**Mathematical Methods in Physics-I**
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**Lecture – 37**
**Harmonic Conjugate**

Welcome student to the next class of our complex analysis. So, in the previous class we started a very important concept of complex analysis which is called the analytic function analytic function is some function f z.

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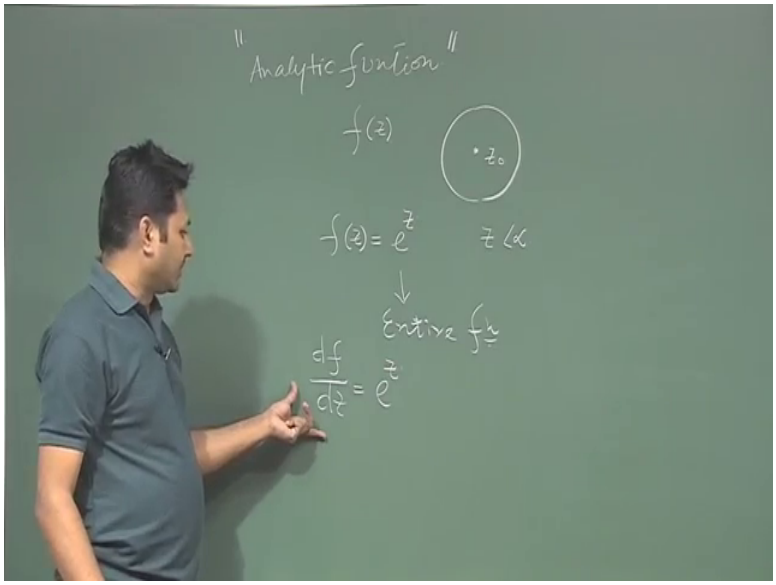


As I mentioned last day that it should have a derivative at some point say z 0 and also in some neighborhood of z 0 like this. Now f z can be such a function, we also give some example like e to the power z, this is a well behaved function and this function should have a derivative in the entire range with the restriction that z should be less than infinity. In this kind of case, the function is called for this kind of case the function is called the entire function entire function

Apart from that there are a few functions are there which should have some kind of singularity that I would like to cover maybe in this class if it is not possible in the next class, but for the timing you should note that this entire function are well behave a function and it just follow; if I do the derivative; for example, in this case; the result will

be simply e to the power z exactly like real function and this all well behave function and the normal polynomial functions like z square zn are following the same rule.

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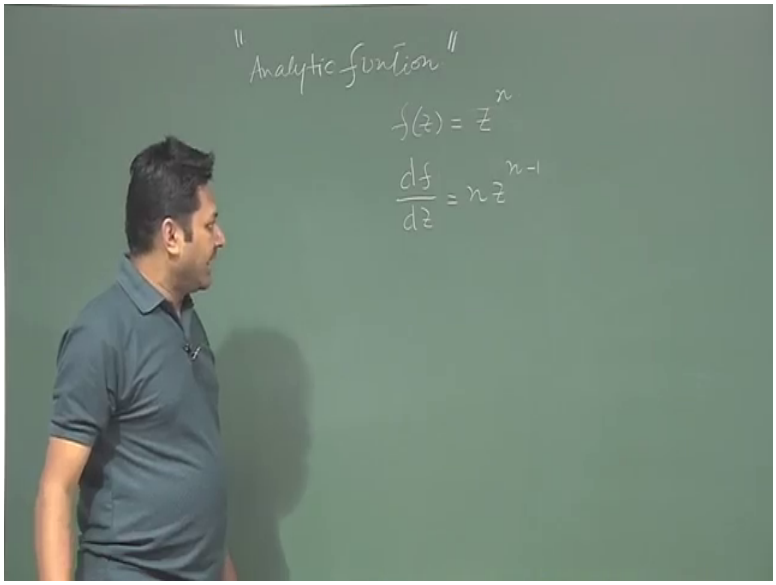


So, let me just give you this things say if it is z to the power n; for example, d of f dz is n z to the power n minus 1 as usual the rule is as usual.

