Mathematical Methods in Physics-I
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Lecture – 36
Analytic Function

Welcome back student to the next class of complex analysis.

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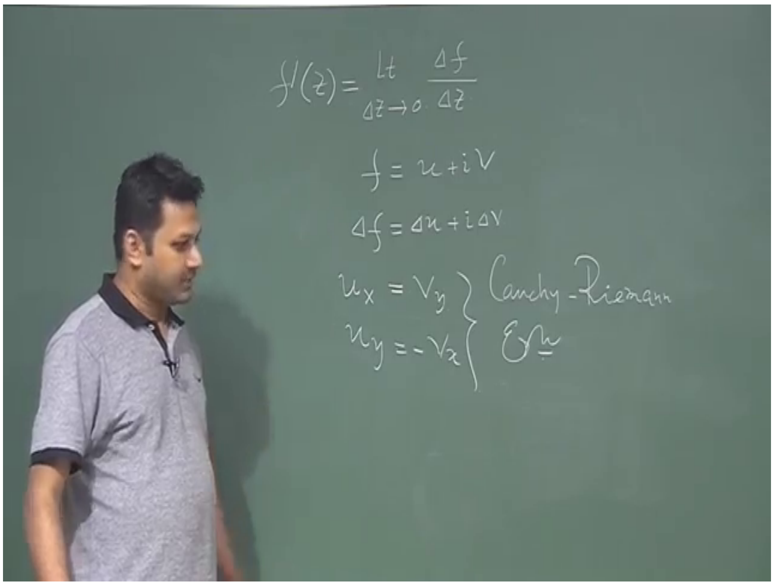

In the previous class, if you remember; we defined the derivative of a complex function fz which is defined as this; like this, f was u plus iv and delta of f was delta u plus i delta v and then we say that this derivative should exist if ux is equal to Vy and uy is equal to minus of vx. This equation was called Cauchy Riemann equation. Sometime it is pronounced at Cauchy it is up to you; sometime is it pronounced Cauchy.

So, this condition is very important condition for a function complex function because if this condition is there; as I mentioned in the last class, this is a necessary condition a function for a function to have the derivative. So, it is not a; I mean it is not a sufficient condition, the sufficient condition which is a very tricky way, if you want to prove that the sufficient condition, it is a very tricky way to prove that, but I can say that the sufficient condition is if these things are continuous in a region if it is equal, then this is fine apart from that another condition, we should impose over that that this is continuous over a region, then we call that this is a sufficient condition to have the derivative.

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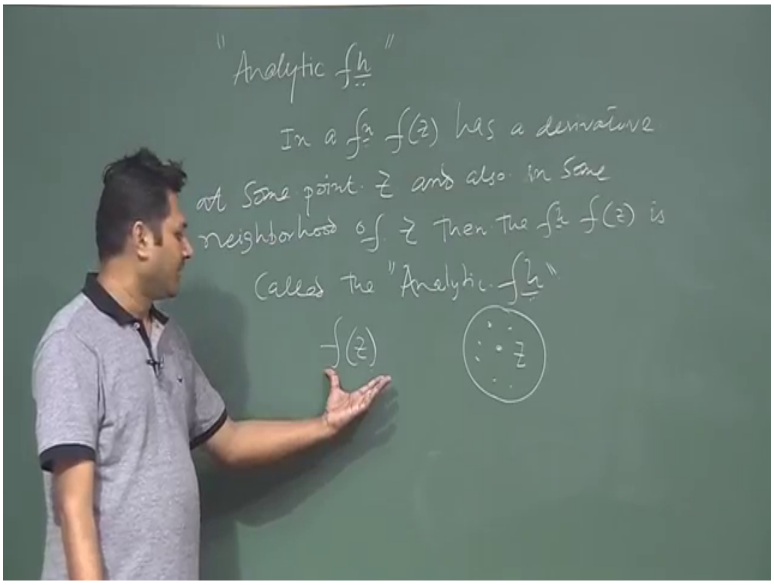

After that; after having the knowledge of these things that a function should have a derivative CR equation or Cauchy Riemann equation is satisfied, then we should define another thing called analytic function which is a very important term analytic; very very important. So, what is let us try to understand first that what is the meaning of analytic function. So, if a function fz has a derivative at some point z and also on some in some neighborhood of z, then the function fz is called the analytic function a very straightforward definition you will find this definition in any book, but you should understand that if a function fz has derivative in some point z.

