Mathematical Methods in Physics-I
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Lecture - 04
Linearly Dependent and Independent Vector, Basis

Welcome back student. Last class we started some very important concept called span and the span is related to something called linear combination.

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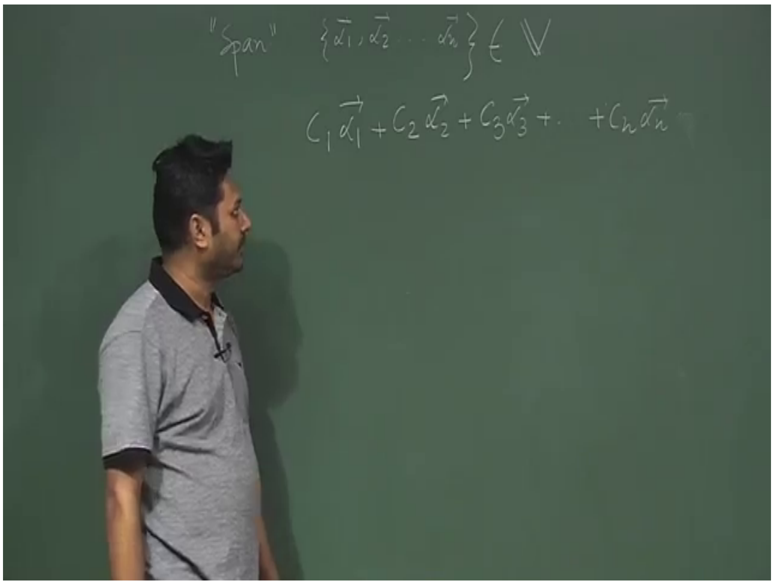

So, suppose alpha 1, alpha 2, alpha n these are the set of vectors belongs to some vector space V, then C 1 alpha 1 plus C 2 alpha 2 plus C 3 alpha 3. So, on plus C n alpha n is equal. This quantity is called the linear combination or whatever the vector is giving to me and C1 C 2 C 3 essentially the scalar quantity in real vectors space, they are real numbers.

So, next we show that I can choose a set of vectors which can span.

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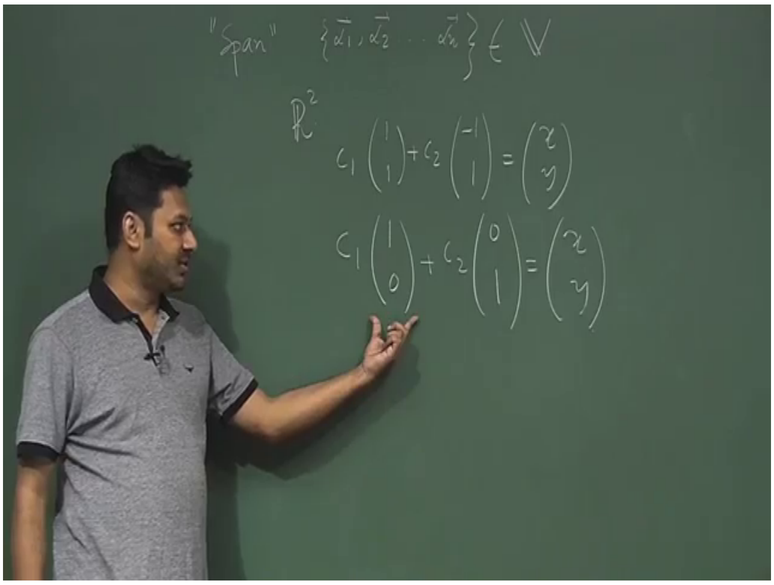

For example in R 2, I can choose a vector, some set of vector 1 2 and say minus 1 1 and if I make a linear combination of these two, I can generate any vector I like and we call that if I generate any vector, then we call that this span, this set span the inter vector space. Also we show that there is a choice that if I choose these things, life will be more easier because in that case I do not need to write what should be the value of C 1 C 2 in terms of x y readily we can find that C 1 is x 1, C 2 is x y. So, these things will be much simpler and we call this is a natural choice and we call this as a natural basis.