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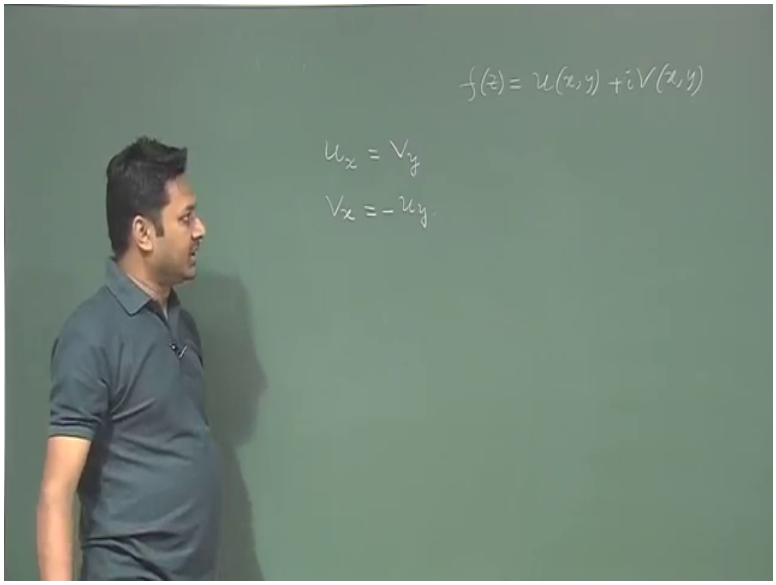


Now, the next thing is that for this analytic function they should obey if the function is represented as u and v both are function of x and y. This is the standard form, then u x is equal to V y and V x is equal to minus of u y. These 2 equations simultaneously satisfied

if the function is analytic and also these 2 functions should be this 4 quantity u x, u y, V x, V y has to be continuous in some domain over x this is a sufficient condition.

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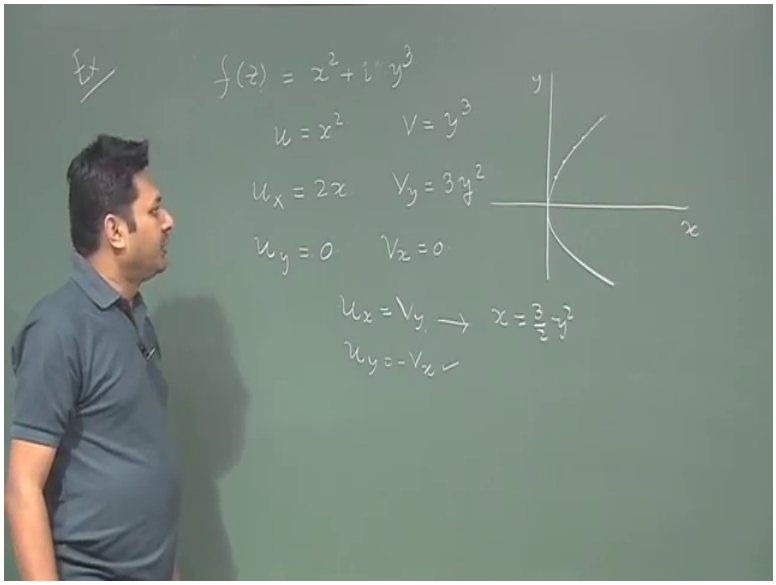


Let me now give you few more examples to. So, that you can understand exactly what is the meaning of these things examples say f z is x square plus I 3 y cube or just 3 y cube is this function is analytic x square plus iy cube.

Now, the standard procedure to find a function is analytic or not. So, here u is x square and v is y square this kind of example also in the last class we did, but let me do it once again. So, u x is 2 y and V y is 3 y square this is one equation second equation is u y is 0 and V x is 0 because u is not a function explicit function of y. So, as V is also not a explicit function of x. So, that s why these things are same 0 and 0 so; that means, I can write that if I try to find out ux is equal to V y, if I try to correlate with this whatever the expression; we have with the Cauchy Riemann equations, then we find that this case, it is valid because u y is 0 vx is 0. So, this is, but these things is not except the fact that if x and y follow certain rules. So, from this equation, we find that x is equal to 3 by 2 y square; that means, it is not true for the entire case only true when x and y follow some relationship.