So; that means, the function should have some derivative at some point z and also in some neighborhood of z; that means, if I have a function fz and this is the point say z the function has a derivative over this point z, it is not that only the function should have the derivative at that point, but I can have a neighborhood; that means, a region surrounded by a curve or something where z is inside. So, this region is a neighborhood of z; the function also has a derivative at the other points; that means, if the function has a derivative this point; the function should have derivative; here in all the points in the neighborhood.

That means the function should not have the derivative only that point, but the region of this point if that is the case, then we should have; we should call this function the analytic function in that case we should have the relationship after having the 2 definition.

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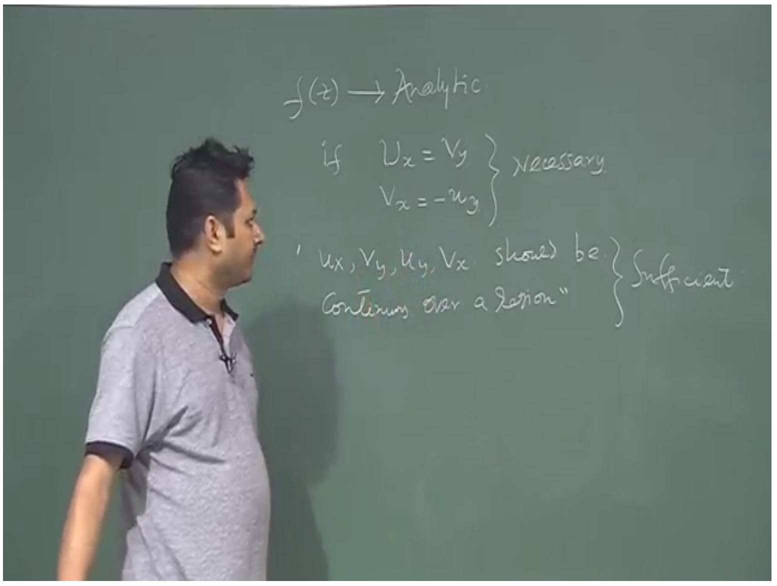

I should say that the function another way that the function fz is analytic, I say it is analytic, if Ux is equal to Vy and Vx is equal to minus of Vy or uy is equal to minus of Vx whatever.

So, this CR equation; so, satisfy; so, I will have a function where the CR equation is satisfied, then I can say if this is satisfied which is a necessary condition, but not the sufficient condition as I mentioned, but we will go with this necessary condition and say that this is; if this is the case, if this relationship is followed, then we can say the function is analytic. So, this is a necessary condition to have the function analytic and for sufficient condition to have the function analytic, we must say that these; this 4 quantity you; this is a necessary condition; let me write it ux, Vy, uy, Vx should be continuous on a over a region; this is sufficient.

So, the necessary condition is that and sufficient condition is there has to be continuous over a region; fine; so, another way to define the analytic function; so, let me first give you a very simple example of analytic function.