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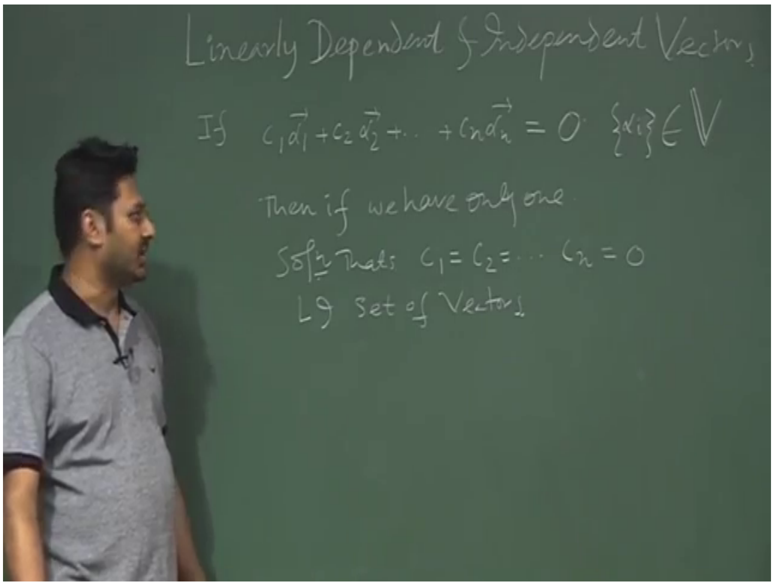

This basis things is still we need to, we do not know and we need to understand before that we need to understand another important thing which I mentioned last day which is linearly dependent and independent vectors. So, linearly dependent and independent vector I have already mentioned in last class. So, by definition if we have a linear combination C 1 alpha 1 plus C 2 alpha 2 plus C n alpha n is equal to say 0, first thing that I will have a linear combination of the vectors and I say that this linear combination is 0. If it is 0, then if we have only one solution that is C 1 is equal to C 2 is C n is equal to 0, then we call this set of vector alpha 1 alpha 2 alpha n is linearly independent vector.

So, the condition is what I have a linear combination of the vectors alpha 1 alpha 2, I choose some alpha 1 alpha 2 alpha 3 alpha n from a vector space V say alpha I belongs to the vector space V and then, I say that I make a linear combination over that all possible linear combination. I say this is a span, but I am making a special linear combination of these two vectors, so that my right hand side is 0. If this linear combination gives me the only solution, it is possible if and only if all C 1s are 0, then essentially we call these set of vectors as linearly independent set of vectors. Now, if any of the C 1 is not equal to 0, then we can say that these vectors are dependent, not linearly independent, but dependent to each other.

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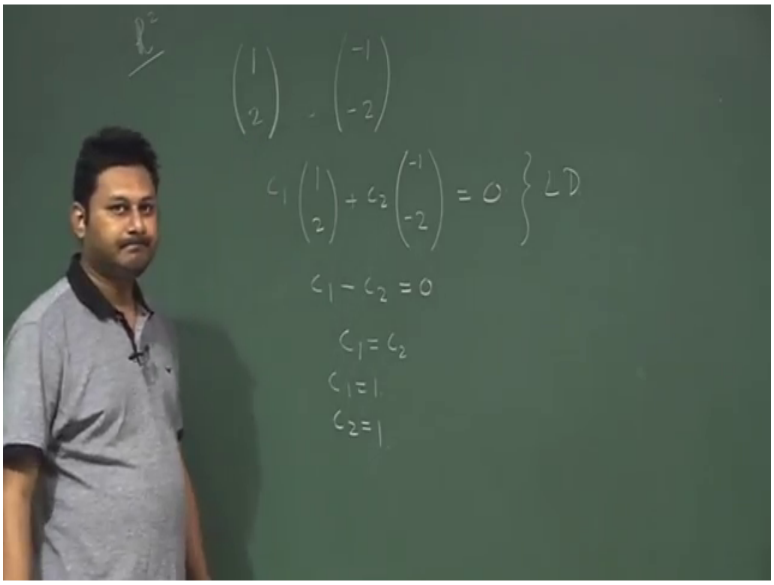

I can give you very simple examples, then you can understand in R 2, I have two vectors say 1 2 and another vector say minus 1 minus 2. These two are distinct vectors. They are linearly independent or dependent. So, C 1 2 make a linear combination C 2 is 0. Now, if I do that, then I will have one relationship C 1 minus C 2 is equal to 0, a relationship I will get from this side. That means, if I put C 1, then equal to C 2. So, I have a relationship between C 1 and C 2. So, if I put C 1 and C 2, both 1 or both 2 or any value, then I will have this equation valid for non zero C 1 and C 2.