Now if I plot this relationship x and y I should have some kind of say parabola like this. So, if I have a parabola like this where this points are following the rule followed by this equation, then what happened then what happened that only over this points the function

is analytic so; that means, there is a restriction over this function. This restriction is I should not say the function is analytic, I must say that over this point where this relationship is valid the function is have some kind of derivative.

But if I say this function is whether analytic or not I should must say that it is not analytic. Because at that point the function has some derivative, because there it is over this line say. And now if I have a domain over that and apart from this line, if I try to find out at this point which is slightly shifted from the first one whether we should have the derivative of this function or not then we should not have because in that case this equation not going to be valid.

So; that means, the basic condition the function to be analytic is not valid here.

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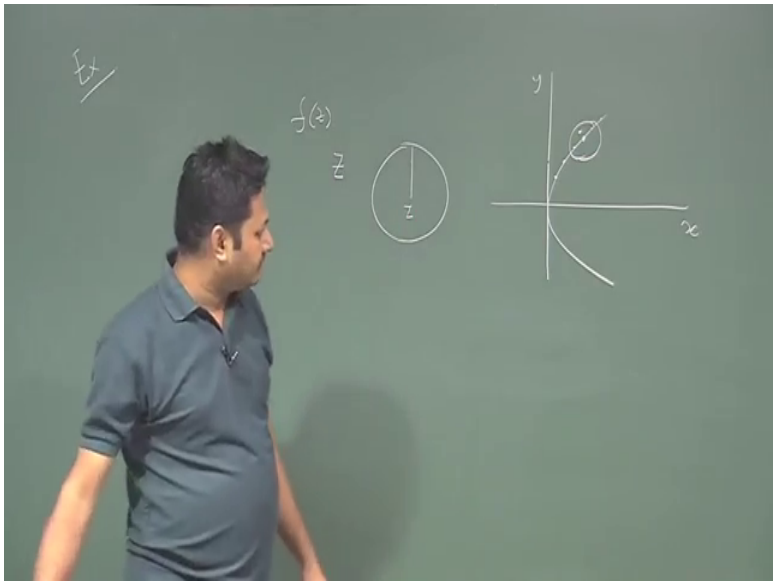


And the what is the basic condition if you remember during the definition I mention that the function fz will be analytic at point z and also some neighborhood of z, here it is analytic, then in some neighborhood of this point also the function has to be here, the function has some derivative and over this region which is the neighborhood of z the function should also have some derivative.

That means, in this region over all region the function should have the derivative in that case, it is not the case because here we have a derivative, but if I go to some other point,

it is not having any kind of derivative. That means, the function critically is not an analytic function rather I should say it should have a derivative over this line.

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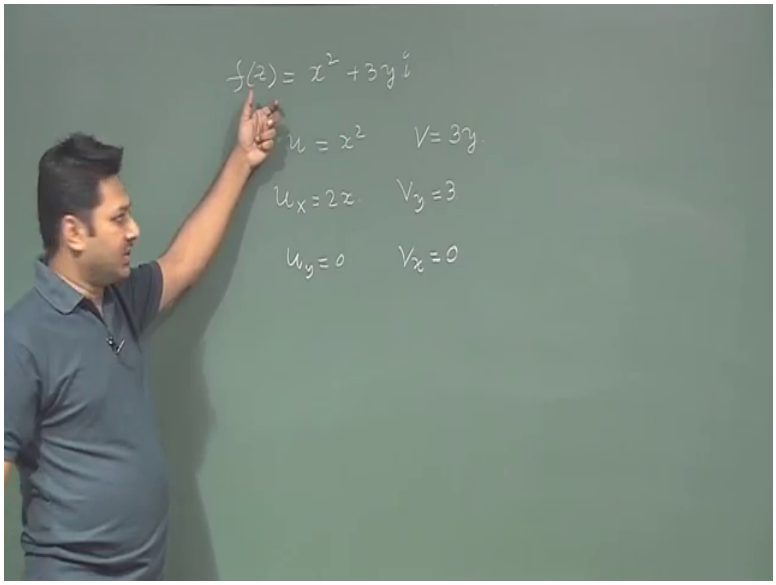


