Mathematical Methods in Physics-I
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Lecture – 38
Polar Form Of Cauchy-Riemann Equation

So welcome back student to the next class of a complex analysis; in the previous class we learned few thing about the Cauchy Riemann equation and analytic function.

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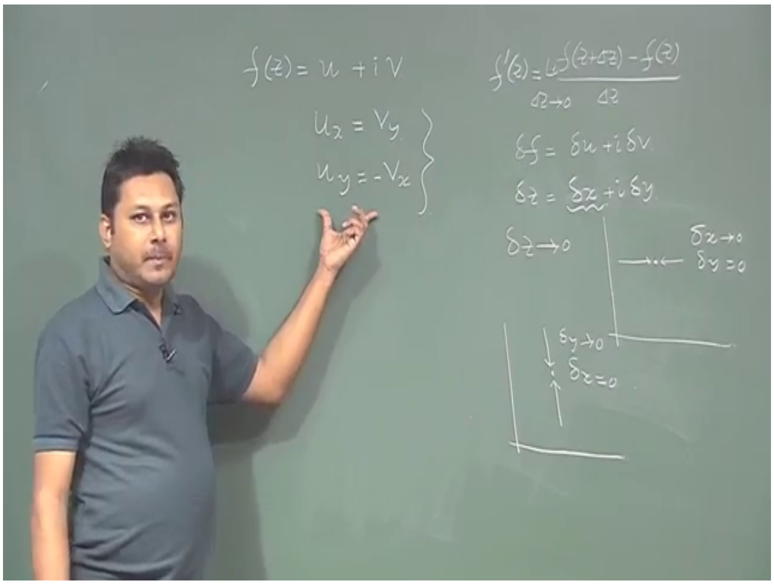

And if a function is given to you, the Cauchy Riemann equation suggests that ux is equal to Vy and uy is equal to minus Vx and if they satisfy this relationship then the given function fz is analytic in nature. Now today I like to show another angle of another way to derive these things, regarding the fact that for analyticity the derivative is important because the analyticity of the function means that if function should have some derivative at particular point not only that in the neighborhood of that.

So, when we do the derivative if you remember f prime z is defined as f z plus delta z minus f of z divided by delta z, limit delta z tends to 0. With this limit I can write the expression of this, during the calculation of this Cauchy Riemann equation we approached this condition in two way, one is delta f was delta u plus i delta v and delta z was delta x plus i delta y and then I mentioned that when delta z tends to 0 and you can do this delta z tend to 0 in infinite way, because it can comes from anywhere, but we choose to prefer direction in one case my delta x was tends to 0 and my delta y was 0.

So; that means, if this is the point I am approaching from here to here and here to here, making delta y equal to 0 that was my one approach and in the second case when I am put delta tends to 0 the second case, our approach was something like this that delta y tends to 0, but delta x is equal to 0. That means, I am not approaching this side this side because I put from the very beginning that delta x is equal to 0 and I approaching this side. So, that this limit can satisfy, for to in order to satisfy this limit I take two preferred condition and this and this and when I do that then I derived this equation, but the question is, is this equation really is this way really gives you the total picture because I ignore all the other side approach, I take to prefer direction and then the calculation is done and from that I am getting this expression in my hand.

So, do I like to show that this equation is essentially gives you the fact that even though I am getting this expression by using these 2 preferred directions approach towards this 0 point delta z equal to 0, it essentially the fact that if I do the same thing this these 2 expression is basically gives you the same result in all the approach in any direction try to show that in a different way. So, I do hope you realize exactly what I am trying to do Cauchy Riemann equation is derived for two preferred direction, now I am saying that the Cauchy Riemann equation is something which is sufficient to show that if these two equation is satisfied even though I am deriving from twos prefer direction it is the same equation that you will get if you will approach whatever the direction you want.