The most simple is let us put C 1 is equal to 1 and C 2 is equal to 1. That means, if I just add these two vectors simply, then I will get 0. So, that means I will have a relationship C 1 multiplied by vector plus C 2 multiplied by vector is equal to 0 and I will have a solution for C 1 and C 2 which is not equal to 0. That means, these two vectors are linearly dependent. So, that means these vectors are linearly dependent vector. What about these two vectors? I just changed this vector.

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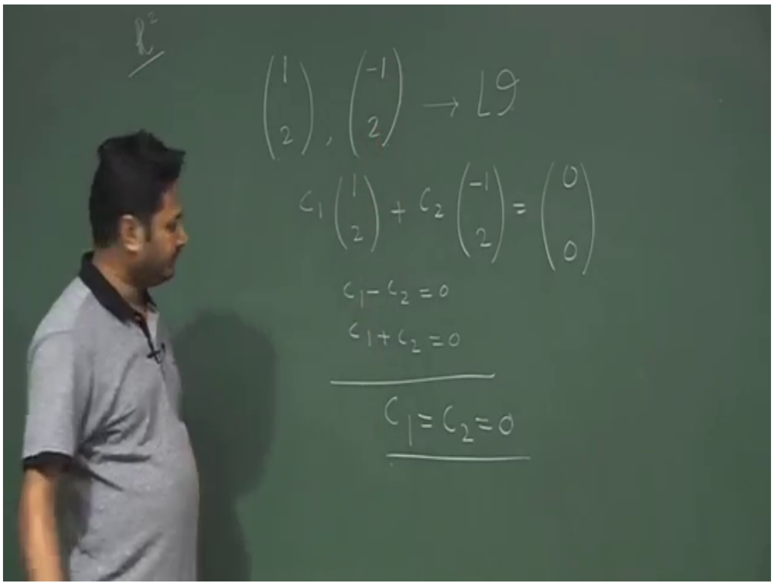

For example, I just changed this vector 1 2 and say minus 1 2, these two are linearly dependent. C 1 1 2 2 plus C 2 minus 1 2 is equal to 0 0. This is my equation. 0 means a right hand side. When I write single 0 that essentially means this is 0 because left hand side is a vector quantity, the right hand side.

When I write the vector, basically I am writing the null vector. So, I should write the null vector in component form which is 0 0. So, now we will have one equation C 1 minus C 2 is 0. Another equation C 1 plus C 2 is 0 2, C 1 2. C 1 is equal to 0. So, C 1 plus C 2 is equal to 0. Now, if I solve that, then readily I will find that C 1 is equal to C 2 is equal to

0. Now, only solution here I find C 1 is equal to C 2 is equal to 0. So, that means these two set of vectors are linearly independent. They are not dependent to each other. That means, these vector cannot be written in terms of this vector. Let us go through in R 2 also. So, what is the meaning of that?

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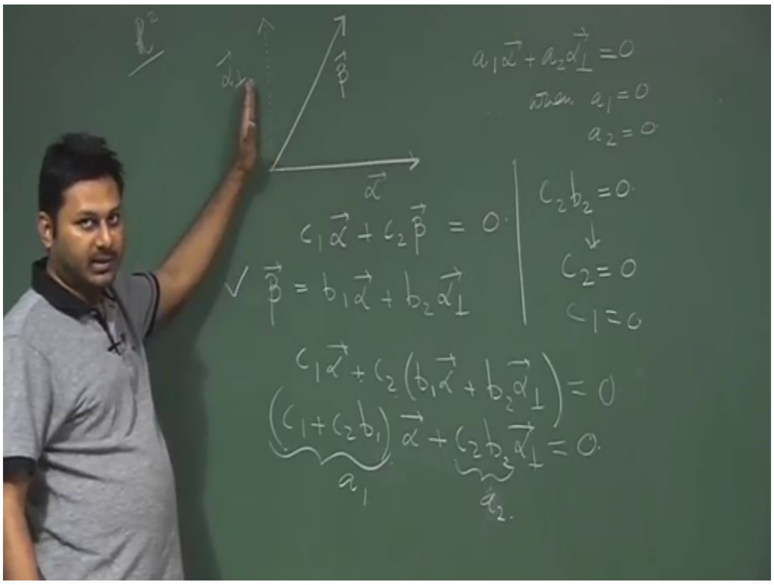

In 2d if I write this vector and this vector draw two vectors say this is my alpha and this is my beta. Is it possible to write alpha and beta or alpha in terms of beta or beta in terms of alpha? The answer is no. If these two vectors in 2 d is forming in this way, then there is no way to write a vector alpha in terms of beta or beta in terms of alpha. That means, there is no linear combination between alpha and beta, such that I can write the equation for example, if let us try to do that and then, things will be clear. So, maybe I can write another, I can make a perpendicular vector of alpha which is I write alpha perpendicular alpha vector and alpha perpendicular vector. These two vectors are always linearly independent I never able to write this vector in terms of this vector.