So, more examples; this is a critical examples where we have a relationship; we can have more examples like fx; say x square plus 3 y, it is a similar kind of 3 y i; it is a similar kind of equation and you can do that in the same way. For example, if I do u is x square, I am doing the similar kind of example. So, that you can understand what is basically u x is equal to 2 y and V y is equal to 3 here uy is equal to 0 and Vx is equal to 0.

So, again this condition is satisfied, but this condition is not satisfied except the fact the critical value x equal to 3 by 2 so; that means, a particular point may be this function has some derivative and if I want to find out whether this is true for other point except x equal to 3 by 2 you will find that it is not so; that means, this function also is not an analytic function in general after having these 2 examples.

So, let me proceed with the next thing. I believe you student now understand what is the meaning of analytic function the analytic function is a function which has some derivative in some point I am repeating once again and it should have also the derivative in the neighborhood of that point and now another important thing is you in your hand that is the Cauchy Riemann equation. So, every time a function is given to you; you just apply the Cauchy Riemann equation and try to find out whether this equation is satisfied or not; if it is satisfied, then you can say it is an analytic function and so on.

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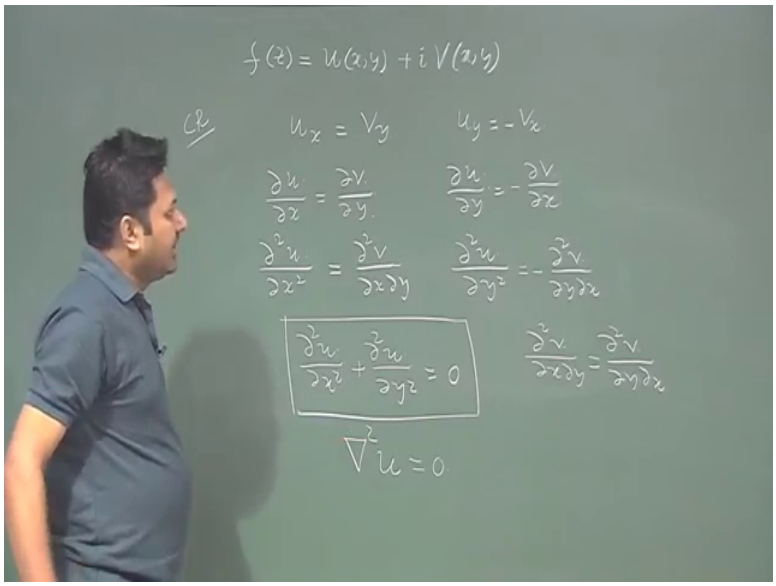


Now, regarding this Cauchy Riemann equation let me write it once again. So, f z is equal to u function of xy plus i V function of xy; this is the standard form of a function in this way I can write and the CR equation suggest that ux is equal to V y and uy is equal to minus of V x; these 2 things is with me, if I write it in just this del notation, it should be something like this. So, let me write it this.

Now, if I make a derivative both side with respect to x in the right hand side, I should have this quantity in that case also, if I do the same thing, but here instead of having a derivative x, I make a derivative with respect to y, once again in the left hand side, the right hand side, I should have now since these two; these two variables are not depend on each other linearly independent rather.

So, I should write this condition and so, once I write this condition from this equation I have a very interesting expression which is this; this is a special kind of equation or I believe you are aware of this sign u is such a function that you should have del square u is equal to 0, it essentially means that u is a harmonic function.