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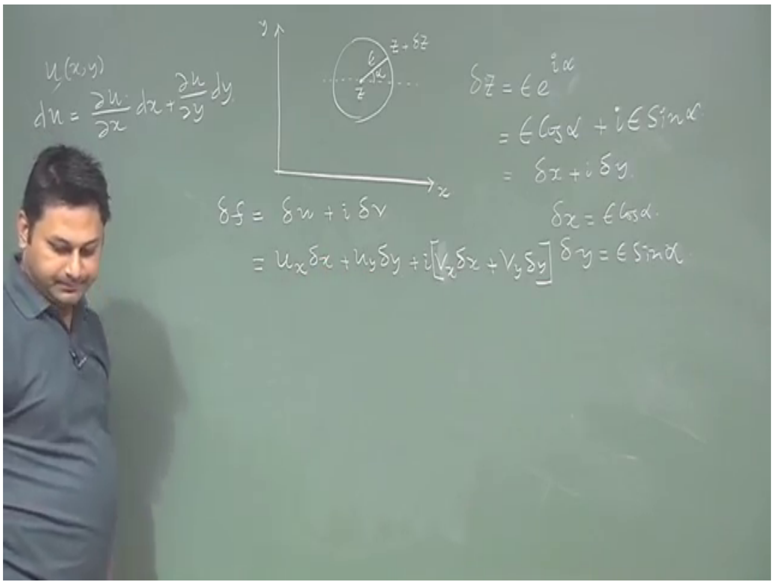

So, let me do the thing here in this way. So, this is the point I have z and another point this point is z plus delta z this angle is alpha. So, from this say here in the complex plane I have 1 point z if I want to approach this point, then if I change this point to some direction with an making an alpha. So, this is my delta z which is the small increment, small increment of this point. So, delta z can be represented as if I write this radius is epsilon. So, epsilon e to the power of i alpha, delta z I can write always because this is the point and I want to put another point delta z with radius is this epsilon and if the alpha is given I can write it.

So; that means, this quantity is epsilon cos alpha plus i epsilon sin alpha which is delta x plus i delta y. So, delta x is epsilon cos alpha, delta y is epsilon sin alpha that I have, these things I have, now what I will do delta f is delta u plus i delta v, uv are function of xy. So, delta u I can write it as ux delta x plus uy delta y plus i delta i vx delta x plus vy delta y, I just expand delta f which is the small change d delta u and small change delta v delta u and delta v are function of xy I believe all of you are aware of these things.

If u is a function of x and y then du can be represented as del u del x, dx plus del u del y dy I am using this simple thing here, when del u del u del x is ux is a partial derivative with respect to x partial derivative with respect to y and the increment ok.

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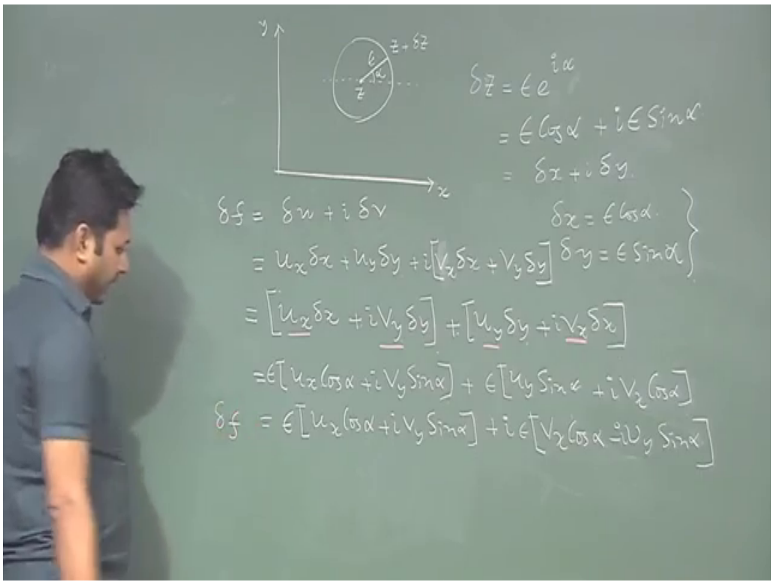

Now, if I write these things this delta x and delta y. So, let me write it first let me write this expression in this way, I just slightly rearrange this equation whatever the equation I have I am just slightly rearranging them.

That is I mean even though I am rearranging that, but you should note one thing carefully that during the rearrangement I take ux and vy which are related to Cauchy Riemann equation uy and vx which is also related to Cauchy Riemann equation and I will separate out these 2 with this form. Now I like to put this delta y delta x and delta y value that we have here. So, it is ux epsilon is both the cases we have epsilon so I can take epsilon outside ux delta x delta x is cos alpha plus i vy sin alpha this is my first expression.