So, they are naturally independent. So, I will write a 1 alpha plus a 2 alpha perpendicular is equal to 0. When a 1 is equal to 0 and a 2 is equal to 0, my goal here is to find out whether these two vectors are linearly independent or not. So, let me write an equation assuming this I write this equation and try to find out whether C 1 and C 2 can be represented in terms of non zero value and alpha can be represented in terms of beta or not. So, what I will do that I will expand my beta into two parts. So, I can divide this vector into two parts. This and this perpendicular, we know how to make it is a component wise. So, beta can be represented in terms of this and this with certain component.

So, say this is b 1 alpha plus b 2 alpha perpendicular. I sub divide this b in these two. Please note one important thing is there if I able to write that, then alpha beta and alpha prime are not linearly independent. They are linearly dependent. That means, beta can be represented in terms of these two vectors. I can do that. I will show once again another thing which gives you the idea how these things are happening. So, now if I put this here, beta I just put this value. So, C 2 multiplied by beta 1 alpha plus beta 2 alpha perpendicular is equal to I will have one equation in my hand.

So, now if I write c 1 plus c 2 b 1 alpha plus c 2 b 2 alpha perpendicular is equal to 0. I rearrange that, but if you look this equation and this equation which is right in the top, then you find that this is nothing, but a and this is nothing, but b, not b here. This is a 1 and this is a 2, but my condition suggest that a 2 is equal to 0 and a 1 is equal to 0. They has to be 0. That means, this has to be 0, this has to be 0, this has to be 0 means I will have c 2 multiplied by b 2 is equal to 0.

From here we can suggest we can have that my c 2 has to be 0. Why? It is because b 2 is not 0 because I expand my beta in terms of b 1 and b 2 that I say these things I can expand this, but this I am not sure about. We need to find out and when I put this find, c 2 is automatically 0 and if c 2 is automatically 0, you will find from this equation c 1 is also 0. So, that means in 2 d I will have these two vectors. These two vectors is always linearly independent to each other, but if I put a third vector, then it is always possible to write this vector in terms to other two vectors and so on. So, that means one important thing I find is in R 3.

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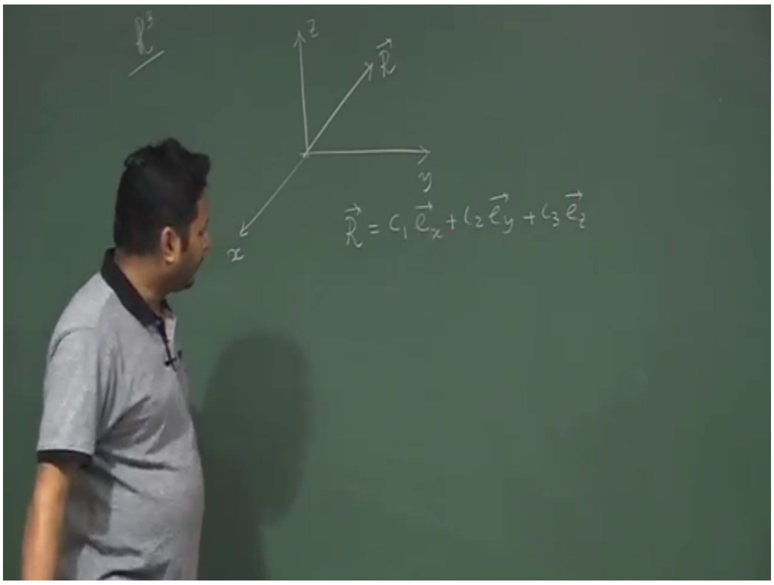

For example, in R 2 I find two vector which are linearly independent, in R 3 this is my dimension x y and z, this is my coordinate system. If I draw any other vector R, I can write R in terms of other three vectors.

So, I can write R in write c 1 say e x is you need vector along this plus c 2 e y. E y is in vector along this and c 3 is a along vector this or in any arbitrary vector I can write. So, that means if I want to write four vector in 3 d, the fourth vector can be represented in terms of other three vectors.