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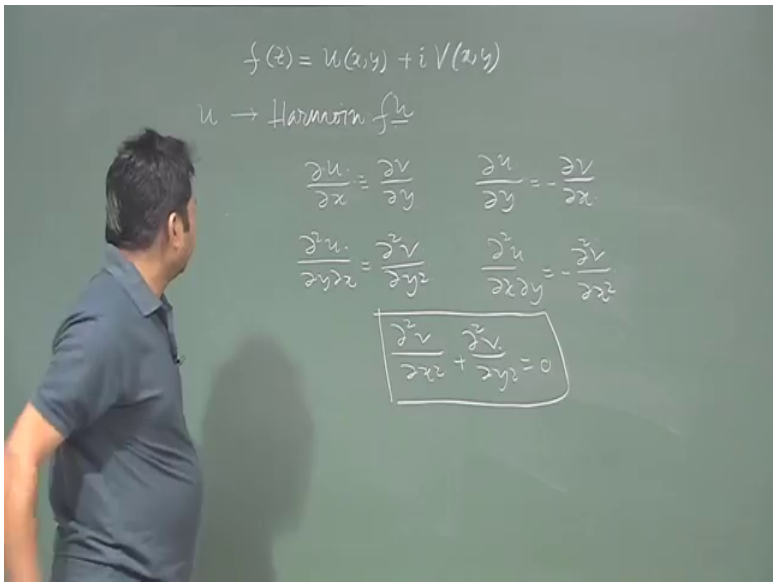


So, u harmonic function if u and v is given to u and if you want to find out what is the nature of u you will find that nature of the u is such that it is following a very nice expression and from this expression you can readily see that u is a harmonic function. So, no doubt about that; what about the b u is harmonic function. So, what about the v if you do that I believe you can do that by yourself ux is e del u del x is del v del y del u del y is minus of del v del x that 2 equations separate equation I have.

In order to find out what is the what is the what is the form what is the differential form of v only to separate out this coupled thing I can do the similar thing here that we did for you quite trivial and from this 2 equation again I can write that d 2 v d x square plus d 2 v dy square is equal to 0 again v is also an harmonic function.

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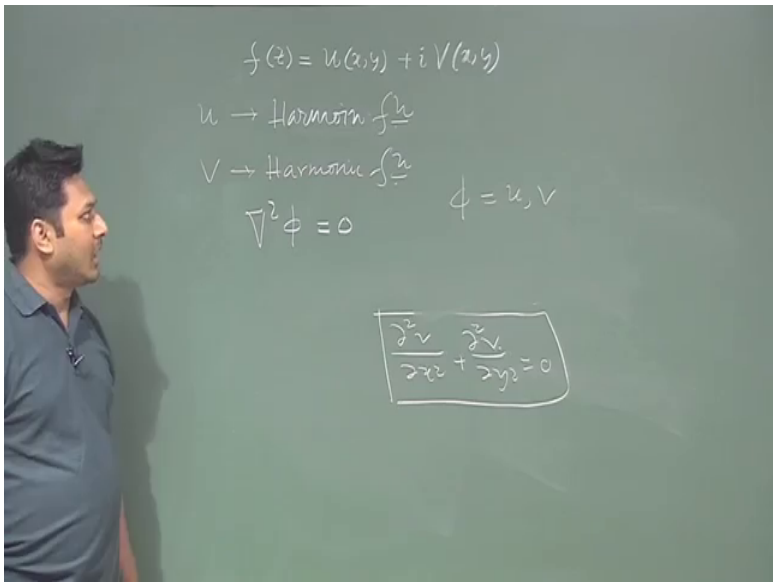


So, u is a harmonic function and also v is also harmonic function. So, Cauchy Riemann equation suggests that whatever the uv you have u and v should follow the expression like this delta square phi is equal to 0 where phi is u or v.

So, after having this small calculation and all these things now we try to exploit this more and try to figure out if u or v is given to you then it is possible to find out the other conjugate so that the entire function should be analytic. So, let me give you one problem.

