Mathematical Methods in Physics-I
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Lecture – 39
Multi Valued Function and Branches

Welcome back student to our complex analysis course. So, today we will like to cover a very interesting thing that we mentioned in the last class, last few classes, but we did not cover it.

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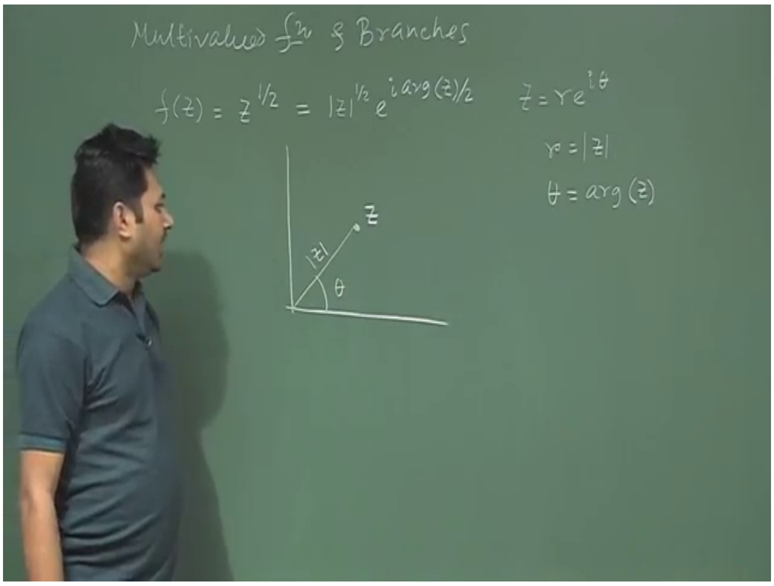

So, that is multi valued function and branches, multi valued function we have already covered this, but branches we did not cover that much. So, let me do that once again.

So, the concept of multi valued function in complex analysis is important because for example, if I have one function z to the power half, we know that this is a multi valued function in the previous class we have done few things related to this particular function, but let us go back to that thing and try to understand more about this thing. So, I can write this function z to the power half as mod of z to the power half, e to the power i argument of z because any complex function I know can write as r e to the power of i theta here by 2 should be here because this is r e to the power I theta.

R is nothing, but the mode of z and theta is nothing, but the argument of z, if the function on the point z is here somewhere with an angle theta then this is angle theta for that particular point z and this is the magnitude and we define with these two we define the location of these thing which is this, the problem is not with; however, with this length the problem is with theta.

So, now let us try to find out what is a z to the power half what is getting so; that means, if I have a point here this is in z plane.

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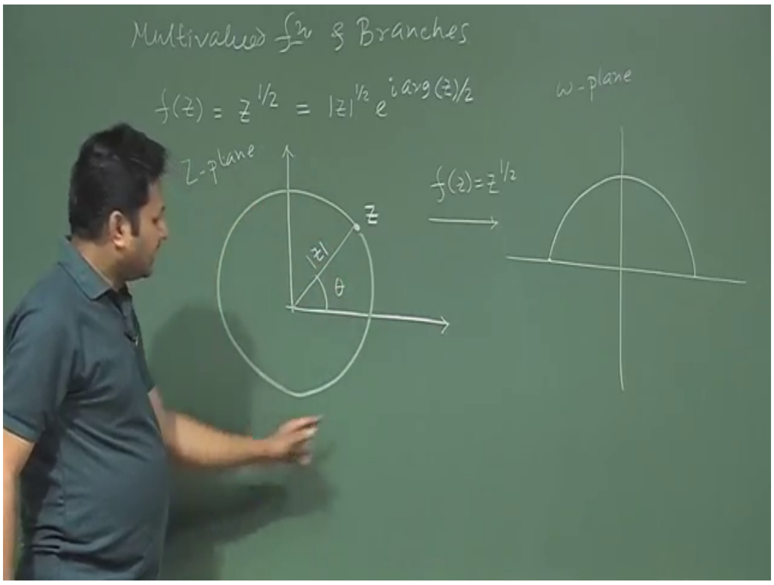

Now, this is related to a function exits which is z to the power half and we know that we can plot this things in my w plane, what this function can be represented at u plus i v in this form, but the thing is that since it is z to the power half. So, if I rotate the entire point to one circle starting from here to here, say z is a point which I am rotating with this path, with this path; that means, essentially I am changing this value of theta in that plane I can only able to cover up to this. Why? Because my argument here is divided by 2, because my functions z to the power half so if I have a full rotation in z plane. In w plane I will have something which cover up to this it will not going to cover the entire lower part only the upper part will going to be covered.