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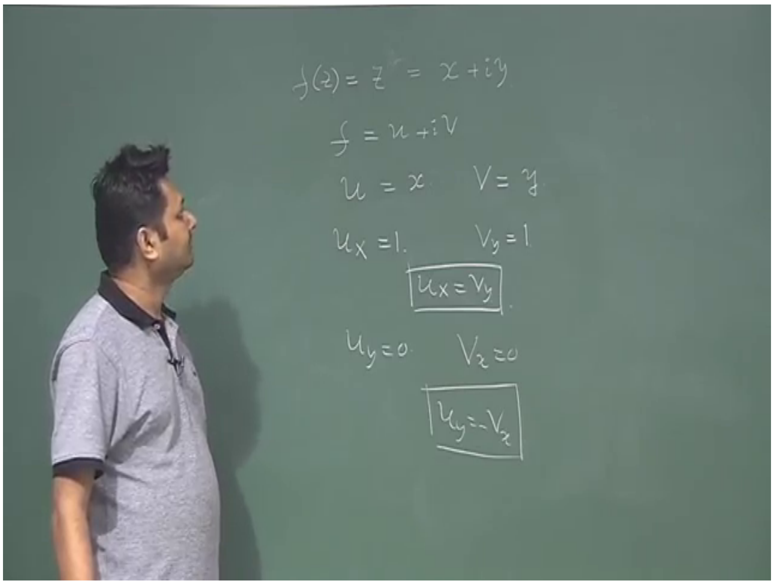

So, fz is equal to z square more simple example is the function is analytic forget about z square; let us start with fz is equal to z is this function analytic. So, let us try to find out it is x plus iy, right. So, I know that my f is u plus iV. So, every time I should write it u plus iV form and then u is my x and v is my y, I know what is u? U is a function of xy. So, here we have u is x, v is also a function of xy I have u is v is y.

So, now I say ux is 1, Vy is also 1 so; that means, my first condition ux is equal to Vy is satisfied what about the next one; uy is 0 and Vx is 0; that means, also in principle my second condition uy is equal to minus of Vx is satisfied; since both the condition is satisfied here, I must say this function is analytic function after having this we will go to more and more examples, but before that one interesting things, I like to show here and that is what is the mathematical meaning of a function to be analytic.

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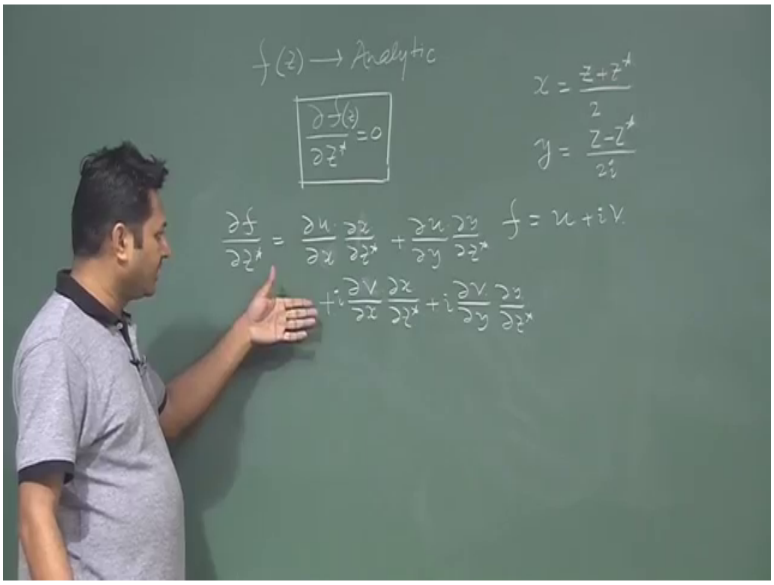

So, fz is analytic means essentially del f del z star is equal to 0, this is an argument I like to mention; that means, if a function f; fz is not explicit function of z prime, then if I make a partial derivative with respect to z z; here z prime and z star means the complex conjugate better to say; this is a z prime not z star not z prime.

So, if the function of fz whatever the function is given to you, if you find it is not a explicit function of z star; that means, the complex conjugate of z, then the partial derivative with respect to has to be 0, I say; I am saying that fz is analytic, if it is not an explicit function of z star, this argument and the CR equation are essentially the same thing we will going to show that. So, what is my x? It is z plus z star divided by 2, I can write my x in this way also, I can write my y in this way z minus z star; that means, I just remove these things and divided by 2.