Second expression is again I can take epsilon outside and uy is here, delta y is how much sin alpha plus I, v of x cos alpha this, now what I will do I will again rearrange this things likely. So, epsilon ux cos alpha plus i vy sin alpha plus i epsilon, I take i common from here vx cos alpha minus uy sin alpha with one i inside here I am taking i common from this side.

So, I will be here there will be no i with this term vx cos alpha. So, vx cos alpha is here uy sin alpha so I take one i outside. So, minus i should be here so uy sin alpha it is something like that. So, this quantity is my delta f, now let me erase this thing do not have the space in the board to go down. So, what is my limit?

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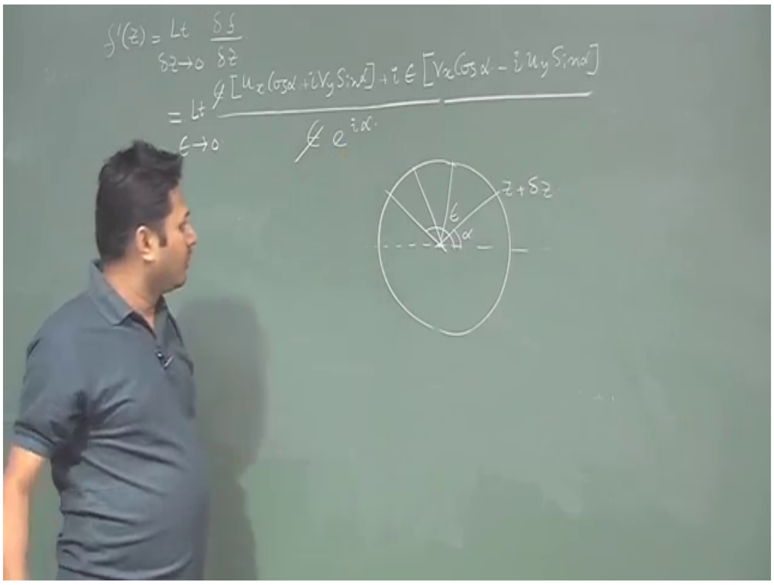

What is my derivative of this point it is. So, limit delta z tends to 0 delta f by delta z delta f I figure out delta z is also giving here.

So, if I write this expression should be something like this, epsilon ux cos alpha plus i vy sin alpha 1, plus i epsilon let me erase this part it is not required. I alpha then vx cos alpha minus i uy sin alpha whole divided by, whole divided by epsilon e to the power of i alpha. Now the limit is delta z tends to 0 delta z tends to 0 limit is essentially the limit epsilon tends to 0 because if you remember the figure, if you remember the figure it was something like this this was the point this was the alpha this was the epsilon is z z plus delta z.

So, now delta z tends to 0 means this length I can come to this point with this direction alpha is the direction, the preferred direction and I am coming to this point by making epsilon tends to 0; that means, this length can squeeze to this point, now you should note a very interesting thing here and that is the reason I derived these things to show. So, now, epsilon tends to 0 so I can cut this epsilon and this entire quantity now depend on one variable which is alpha, one variable which is alpha.

Now, if I say that whatever the approach may be in which direction I go to that particular point if it is alpha independent; that means, my derivative is true for all approach not any preferred direction so; that means, let me try to understand once again, this was the case this is epsilon, this is alpha this is z plus delta z I try to make this delta z 0 and my approach is now depend on alpha here. So, down stair we have one alpha and up stair we have one alpha is it possible to make this expression alpha independent.

If the expression is alpha independent I should not bother that what is the angle of this alpha I can go from here, I can go from here, I can go from here this alpha is not a very important factor here if I make this expression alpha independent.