Now, in order to cover the lower part we need to rotate once again another round and because of that I will get the lower part, but the question is let me erase this, the question is about the value. So, this argument as I mention will create the problem.

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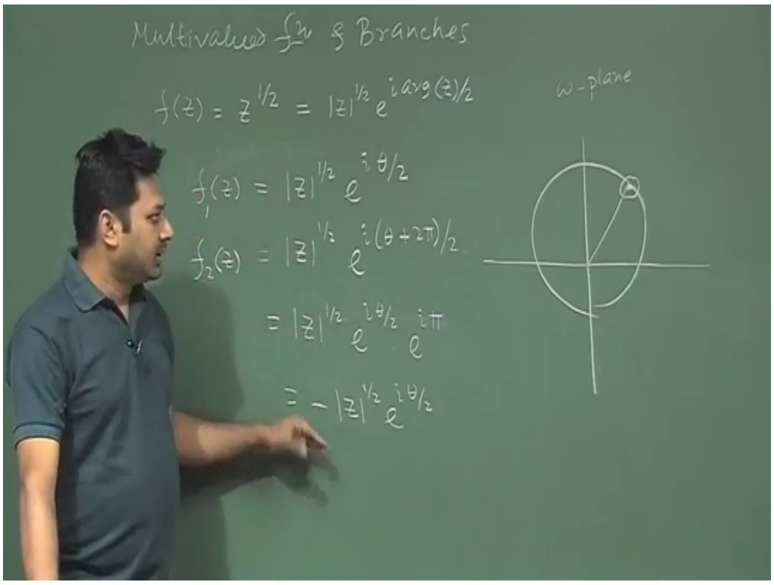

So, fz is equal to z to the power half e to the power of i theta by 2, if I take this theta as my principal value right

Now, the same z point can be represented by this because if I rotate with these if I rotate another 2 pi. So, I will return back to the same point as I mentioned. So, let me draw that once again, this is my z I am rotating these things and return back to the same point. So, my z is same the location of z is same, corresponding to these two, but let us check what is the what is the value of this quantity, it is e to the power of I theta by 2 multiplied by e to the power of i pi.

E to the power of I pi is essentially minus 1. So, minus of, this is a new value that I have for the same z. So, in the first case if I take my principal theta then I will have one value, but if I rotate to another 2 pie so, that I can cover in w plane I can cover the entire range I will have something like this having a negative thing. So, these two things are little bit ambiguous because I will have the same z point here and in the first case I will have one values and the second case I will have one values. So, if I say this is f 1 which gives you the positive value f 2 is something which gives you the negative value.

So, the total function is f and f 1 and f 2 are behaving like 2 branches giving two different values for the fact in the keeping that in mind that theta is now changed into 2 pi so; that means, there are 2 branches because of which I am getting 2 different values for a given function. So, that is why this is called a multi valued function I will come to that multi valued function more, but let us try to understand this basic things.

So, now to how to represent these things because as I mention there is a ambiguity that for a same z point I am having, for the same z point I am having two different values in mind w plane. So, I mean what is the procedure to do that. So, in order to understand that I we understand this roughly by the way this f on1e and f 2 are 2 branches as I mentioned.

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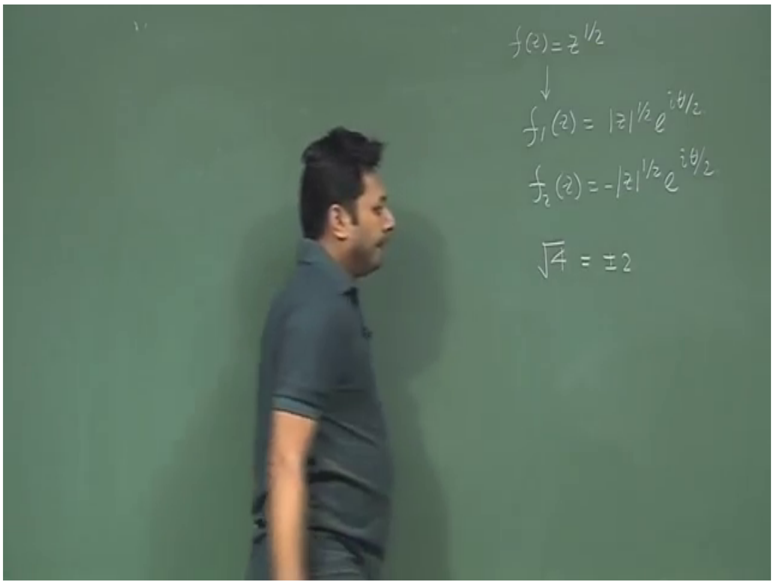