So, now I like to write these things in terms of x and y and then find then try to find out what is the meaning of that. So, if I do so; how the things will be there. So, del f del z star if mind it is u plus iV. So, if I try to write this partial derivative f is a function of u and y. So, I can write it in this way del u del z star say del; del u del x del x del z star plus del u del y del y del z star plus del u del v with a i because this is a variable, this is the variable where f is defined by x plus iy.

So, if I do that then I should have this quantity x del x del z star plus i del v del y del y del z star the separation of variable; you can do that by just write del f del x and del f del y and del f del x again you can write in terms of this. So, it will come to the same form. So, I mean I just do that at a stretch; if you write for example, let me write it here.

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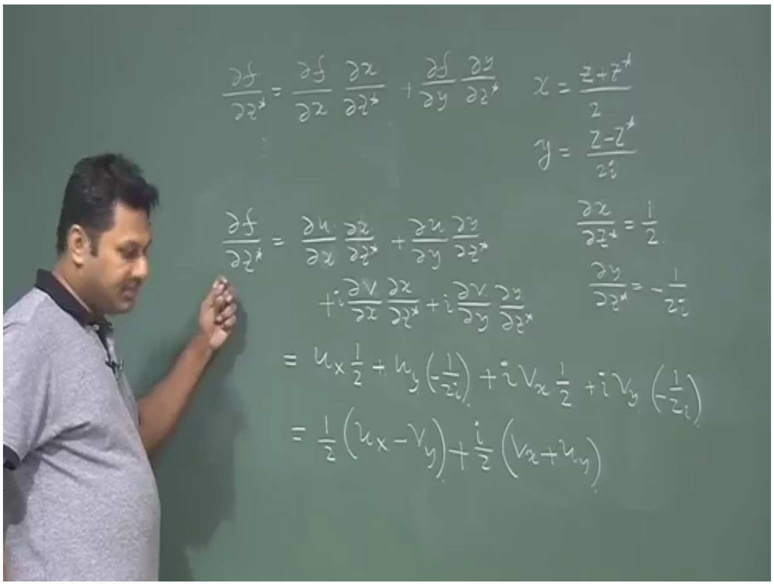

So, del f del star I hope you can understand this, but if you do not; if you have any confusion.

So, let me write it once again what I am trying to do it may be helpful for you if you do not get it and then del f del x; I can take it from here. So, it will be simply del u del z thing and plus i del V del thing; these things and these things. So, now, I can write it as ux and del x del z because I have the relationship between x and z. So, I can take out this quantity from here. So, this gives you that del x del z star is equal to half in a similar way del y del z star is equal to minus of 1 by 2 i.

Now, I just put it here. So, del u del thing is half plus this is uy and I should put a minus of 1 by 2 i here because del y del z star is this quantity plus i del v del x is Vx; these things again we have half and another thing is here which is plus i V of y this things is minus of one by 2 y. Now, I can combine the real and imaginary part because this is real, again, we have a real. So, if I rearrange this term, I should have half ux minus Vy.

If I multiply i here; so this term is be it will be simply i by 2; I will have Vx plus uy, this 2 equation; I will have when I am making del f del z star, then I will have these things; now you can readily understand that if this is 0.