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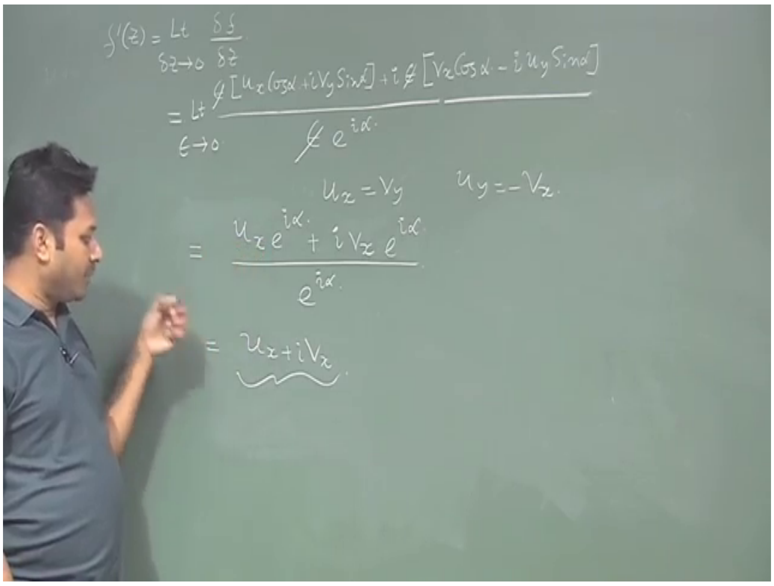

Now, is it really possible to make these things alpha independent. In fact, if you look very carefully when I put this ux is equal to vy and uy is equal to minus vx we will suddenly have something special here. So, what I am having u vy I replace with ux.

So, I will have ux e to the power of I alpha cos alpha plus i sin alpha I will have this plus I, alpha alpha and this alpha has already gone. So, alpha so now what I will do, I will replace this vy, I will replace this ah vx to minus of uy. So, if I do that minus of uy. So, it will be something let me check once again. So, I replace ux here and now I am going to replace it is I, I am going to replace uy as uy as minus vx. So, just I replace uy as minus vx, if I do then I will have vx uy I just replace minus vx. So, this term will be plus.

So, eventually I will have e to the power of I alpha divided by this quantity is already there e to the power I alpha. So, result is ux plus ivy vx, this quantity which is alpha independent. So, when this quantity is alpha independent when I put the Cauchy Riemann equation and plug it to this expression. So, Cauchy Riemann equation when I plug this to the to the original expression gives me the fact that my f prime z is alpha independent; that means, whatever the approach I go to that particular point this is I mean it is independent of this alpha; that means, every time it is if I go to whatever the direction I want.

So, Cauchy Riemann equation gives you the fact that it is true only when the Cauchy Riemann equation is satisfied; that means, it is a cross verification that Cauchy Riemann equation is sufficient to gives you the fact that the given function is whether analytic or not and the derivative is exist or not. If the derivative is exist meant the limit should be exist in that case and the limit should be approached from all possible directions and here I find that the direction is independent, the result is direction independent when I put the Cauchy Riemann equation in the expression. So, this proves that Cauchy Riemann equation is sufficient.

This is the another way to cross verify that how Cauchy Riemann equation is important and it is sufficient at that point to find out whether the function is analytic or not. The next important thing I like to put it here that what should be the polar form that I mentioned in the last class for the Cauchy Riemann equation. So, the polar form is something which is important also because the function z can be represent in re to the power i theta this form which is polar.

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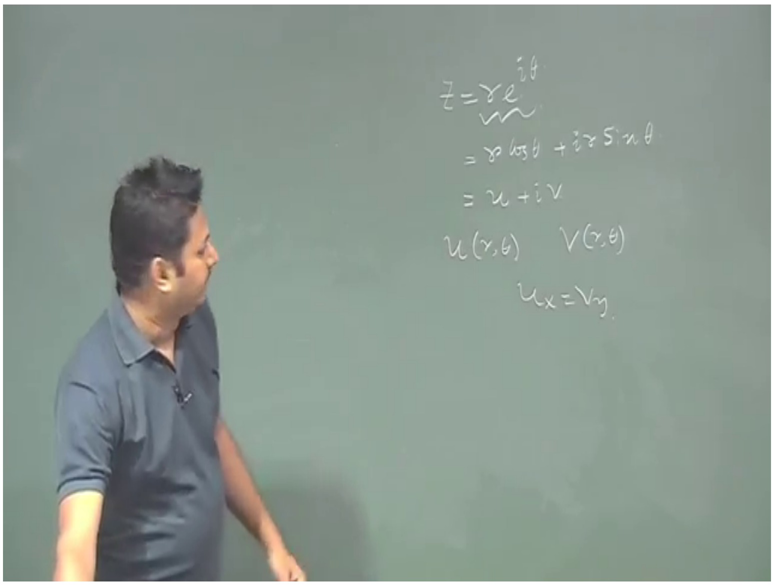

So, now re to the power i theta can also be represented as r cos theta plus ir sin theta and if I write this now u plus iv then my u will be function of r and theta and v is my function of r and theta as well, since u and v are now function of r of theta. So, it is important to know what should be the relationship between the r theta in the in polar form. So, that I will have a I mean ux is equal to vy this is the Cartesian form of the cr equation, but I need to find out the polar form.

