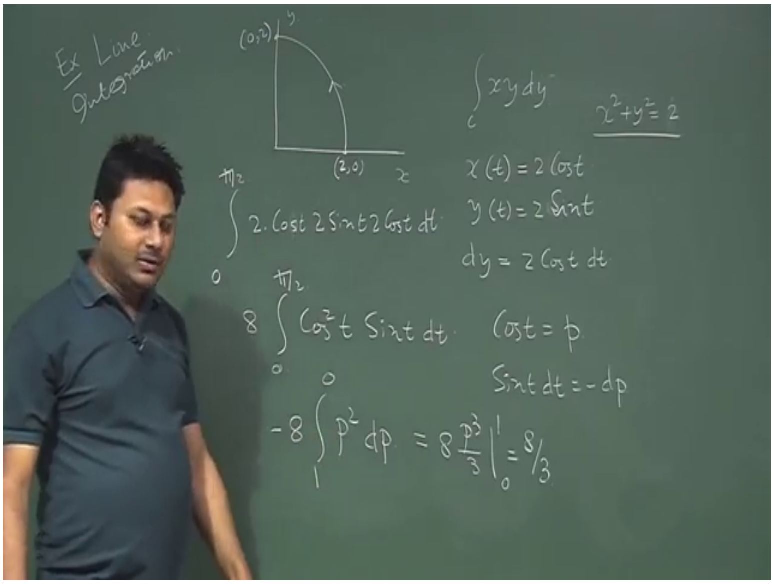

So, let me give you one example about the line integral thus, first example one line integration, how to do that that is the thing. So, for the line integration first thing, that a contour should be given to you. So, it is a real integration for first I start with the real integration and the integration is giving to you something like this xy, dy suppose xy ,dy this is xy my function and I want to find out this is the region that is why two variable are associated with that exactly like the fz and here I want to integrate say from point 2 0 to 0 2 over this line this is my x this is my y over this line.

So, the thing is that here when I do this calculation if you do try to do this calculation you need to know every time how they are related, how they are related you need to know what is the relationship between x and y because you need to put these things to a single variable that I show just in the previous I mean derivation. So, we need to use some kind of parametric variable to understand how this is changing. So, here the parametric variation is x say function of t, t is my parameter is such that 2 cos t and y t is 2 sin t as I mentioned my x and y should be the explicit function of t if t is my parameter here I changed here I wrote these 2 things in this way these 2 variable to make it a single variable.

So, now you can see that when this point which is 2 0 is essentially means x is equal to 2, y is equal to 0, x is equal to 2 means my t here is something like x is equal to 2, let me change it to sin t and cos t you can do that with cos t and sin t no problem with that. But your limit will go to, no it was initially it was I think it was correct because when my theta is 0 yeah cos t sin t its fine I mean when my theta is 0; that means, t is 0 t is behaving like a theta here when t is 0 then x is 2 and sin y is 0 when it is 90 degree if t is behave like a theta. If it is 90 degree then what happened x will be 0 and y will be 2 and also this is us this is a part of the circle. So, the relationship between x and y is already we have is equal to the radius here is 2. So, it is something like this if you calculate from this parametric variation then you will get back to that thing also.

So what is my d y then 2 cos t, d t if I make the d y then it will be something like this. So, what is my integration then, now this is defined because I am changing one variable and now t is changing from 0 to pi by 2 if I want to write in form of t, x y is 2 into cos t 2 into sin t and d y is 2 into cos t d t. So, this there are many two's are here. So, 8 integration 0 to pi by 2 I will have cosec square t into sin t d t this 1.

Now, I need to change say I put cos of t as another variable p. So, sin of t d t is minus of d p if I take a derivative of cos I will have minus sin of d t. So, I will put as a minus dp. So, if I change it will be cosec square means p square sin square d t is with negative sign dp again the limit will going to change because now I am doing that in terms of p. So, the limit when t is equal to 0, t is equal to 1 and when t is equal to pi by 2 p is equal to 0, this negative sign now going to absorb. So, if I do directly it will be 8 p cube divided by 3 0 to 1 which is seems to be 8 by 3. I will have this value something like 8 by 3 now if the path is changed I am not going to go from here to here in this way, but in different way. So, the value of this quantity will be same the standard answer says may not be the same thing, let me do that.