So, f z is z to the power half that was my original function and from that I figured out one value is mod of z whole to the power half e to the power of i theta by 2, another value is minus of these it is exactly same like in the real case plus minus. If I want to find a route for example, root over a four is result is plus minus 2, one branch is for plus another branch is for minus, but in complex case I need to be more careful because things are slightly different here.

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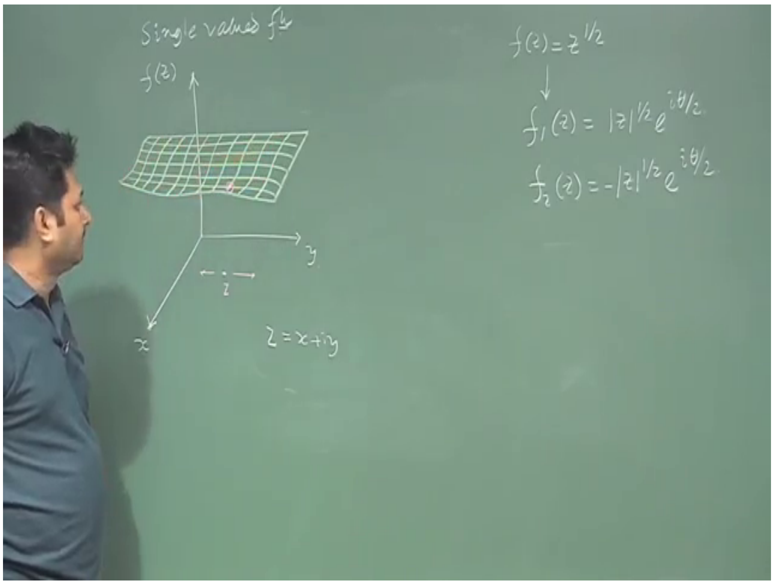

So, let me try to give you the essence of this multi valued function in terms of some kind of projection or some kind of topology for example, this is my x, this is my y and this is my x y plane and perpendicular I have f z if it is a single valued case whatever the function if the function is single valued. So, for single valued function, for a single valued function whatever the z is in this plane so x and y z is x plus I y. So, somewhere here the z is located by this point.

Now, it is moving. So, it is a variable. So, it is moving. So, every time for 1 z I will have one function functional value, f z and if I plot that it will be a surface because z is moving over the surface. So, for each z assume that for each z I will have one single value here somewhere and based on what is the function we will have the surface. So, the surface is looking something like this, just I try to draw I do not know exactly what is the function, but if I plot that it will be something like this, these are the may be the coordinates I am plotting to show you the essence that how this things may work, some sort of wavy planes are there. So, I have one point here and for this point suppose I will have 1 point by with mark say, I will have one point here over the seat f z then if I shift this point to somewhere here or somewhere here I am moving this point in exit and I am making the surface, whatever the surface is drawn here this is an arbitrary surface.

Since f z is a single valued function for one particular point I should have one surface like this, because this is a single value 1 point in this is z 1 point in x y plane and as a result I will have one particular point over the sheet fine.

Now, I will do the same thing for multi valued function. So, how it will going to differ. So, let me do that, I believe you really understand then how this things are working.