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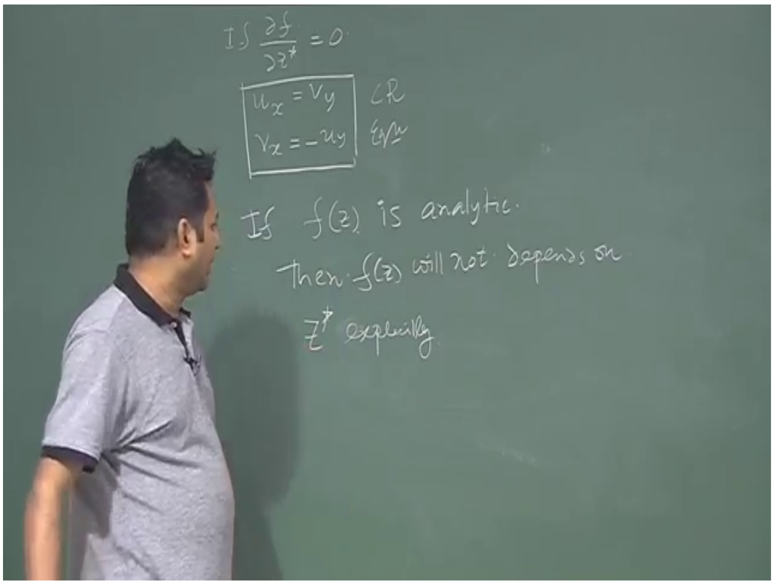

If this quantity is 0 that i from where I started, if del f del star is 0 that essentially means this is 0 and this is 0 because this is the real part and this is the imaginary part.

So, this real part and imaginary part should 0 individually if this is the case, I can readily have ux is equal to Vy and Vx is equal to minus of uy which is nothing, but our old CR equation; this is Cauchy Riemann equation. So, the Cauchy Riemann equation whatever I am saying is nothing, but a mathematical argument over the analytic function and it suggests that the analytic function should not have an explicit dependence of z star fz, if fz is analytic, then fz will not depends on z star explicitly; that means, fz should not be explicit function, fz should not be the explicit function of z star this is nothing, but the mathematical expression we show in the top that is del f del z star is equal to 0.

So, we find 2 interesting aspects of the analytic function first of all if a function is analytic we find a necessary condition that CR equation should satisfy that is one and also the sufficient condition that they should be this; this quantity should be continuous, it is not that they are just a single point they should have the value equal second thing is that the function f the function given function fz is should not be the explicit function of z star, if it is explicit function of z star, then this equation will not going to satisfy if this is explicit not f an explicit function of z star, then eventually the CR equation is satisfied.

So; that means, CR equation is satisfying for the function when z star is not an explicit dependence of that; let us now do some straightforward example and figure out that is all; I mean now we are in a position that we know how the analytic function behave and what we will do that we will just find out how the different; I mean how the different function behave with that.

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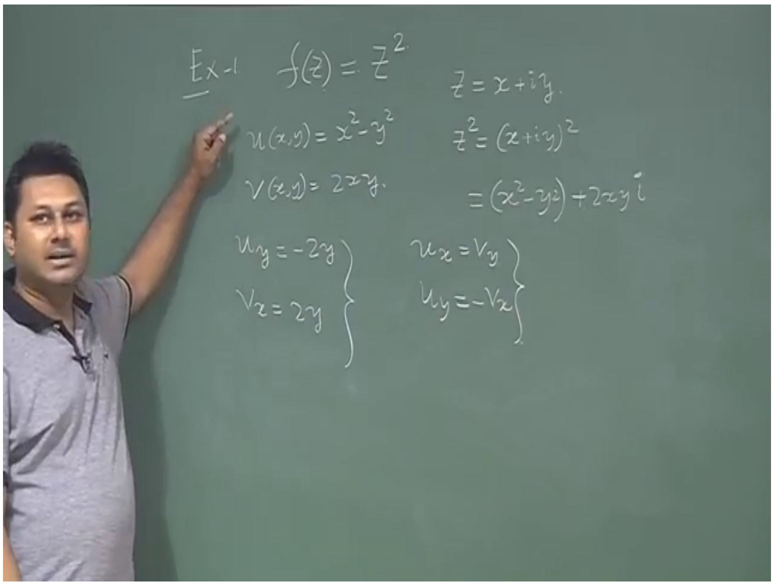

So, let us try to start with example one; we wrote it earlier that z square; is it a analytic, is it an analytic function or should we have a derivative over that. So, first thing you should note that this function is not an explicit function of z star you will not able to write this function in terms of z star at all if you do not, then readily you understand that this function whatever the function is given should be analytic, but we need to prove with the by using the CR equation.