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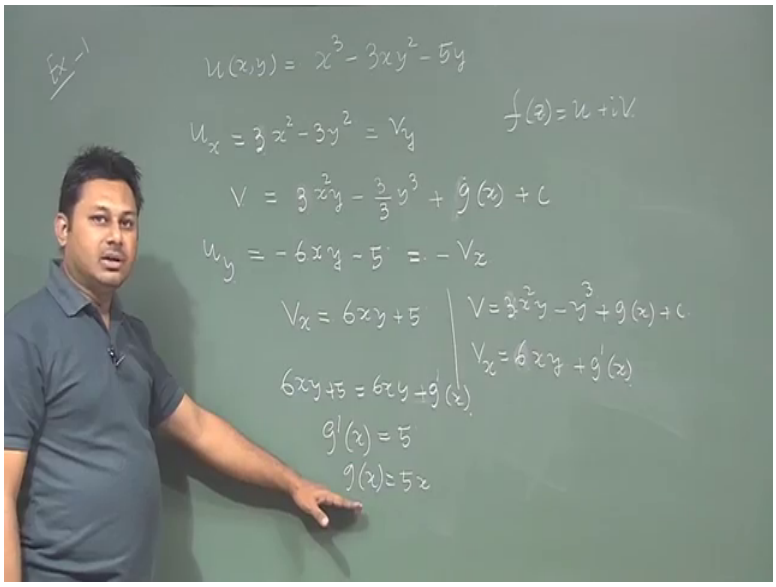

So, I already have one problem; the problem is u is given to you which is a function of u x y exercise one where it is given like x square let me write it minus 3 x y square minus 5 y.

Now, if u is giving to you like this way, a just a functional form now the next thing is that you are asked to find out what should be the v for this u. So, that you will have u plus iv a form which is analytic in nature the problem is that. So, u is given to you; you need to find out from u because your function total function is u plus iv u is given to you; you need to find out what is your f z. So, that the function is analytic; that means, eventually you need to use the Cauchy Riemann equation to find out what is your v. So, let me do that then maybe you can understand how the things is going on.

So, let me first find out what is your ux u x is the partial derivative of this quantity so; that means, 2 of x square minus 3 y square. Now this quantity u x if the function fz is analytic should be equal to v of y now if I write v from this equation v of y is the partial derivative with respect to y so; that means, v if I try to find out should be the integration of this quantity. So, it will be just 2 x 2 x square y because x will behave like a constant here and this quantity is minus of 3 if I want to make the derivative the integration then one 3 will come y cube I am having.

When I am making the integration you should remember that I am integrating with making y making x as a constant here. So, x here is a constant and since I am using x as a constant like making the deri partial derivative with respect to x I when I do that the when I do the partial derivative with respect to x what we will do that we make y as a constant here I am doing the opposite thing V y is given which is the partial derivative with respect to x I am integrating it to find out what is my v when I am integrating it my x should be constant. So, I am doing the same thing here. So, when I am doing this I will have these things. So, I should put another function of gx as I should put another gx function gx as a constant, because as I mentioned this and also I can put one c here. So, that I can say my general form of v is something like this.

Now, once I have this also the Cauchy Riemann equation gives me another expression. So, another expression is uy uy is the partial derivative of this quantity with respect to y. So, that gives me something like if I do the derivative here. So, it will be minus of 6 minus of 6 x y minus 5 this quantity which is this quantity is equal to minus of V x this

quantity is minus of V x. That means, vx, I know this quantity. So, vx is my 6 xy plus 5 from here I already have my v. So, my v is 2 of x square y minus y cube plus g of x plus c.

Now, if I compare these things and these things I need to make a. So, if I make a derivative if I make a derivative of this quantity then I should have vx and vx should be something like if I make a derivative with respect to x it should be 4 x y this is a function of y. So, I am making a derivative. So, this quantity is not be there plus g prime x . So, I will have this quantity and this quantity which suppose to be the same the goal is to find out gx, because already I already find what is my v here. So, v is an expression where I have some unknown quantity gx and some constant arbitrary constant c, but this function of g I need to calculate and when I try to calculate this I need to use the equation another set of equation.