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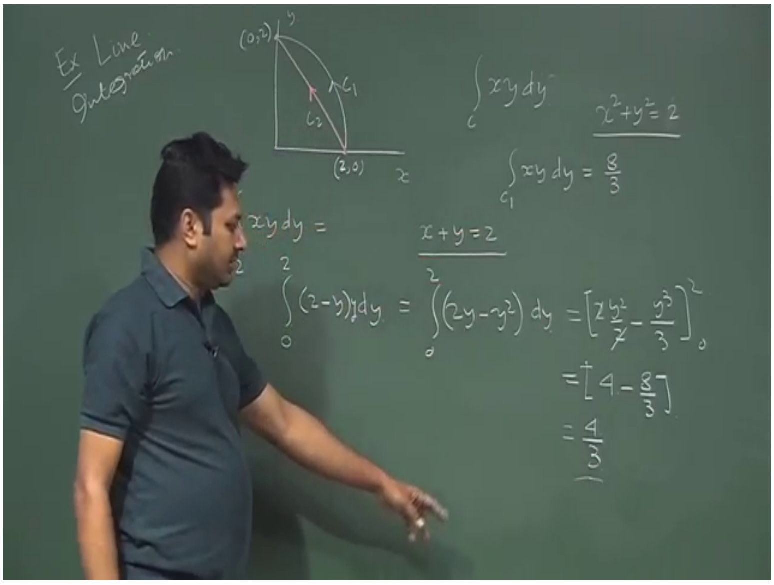

So, now what I will do I will go from this point to this point in this path I will change the path, if I change this path from this point to this point then this integration c is now changed. So, in first case if this is my c 1 path, this is c 2 path I calculate that c 1 xy dy comes out to be 8 by 3 when the path is define like this. Now I am doing this problem xy dy now try to find out what should be the value of this things since you have already done this things I believe it is it is set for to you to find out this. So, what is the relationship between x and y. So, x plus y is 2 this is the relation here for this red line, every time if I change x y is changing and the relationship between x and y should be 2 when x is equal to 0 y is equal to 2, when y is equal to 0 x is equal to 2. So, this equation gives you the same thing.

So, now what I will do now things become simpler I will just change this 2 here I just replace x in terms of y, also now since it is a single variable now and I know what is the path now I can put my limit here. So, what is the limit of y I am going from here to here. So, limit is y 2 to 2, 0 to 2. So, now, this calculation, if I do the calculation 1 y multiplication is there. So, I need to multiply that also. So, it will be 0 to 2, 2 of y minus y square d y which is 2 y square by 2 minus, y cube by 3 with the limit 0 to 2, this 2 2 cancel out. So, I will have something 4 minus 8 by 3. So, it is a 3 into 4, 12 minus 8. So, 4 by 3 so for c 2 the path c 2 this value is now different. So, there is it. So, for c2 c1 this value is this and for c 1 path this value is seems to be 4 by 3.

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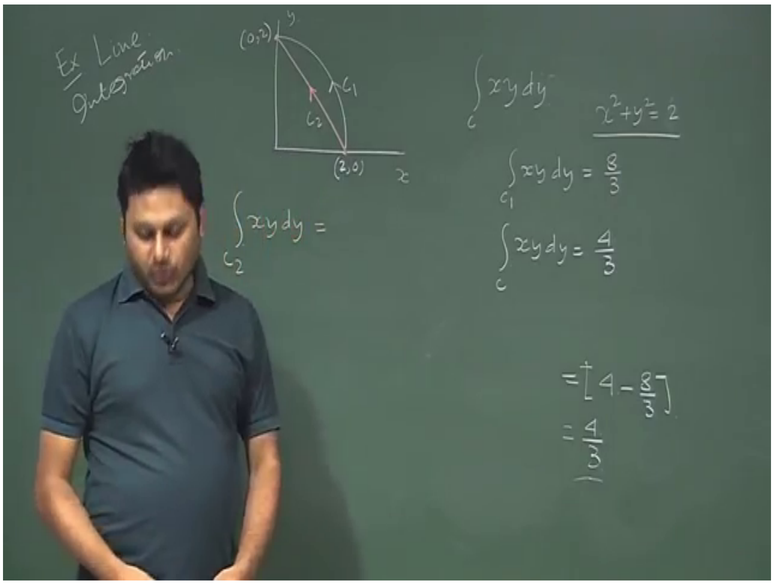

So, today I would like to conclude here because in the next class is the very simple example how one should do the line integration in the real case, but imaginary calculation will be equally same. It will be exactly the same way we have done here only thing is that one I will be somewhere in the expression only that is the thing and the variable that the way it is vary x and y will be same because they are also we will do the same thing x and y it will be varying in this way.

So, next day we will try to find out that here one interesting thing I find that for this particular function in real case if I go from this path to this path and calculate the line integration and if I go this path to this path and calculation the line integration the result are not same. Even though I am reaching the same point starting from the same point to reaching the same point but in complex analysis we will have a very beautiful thing regarding that.

If the function is analytic then we will find that it is irrespective of the path we were going, if we try to calculate the integration line integration of an analytic function we will show that it will be independent of the path that is a very very interesting outcome that we will have for analytic function specially and we also try to prove that it is it will be independent of the path rather depends on only the initial and final points. So, with that note let us conclude that. So, see you in the next class where we again discuss this line integration, but using some kind of complex variable with that.

Thank you very much for your attention.