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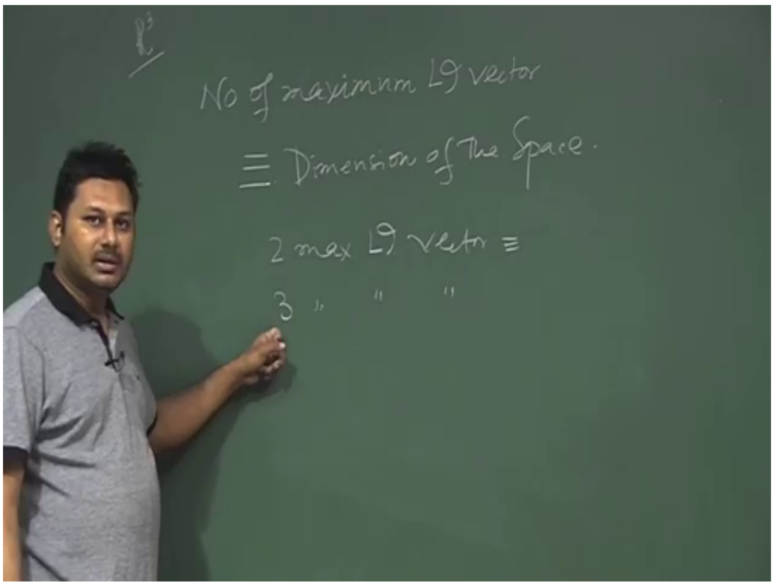

So, from here I can conclude one very important thing which is number of maximum linearly independent vector is equal to the dimension of the space. Dimension of the space means essentially dimension of the vector space. So, number of maximum linearly independent vector if I have two maximum linearly independent vector, then I will have this is for two maximum linearly independent vector. I can say the dimension of the space is 2 for 3 maximum linearly independent vector, dimension is 3. On the other hand, if I say that if my space is 2 d, then I will have maximum two linear independent vectors. If my space is 3, then I will have three maximum independent vectors and so on. So, let us once again show that by geometry how these things are happening.

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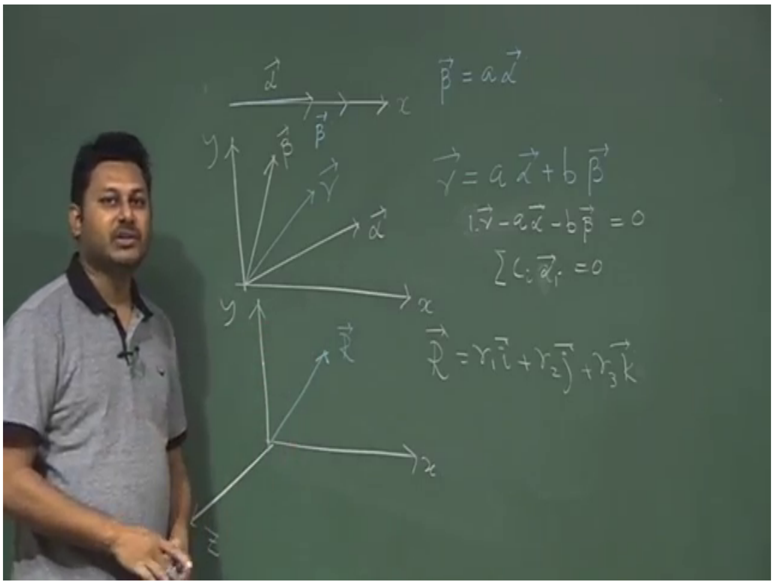

So, this is 1 d, this is 2 d and this is 3 d. I draw this figure like this in 1 d. I will have one line x, this is x y, this is say x y z 1 d 2 d 3 d. How many linearly independent vector you will have? Only one for example, this is my linearly independent vector and I call it alpha. If I draw another vector, let me take another chalk say this is my another vector and I call it beta. I can always write beta in terms of some multiplication into alpha. That means, beta can be represented in terms of alpha.

So, how many number of linearly independent vector in 1 d? It is 1 which is alpha or any other. In 2 d, I have one vector here alpha, I have another vector here beta. If I have another vector gamma, then gamma can always be represented in terms of alpha and beta with some coefficient. Gamma alpha beta can be represented in this particular form. So, this particular form is nothing, but if I write clearly, this particular form is nothing, but this form where phi let me write it 1 into 2 gamma minus a alpha minus b beta is equal to 0 c 1 vector minus plus c 2 vector plus c 3 vector is equal to 0, c 1 c 2 c 3 naught equal to 0. Here 1 is not equal to 0; a b is not equal to 0. That means, I will have a equation like this with the coefficient c i not equal to 0.