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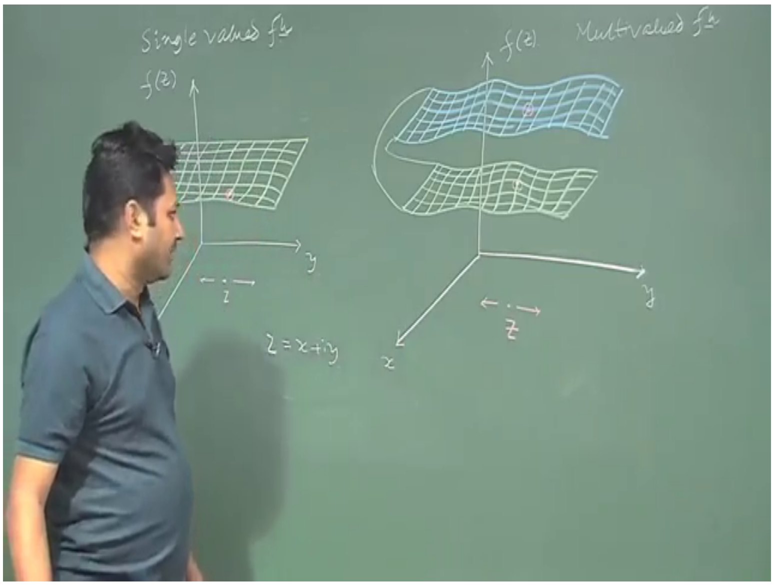

Again I will draw a coordinate system like this, where x and y is here and suppose I am plotting fz or some real or imaginary part of the fz because fz itself contain real and imaginary part, but it is just a rough way to understand how the things is working here.

Again for a same for a point z here I should have 1 point like this, but since this is a multi valued case. So, this is a multi valued multi valued function. So, the basic difference is I will have 2 points here and here somewhere. So, first I will have some point here since is a multi valued I should have another point also for the same z that is the case we have in the previous example we have root over of z and we find that for a single value of z I should have 2 values one is positive and another is negative. So, here if I again draw the surface let me draw it properly.

So, that you can realise it is very difficult to draw 3 d, 3 dimensional object in this way, but I believe you understand how these things are working. So, this is surface one. So, for this value I will have some here, but I should have another surface because this is a multi valued function. So, another surface somewhere here like a sheets of paper fine.

So, here you can understand that in this case if I have one sheet of paper because this is a single valued function, for multi valued function if this is my z point, if this is my z point one z corresponds to a single point here also some point here and since z is changing and z is changing with a function f z a given form and based on that I can plot these 2 surfaces. Now the question is for the multi valued function these two surfaces are not separate like this, some point they are merged to each other in a complicated way, it depends on the function they are merged to some point in a complicated way.

For example in some if I extend this things somewhere in for some value of z it will be, it will merge to each other I do not know in which way it will going to merge, but maybe it is like something like this it in merge in some like this they emerging. There is a common point where the 2 values are same that particular point the common point is called the branch point and if there are many branch points are there they are called the branch line because we are making the lines. So, I will show you something. So, after having a rough knowledge about how the multi valued function visually look it will be much easier for you to realise the thing now let us go back to our original problem that I start with.

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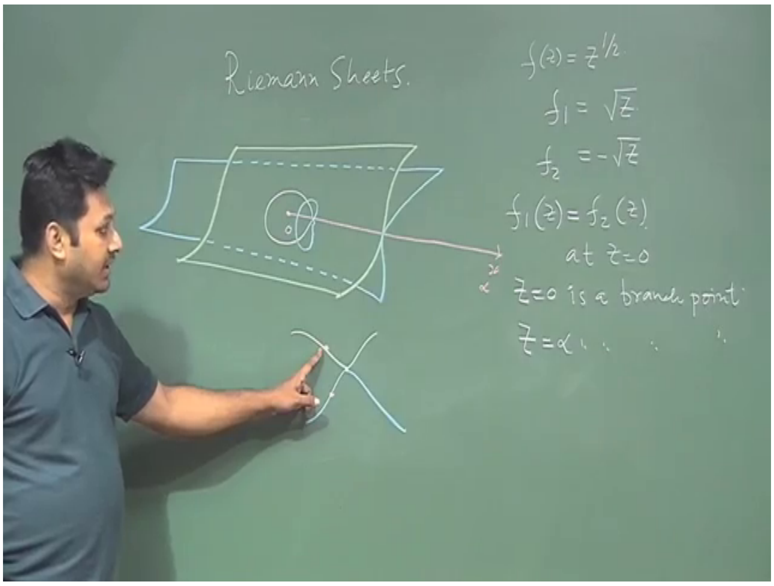

That f z is z to the power half. So, one f 1 value is root over of z whatever the value and another f1, f2 another branch say it is minus of root over of z, after rotating 2 pi it comes down to a negative value. So, how I realise that, again I can draw very interesting sheets like that since is a multi valued function. So, I can draw 2 sheets which is called let me first draw the sheets then I will explain how things are going on so.