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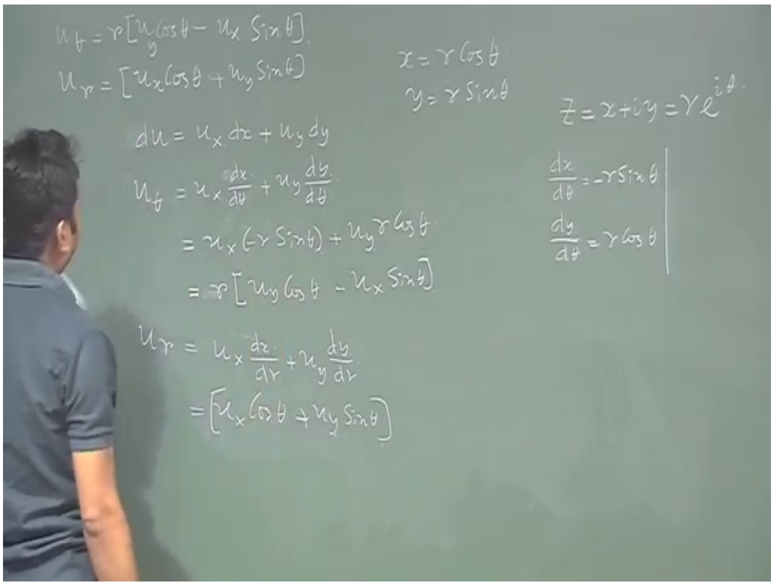

So, let us do it with this. So, u so x is r cos theta and y is r sin theta, that we know this is the starting point because we know that a particular point z when its represented in terms of x plus iy and is represented re to the power I theta these are the relations is just before I derive it. So, now, u is a function of x and y so if I write du it will be ux dx plus uy dy.

So, now from this equation I can write what is my u theta partial derivative with respect to theta it should be ux, dx, d theta plus uy, dy d theta. From here if I do dx d theta is r minus r sin theta and dy d theta is r cos theta this is one expression because this relationship is required here and other expression also I required I will do that, but let me finish it first. So, it is ux this quantity I will put which is minus of r sin theta plus uy r cos of theta u theta is this.

So, it is say minus of say r if I write it uy cos theta minus ux sin theta, I will have one expression in my hand it is something like this. In a similar way exactly in the similar way I can have ur, ur is ux, dx dr plus uy dy, dr, ux, dx ,dr dxdr is from there I can readily write it is the cos theta and uy dydr is sin theta. So, two expression u theta and ur I find. So, I should write I should write it somewhere because I need to use this again.

So, u theta is r u cos theta uy cos theta minus ux sin theta and ur is ux cos theta plus uy sin theta, I will do the same thing for now v. So, if I do then I will have 2 relation and then compare them it is done.

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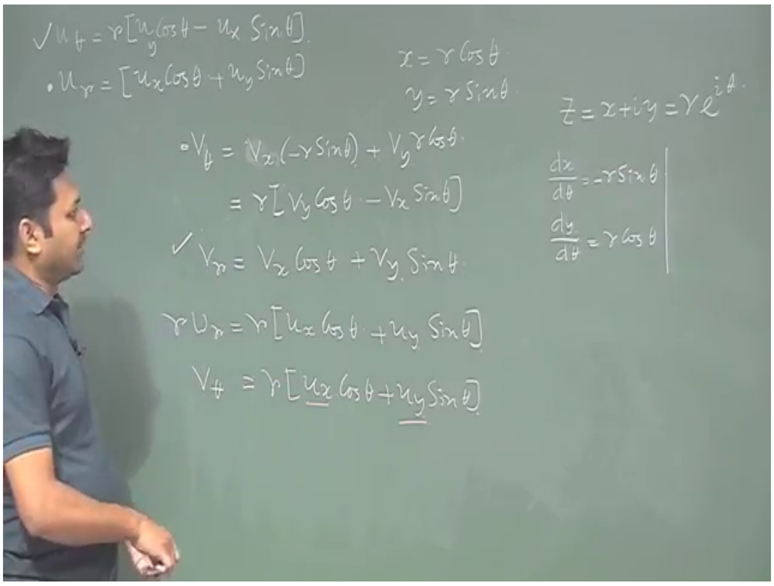