So, let me go back once again and check whether everything is correct or not. So, my original function is given x square plus minus 3 x y square minus 5 y. So, ux is the derivative of this quantity with respect to x when I do that here I am making one mistake it should be 3 3 x square, when I am making the derivative in hurry I am making some mistake. So, 3 x square minus 3 y square, let me check whether any other mistakes are there or not otherwise we will get a different answer. So, it will be V y because the Cauchy Riemann equations such as it will be Vy. So, v should be equal to the integration of this quantity.

So, integration of this quantity gives this quantity should be changed by 3 and other thing I hope other things are. So, it will be y cube divided by 3 and because of the integration I will have gx plus c. So, uy, again I can do the same thing with respect to y I can derivative when I make a derivative with respect to y first term will not be there second term will be 6 of xy minus 5 which is equal to minus of v x. So, Vx is this quantity v here I have 3 here. So, essentially when I make a v this should be 6 x y now seems to be fine.

So, now I come, if I compare these 2 things then I should have 6 of xy plus 5 is equal to 6 of xy plus g prime x this equation I have right now. So, if I write. So, g prime x is 5. So, what is my g gx is my 5 x. So, that is my goal I need to as I mention I need to find out what is the function here what should be the function here because this is this gs of

issue I need to put a function of x, because when I am integrating to find out my total v I consider x as my constant. So, I need to put this function here with some arbitrary constant c and using another equation; I should have this equation where my g is given. So, g; I can find out.

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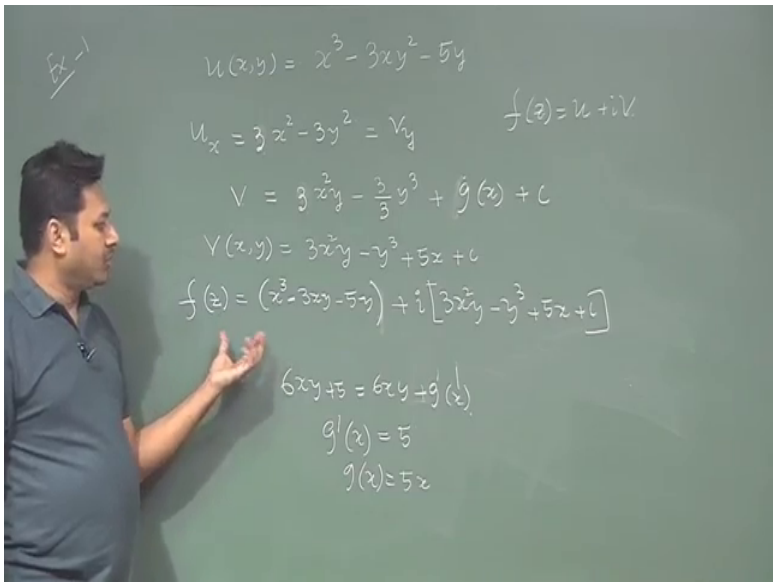


Now, if I put it here then the total expression is complete. So, my v finally, v of xy finally, comes out to be 3 x square y minus y cube g x; now I figure out. So, it is 5 of x plus c and u is already given you is already given. So, u and v I figure out separately. So, the entire function which is analytic for the given u should be something like x cube minus 3 x y minus 5 y plus say I can put this c here plus c plus or c is a constant. So, you can put this in the real part or in the imaginary part that does not matter.

So, right now since c is c is with v. So, let me write it in this way only plus I 3 x square y minus y cube plus 5 x plus c . So, my fz the complete f fz, I figure out which is u plus iv and this u is given to me I was asked to find out what should be the corresponding v. So, that the entire function should be analytic and I figure it out and it comes to be something like that.

Now, another thing from that whether this u is given is the harmonic or not because harmonic as I mentioned that you.