Say this is and another sheets may be here, it is something like this. So, let me draw one important thing that, this function fz, f1 of z will be equal to f2 of z at z equal to 0. So, z equal to 0. So, z equal to 0 is a point which we called the branch point and also z tends to infinity is also another branch point. So, now, if I since the 2 branch points I have if I glue this to branch point, say this is my first z equal to 0 point and this is my x axis positive x axis say which goes to infinity I can go infinity in any direction mind it, but let us take the positive x axis as my direction of infinity for this case. So, where you can if I glue these things. So, the paper is joint here in this from here to here from 0 to infinity and other part is open. So, this is called the Riemann sheets.

So, in the Riemann sheets now I can represent my points, in the Riemann sheets what happened in the first branch if I rotate this z point to 0 to pi. So, it will be like this, it will give you the it will be on the first Riemann sheet and then when it crosses these things after rotating 2 pi it will just slide down to the next sheet it will slide down to the next sheet and again I will have something.

So, in the first rotation it will be in the first sheet and when it crosses this line this branch we called is a branch cut, when it is in branch cut there is a discontinuity because we have a negative sign here and its lies down to the next sheet. So, this lower sheet corresponds to say minus and the upper sheet is plus. So, it is in the upper sheet for the time being when the theta is up to 2 pi and then it goes to the next exactly the next one when it crosses this branch like. So, it lies down to the next thing.

So, if I look this side. So, obliviously it will be something like this, from if you look this from this side to it will be something like this. So, this let me draw this in this way, this is my first sheet and my lower sheet is something like this here it is glued. So, the point was initially here when it is rotate and cross this line it will slide down to the next one, that is the thing the point was here it is rotate and when it cross because it is glued when it cross it go back to the next part. So, here to here if this value is positive after rotating to pie it goes to the next one.

So, now it is clear that when the point is here and when the point is here 2 different values I should get now the ambiguity that the same z for same z I am getting 2 things is now gone because now these 2 points I can identify clearly that it crosses the branch cut and it slide downs to the next point, when it slide downs to the next point I will have these things.

So, now we have a rough idea about what is the meaning of branches and multi valued function.

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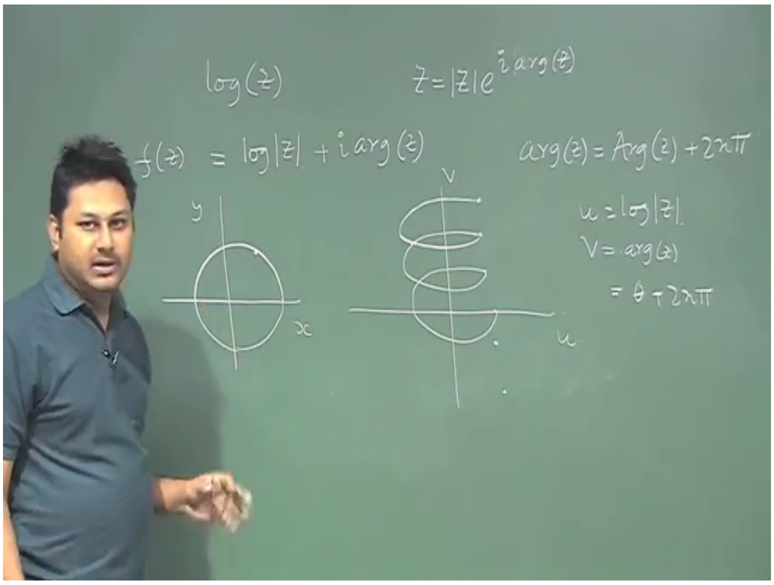

So, let me go on with that with another very interesting example that is important, that is log of z this quantity, again z is mod of z e to the power of i theta if theta is a principal angle or rather I should write it as argument of z. So, this quantity is how much log of mod z plus i argument of z this quantity.