So, in a similar way if I write v theta which is ux then sorry it is vx then dx d theta. So, minus of r sin of theta plus vyr cos of theta. So, it is say r vy cos of theta minus vx sin of theta, what about vr, vr is vx into del x del r. So, I am doing the same thing. So, that is I am not writing explicitly I believe all of you will get because the same way I calculate this u, I am doing the same thing for v expression is same and then ur drd thing; that means, it is sin theta.

Now is a time to compare these 2 things and then we will be getting something. So, let us start with this if I. So, it is ux uy. So, here so now, if I compare this and this and this and this I will get something, let me first compare these and this if I write this equation r ur it will in the right hand side it will be r ux cos theta plus uy sin theta, now if you look this equation v theta and now v theta if I write it here v theta r vy is v ux is vy. So, vy is ux.

So, ux cos theta and vy is minus of vx so plus vuy, uy is minus of vx I am using the Cauchy Riemann equation right now. So, replacing vy and vx in terms of ux and uy if I do that then I will find both the things are same, these things and these things are same mind it this is v theta I just replace this quantity and this quantity this and this quantity by using here, here to here and here to here by the Cauchy Riemann equation. So, these 2 things are same. So, when these 2 things are same I can write the expression between.

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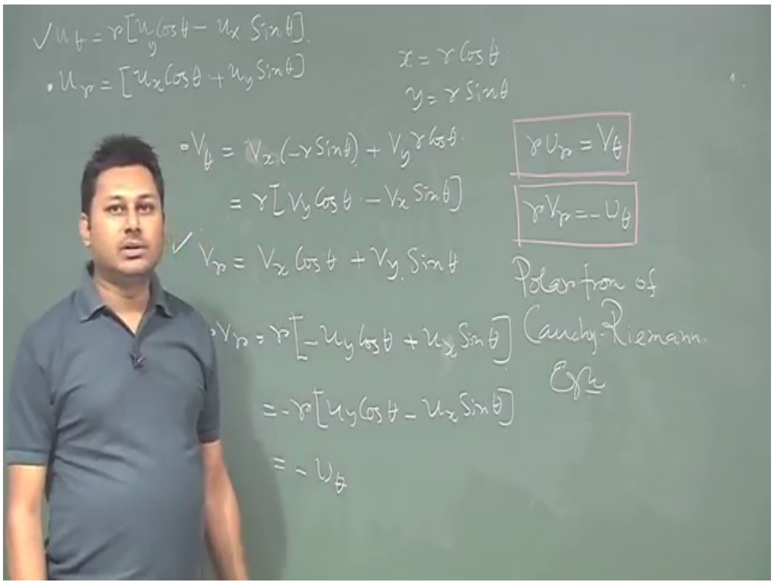

So, this is one expression we have and the expression is r of ur is equal to v theta this is the polar form of Cauchy Riemann equation one equation.

And next I will compare the other one and the other one is this and this. So, again if I multiply this with r so r vr is r, I will change this v to u. So, vx is minus of uy cos theta and vy is ux of sin theta if I do that this quantity this quantity is here, uy cos theta and ux with this negative sign. So, I can write it like minus of r uy cos theta minus ux sin theta, this quantity from here I can write minus of u theta.

So, my second expression is r vr is minus of u theta. So, these are the 2 expression that we have this is the same expression Cauchy Riemann equation, this is the same expression or the same relationship that we have in polar Cartesian coordinate. So, this equation now in polar coordinate, this is the polar form of, polar form of Cauchy Riemann equation. So, polar form is Cauchy Riemann equation is this and Cartesian form that we already know.

So, here I like to stop because after that we will go to the next topic. So, before going to finish let me quickly do one easy thing. So, that you will understand the consequence of this equation so let me try whether I can finish within the time frame. So, this is my equation Cauchy Riemann equation.