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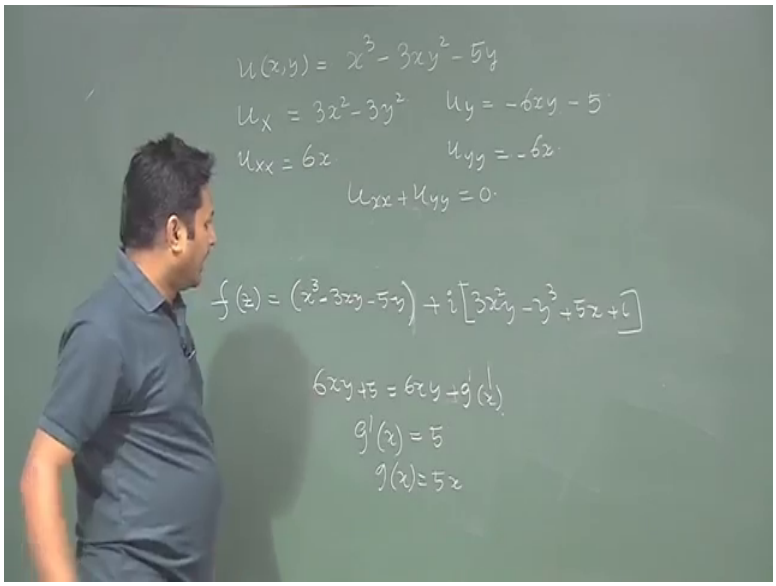


If the function is analytic and following the Cauchy Riemann equation then whatever the you I am getting should be harmonic in nature so; that means, ux is again 3 x square minus 3 y square u xx is 6 x; this quantity uy is how much; uy is minus of 6 x y minus 5 uy; y is this is constant. So, it will be minus of 6 of x. So, from here and here I can write that u xx plus u y;y is equal to 0; that means, u the function that is given to me is harmonic in nature.

Now, if this is analytic then v is also we should also harmonic and you can like you can take it as an homework to figure out whether this v whatever the v I am figure.

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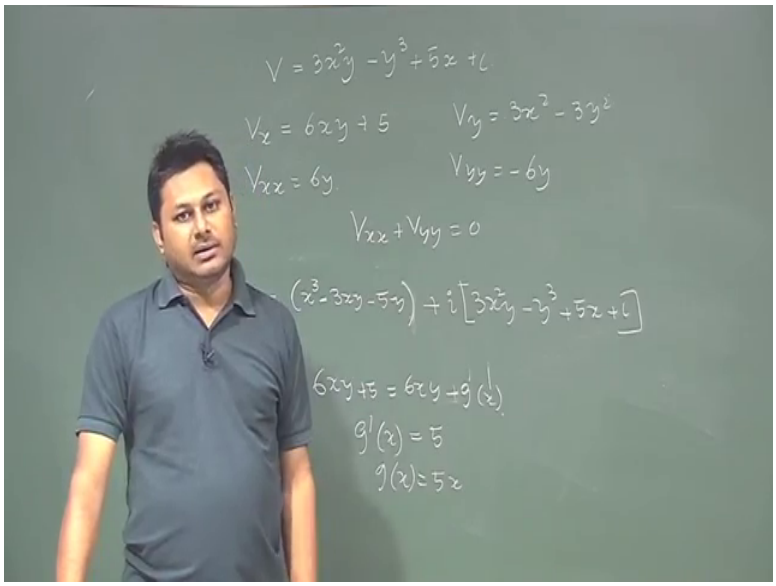


So, let me do that. So, v here is 3 x square y minus y cube plus 5 x plus c. So, V x is the derivative with respect to x. So, it is 6 x y plus 5 V x x is 6 of y V y is 3 x square minus 3 y square v yy is minus 6 y. So, v xx is 6 y v yy is minus 6 y. So, vx plus v yy is equal to 0. So, v is also harmonic just to show that whatever the function is generated through the Cauchy Riemann equation is also following the fact that they are harmonic to each other well at that point, I like to stop this class, I like to finish here in the next class, we will go forward with this and try to find out a few other important aspects of analytic function and also figure out in polar coordinate how this form will look like. So, far we are dealing with the Cartesian coordinate and in Cartesian coordinate the CR equation looks something.

So, in the next class maybe we will try to find out in the polar coordinate how it will look like and few other important things also with that note. So, let me conclude here.

Thank you for your attention.