So, z is equal to x plus iy z square is x plus iy square which is x square minus y square the most simple example we have right now 2 xy with multiplication of i. So, from here we know that for this function my u xy is x square minus y square my v xy is 2 of xy these 2 things. Now I will try to find out what is my ux? Ux; this quantity is 2 of x; what is my Vy? Vy is the partial derivative with respect to y of this function v which is 2 of xy. So, it will be just 2 x and they are same.

So, from here; we manage to find out that this is ux and Vy are same one condition what about the second one second one is uy; what is uy minus of 2 y what is Vx? Vx is 2 of y; now again the second condition satisfied because the second condition says that uy should be equal to minus of Vx. So, uy here is equal to minus of Vx; these 2 condition readily gives us the idea that this function is an analytic function after that.

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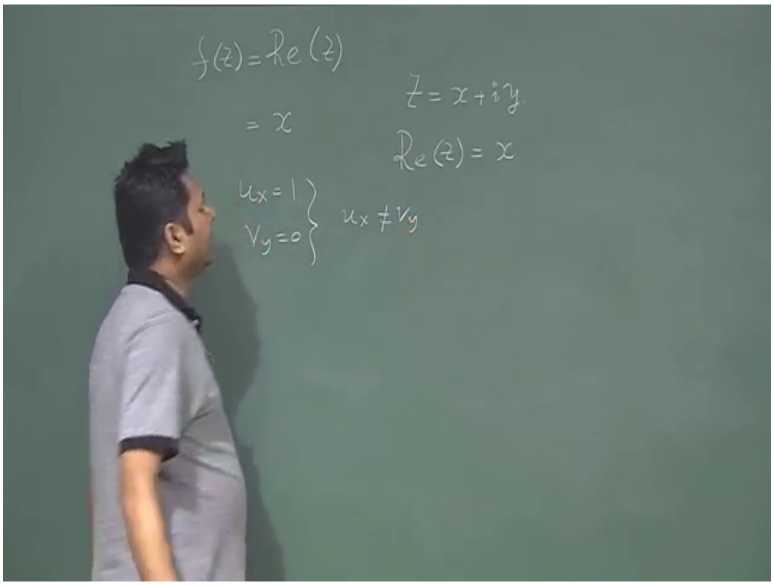

Let us give some fz is real of z if somebody gives you this these things f z is the real of z; what is the meaning of that real of z; what is the function although function will look like? So, real of z is very simple x is equal to xy z is equal to x plus iy. So, real of z is equal to x. So, this quantity is simply x. Now the question is it is analytic function. Now 2 way you can check; first you can use the Cauchy Riemann equation. So, this is you; if you write u plus iV, then you will find that u is your x and V is 0.

So, readily you will find that ux will be 1, but Vy is 0 it is not matching so; that means, they are not same ux is not equal to Vy so; obviously, this is not an analytic function, but apart from that also you can find out this is non analytic function because these the these the next thing.