That means, this vector has to be linearly dependent because I have non zero c 1 which satisfy this equation in 3 d also. In 3 d also if I write 1 as I show R, R can be representing terms of the other three vectors. So, R normally we do that R can always be represented as R 1 i plus R 2 j plus R 3 k, where i j k are special vector which we call basis. So, let me now go back to the basis thing which is important. So, now we are in a position to define something called basis which is basis.

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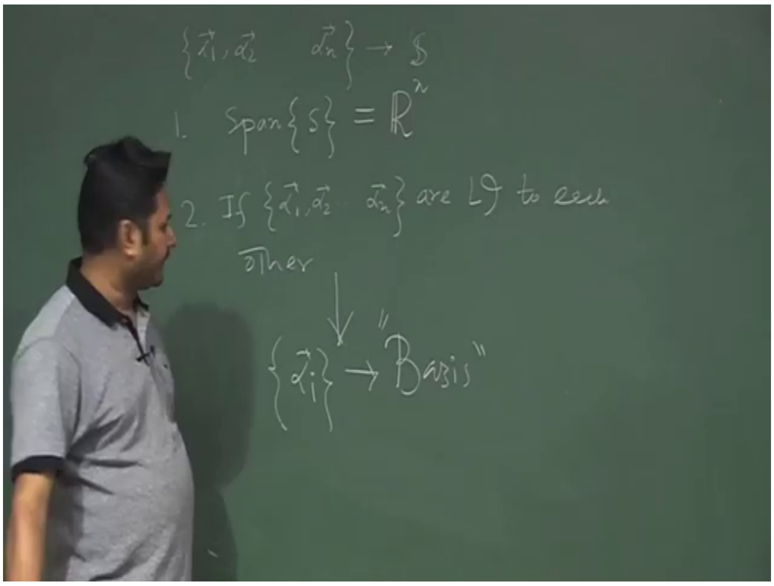

So, first I know that these are the set of vector that can span. First thing is that this belongs to some set s. So, now I am saying that span of this set is equal to the entire if it is n number of vector. So, I will say the span of s gives me the entire vector space element of entire vector space in R n.

This is the first quantity. Alpha 1 alpha 2 alpha 3 alpha 4 alpha n are the few vectors that are taken. This is the elements of vector space in R n and this is taken from that and I make a span over that and when I make a linear combination of these things, span is essentially the linear combination I can generate any vector possible. So, this is first criteria. Second thing is that if alpha 1 alpha 2 alpha n are linearly in depending to each other, if these two criteria is satisfied, then alpha i this set is called the basis that means, any vector can be represented in terms of alpha i in R 3. So, they have to be linearly independent which is a prime criteria that we have already shown that in the previous class and they should span entire space.

So, let me now give you the idea how to find the linearly independent set of vector and all these things. So, that means a vector whatever the vector I say vector V.

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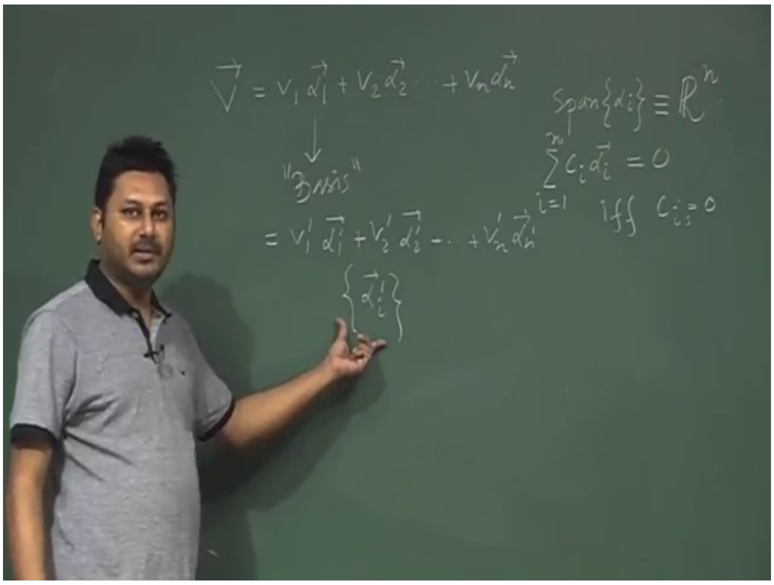

I am using this notation, but any way let me change this notation to just V. So, this V can be represented in terms of the spanning set. So, now I will write V 1 alpha 1 plus V 2 alpha 2 plus V n alpha n alpha 1 alpha 2 alpha 3. I show that they span the entire vector space. So, that means span of these things are equivalent to R n. Any vector can be represented. For example, m in terms of this and they are also linearly independent.