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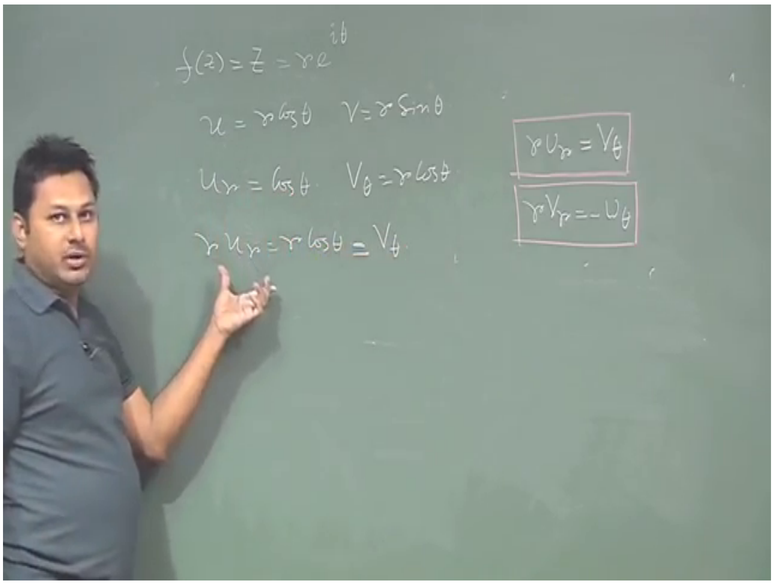

Now, a function is given fz is z I know this z is the analytic function z is r e to the power i theta. So, what is my u here u is r cos theta and my v is r sine theta.

Now, if u and v are given are they satisfying this equation because this is a analytic equation, analytic function. So, it should be this satisfying these things. So, let us do that, ur is how much cos theta ur is cos theta and u theta is let me r ur is how much r ur is r cos theta, what is v theta, v theta is r cos theta. So, this is v theta. So, first equation is satisfied quickly first equation. So, r ur is v theta they satisfying from. So, z I the most simple function I am taking because it is an analytic function if it is an analytic function whether they are satisfying this Cauchy Riemann equation in polar form or not I am taking this as a polar form.

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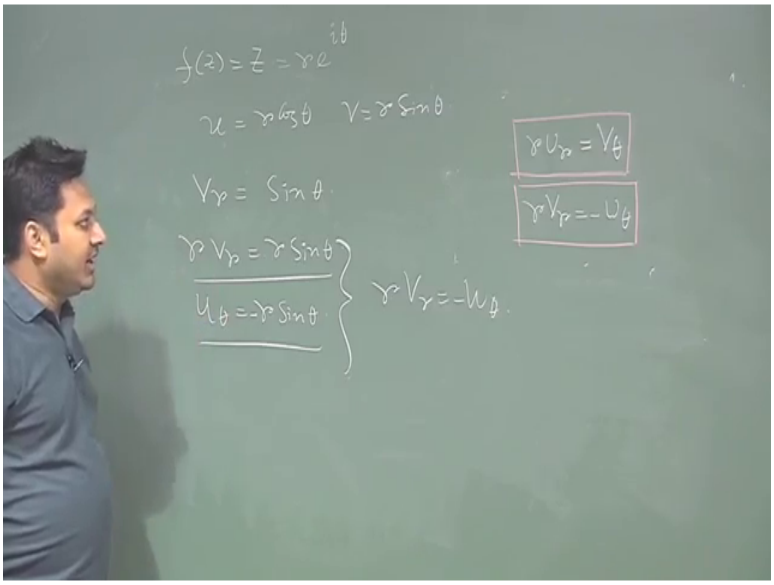

So, first thing I check what about the second one r vr is minus theta or not. So, rvr so vr is sin theta, rvr rvr is r sin theta and what is your u theta, u theta the derivative of this, r with a negative sign minus sin theta. So, these two are again same. So, the relationship rvr is minus u theta is satisfied here. So, you can readily check that when a analytic form is, when a analytic function is given in the polar form this is the way you can readily identify whether the given function which is given in the polar form is really analytic in nature or not.

So, with that I would like to stop here in the next class we will start a very interesting thing and very important thing which is the integration, the line integration and all these thing the concept of complex integration will come gradually come with this concept to. Next class onward we will go to the concept of this as I mentioned integration and all these things and also we will like to know something about the singularity what is the meaning of singularity and all these things. So, with that I would like to stop here.

Thanks for your attention.