Now, this argument of z I know it is big argument of z plus 2 n pi; that means, if z is given to me I will when I make a log of that I will get different values. So, if I now plot that this is my x y some point z is here, this is my u v this is my function f z which is this quantity. So, my u is this quantity and v is that quantity. So, my u here is log of z and my v here is argument of z which is essentially theta plus 2 n pi if theta is my principle angle this quantity.

Now, for a single point here how I plot that. So, my u is some fixed value here and v is what v is theta if n is 0. So, somewhere some value here, but if I now go for different n values it will be some other points also. So, I will get another point here also, another point here also, another point here also, here also and so on so; that means, for a single point I will have 2, I have more values depending on the values of n.

So, now how to represent these things, how to find out that how these things are there is a way you can do that, if I now if I change my theta say like this in the previous case I am now rotating my z. So, that I am changing my theta and now after one rotation I am returning back the same point, but here after one rotation means my n value is now 1, after two 2 means my n value is 2 and so on. So, now, if I if I see some where here the value. So, when I start from here x equal to 0 and y equal to, x equal to that point and y equal to 0; that means, I am having somewhere here and now I am rotating. So, when I am rotating I will go back to this point, then again I am go back to this point, again I am go back to this point and so on; that means, with the rotation here.

I am rotating in such a way that I am having the same point and not on the this is not a single curve this is a continuous value because z can take any values from here I am taking only for a fixed z here mod of z, but mod of z can also vary as well as the theta. So, entire for entire range whatever the z are there they can change, but if they have a one radius. So, the value is rotating like this and for each case I am having the same point and this points are corresponding to this theta plus 2 n. So, this is also an interesting thing this is also an called the Riemann surface. So, every time we have a multi valued function the point is every time if you have a multi valued function, if you try to find out the Riemann surface you will you will get this kind of weird looking surface every time.

And it is interesting because with that you can distinguish the case for example, when I have this point which correspond to the same point then I understand that this for this z the theta is theta plus 2 pi for this z the theta is theta plus 4 pi and so on and every time I will change the theta. So, that is why it is like this. So, this is interesting thing, I will suggest the student to just looked in the internet or any other books just Riemann surface. An Riemann surfaces is you will find there are ample amount of examples.

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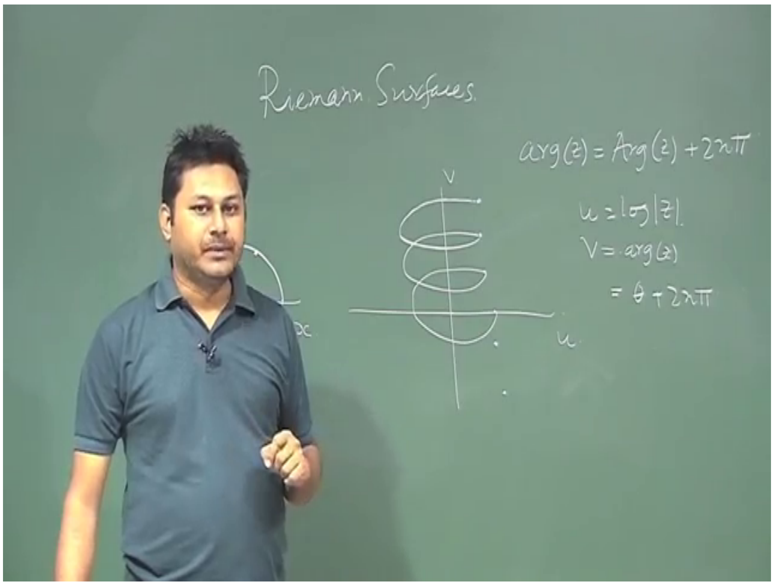

And the visualisation of the Riemann surface for different, different kind of multi valued functions. So, if you see that in that books or form that references or from internet you will easily understand the how the surface are formed, but you should understand first the basic thing that for multi valued function I should have some kind of layered thing and if one layer we have one value the second layer we have one value and so on. So, they are called essentially they are called the different branches.

So; that means, for each branches I will have one value when I change the branch I will go to the next branch; that means, a next set of values and so on.

So, with this I would like to conclude today's class today you will learn I mean you have learned a very interesting thing multi valued function and branches in different part of the complex analysis this concept use and it is important also to understand how these things are working specially when z is changing there are 2 parameters associated with that one is r and theta. So, that is why this kind of things appear in this way. So, with this note let me conclude today's class see you in the next class.

Thank you for your attention.