So, another condition is that summation of c i alpha i, i is 1 to n is equal to 0, if and only if c i are 0. These two conditions they follow for. If they follow any vector can be represented in terms of this and we call this as basis.

Now, the next obvious question arising in our mind that how many different alpha 1 alpha 2 set? One can form the answer is infinite number of sets. This is not in alpha 1 alpha 2 alpha n is not infinite, the set is infinite. For example, I can write same V with this notation. My coefficient is going to change, but I can form another set alpha 1 alpha i prime which can also form the basis because they are linearly independent and they are spanning. The entire set what will be the example in R 2. This is very important for you to remember that there is not a one choice, but there are different choices. In fact, there are infinite number of choices in R 2.

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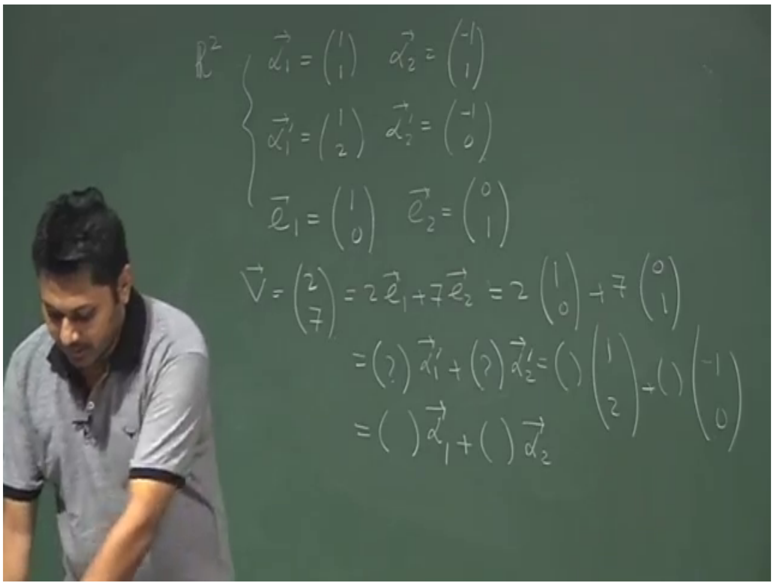

For example, R 2 I choose 1 alpha vector 1 1 minus 1 1. I can write any vector in terms of these two. Also, I can write another vector, another set prime 1 2 prime minus 1 0. This can also form a span. So, you can write infinite number of vectors which are linearly independent and they can form this span and one vector can be represented.

For example, this is my 2 d vectorially and if I write one vector like this alpha 1, another like this alpha 2, this alpha 1 and alpha 2, they are linearly independent and they can span the entire vector, entire vector space, but I can choose another prime, also another set. This alpha 1 prime and say alpha 2 prime. The blue one, they can also form a set of linearly independent vector because they are linearly independent. Not only that they can span the entire vector space, they potential also form a basis. So, number of choice of basis is in infinite. The number of choice of basis is infinite. I can choose the set containing alpha 1 and alpha 2, alpha prime and alpha 2 and so on.

The number is infinite, but what is the natural choice? I already mentioned is this e 1 vector which is 1 0 and e 2 vector which is 0 1, I believe now you understand which is why this is a natural choice because any vector can be represented in terms of this and this, also this and this also, this and this, but the vector, the coefficient of the vector can be easily identified. So, let me write for this.

Let me write a vector V which is say 1 which is say 2 7. If I write this in terms of e i, it will be 2 of e i plus 7 of e 2 like this which is 2 1 0 plus 7 0 1 readily, but if I want to write these things which we can in principle, then I do not know what is the coefficient alpha 1, I do not know what is the coefficient alpha 2 and this alpha 1 and alpha 2, I do not know, but this I know 1 2.

This I know minus 1 0, but in principle you can find out what is the value here, what should be the value here. In natural basis I do not bother about this value because if the vector is given I can readily put this, but here I need to find out what is the value and this is easy if you put c 1. If you put c 2, you can solve this equation because 2 7 is given to you. The c 1 multiplied by 1, c 2 multiplied by minus 1, it is equal to 2 and other thing is 7. So, you have two equations and you can figure out what is the value of this which is given as this question mark.

So, you can write another thing, another in terms of alpha 1 and alpha 2. This is alpha prime. So, you can also write alpha 1 and alpha 2. So, here also you can write another equation same thing in terms of alpha 1 and alpha 2 and obviously, this coefficient will be going to change. So, here this coefficient is 2 7. Obviously, this coefficient will not be 2 7; it will be different and this coefficient will be some other value. The important thing is that here you know what is the basis. Basis is something using which you can define a vector, but the coefficient will differ when you choose different basis. So, it is called basis transfer. We will do this in the later class, but you should know one thing that there is no unique choice of basis. You can choose infinite way to basis. Only condition is that they should be linearly independent and second thing that they can span entire vector space.

For example, here merely I show three examples. For R 2, you can show different example, so that every vector which are linearly independent can be well enough to define a basis, can be well enough to use as a basis, but here natural choice is this. In 3 d, it will be 1 0 0 0 0 1 1 0 0 0 1 0, then 0 0 1 and in 4 d, it will go on and you will have more and more order, but the concept is same that if you want to define a vector, a vector can be defined in terms of basis. This is natural basis, so that I can have the vector coefficient quiet easily. If the basis is not natural, basis is some arbitrary basis, but still it is a basis. You need to find out what the value of the coefficient of that vector is.

So, in the future class we will learn more important things. For example, what is the relationship between these and only this is linearly independent. Apart from that there is any other property associated with this, we will find. So, the question is let me before concluding this class, let me quickly show you one very important thing by drawing something, so that you can readily understand. So, this is 2 d, this is my coordinate system. I choose this coordinate system perpendicular because it is easier for me.

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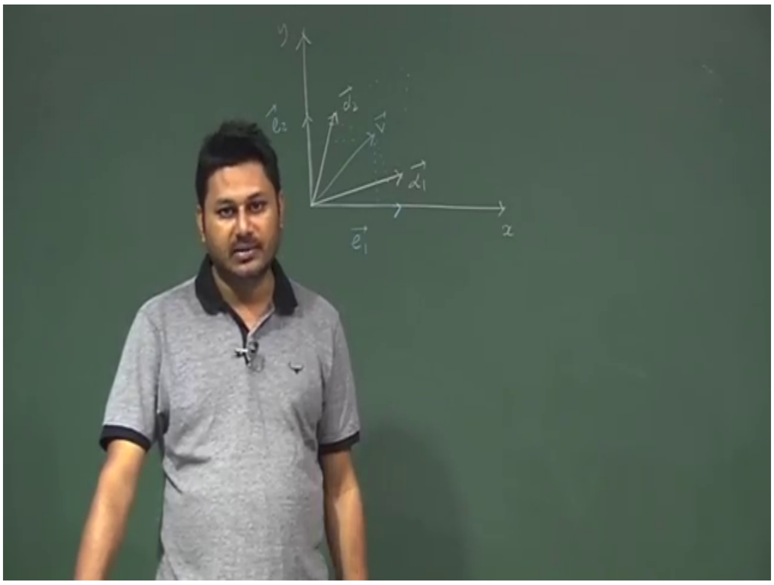

So, if I write this is my e 1 and this is my e 2, but I can have other basis also as I mentioned these and this is my alpha 1, this is my alpha 2. For example, now e 1 and e 2, there is a special relationship between e 1 and e 2 and this special relation is nothing, but they are perpendicular to each other, they are orthogonal to each other, but for alpha 1 and alpha 2, they are not orthogonal to each other.

Because of that what happen if any vector V is given to you, you can divide V very easily in terms of e 1 and e 2 because the coefficient is nothing, but the projection of this vector over this vector over there, but this vector can also be represented by making a projection over that, but this projection value is difficult to figure out and that is why this is not a very convenient way to represent this vector, but you should know that this also can be used as a basis, but you can appreciate with these figure that the choice of e 1 and e 2 is more useful because I can readily find out many things.

Student with this let us stop my class here. In the next class, we will start a very important thing as I mentioned since it is an orthogonal. So, I need to find out how to find out what is the procedure to find two vectors or orthogonal or not.

So, to do that we need to do something called inner product to concept of the inner product and all these things. You will learn step by step and understand what is the other property of this basis. This natural basis is natural basis of few other properties. So, we need to also find out how the other properties are emerging with the fact that they are orthogonal to each other. With this let us conclude this class. So, see you in the next class where we learn more about inner product and other issues that I mentioned.

Thank you very much for your attention.