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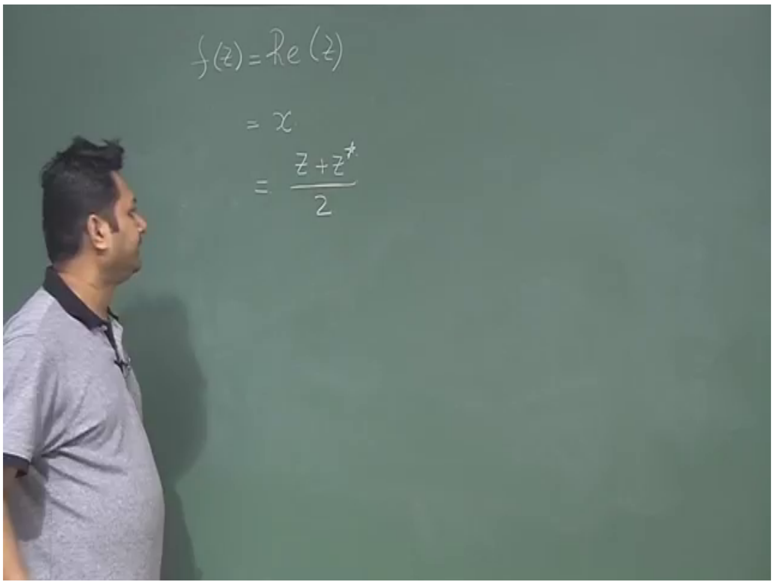

I show x can be represented as z plus z star divided by 2; that means, function of z function of f, it is the is an; it is an explicit function of z star, if it is an explicit function of z star; that means, I never say that this is an analytic function that we already prove.

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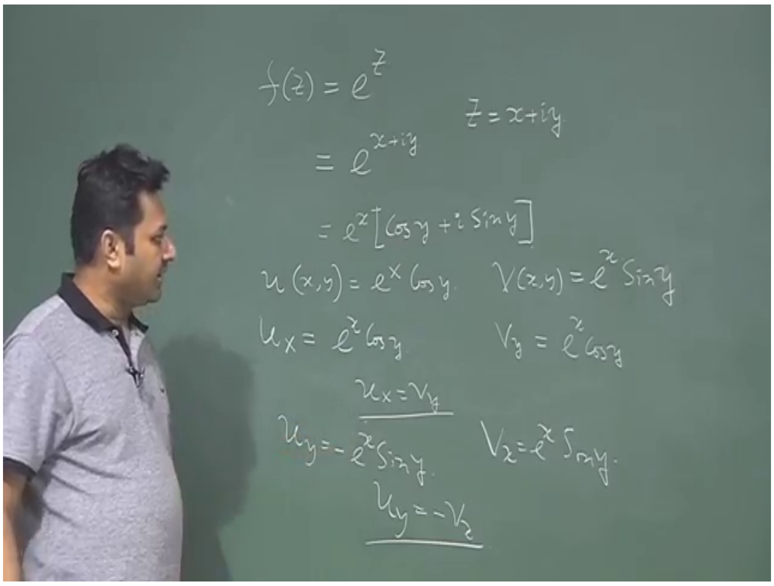

Let us go to another example that function of z is e to the power of z e to the power of x we know e to the power of z is given to you, but you should not worry about these things because z is equal to x plus iy. So, I can write these things here x plus iy and from the function, we can say that if I expand e the normal way, we expand in real casem it will be a power series over z. So, there is no way that z star will come into the picture so; that means, it seems that this function should be analytic function it seems, but we need to prove that. So, I will just divide in these 2 things in this way.

So, now I can write e to the power ix multiplied by e to the power iy e to the power x multiplied by e to the power iy e to the power iy is just cos y plus i sin y. So, from here I can write my u xy is e to the power x cos of y and my v of xy is equal to e to the power of x sin of y when we have u and v in this form, then we are in the position to write what is ux; ux is a partial derivative with respect to x; if I do it will be simply ux cos y, what about the Vy? Vy is the partial derivative with respect to y, if I do this, here I should have e to the power x sin y is cos y and from here I can write ux is equal to Vy.

So, now another equation, I need to find which is uy; uy is minus of e to the power x sin y and V of x is e to the power x of sin y uy and Vy are related with fact because of this negative sin; uy should be minus of Vx again the second equation satisfied. So, the first one and second one are satisfied together; that means, the CR equation is satisfied when the CR equation is satisfied you can say this function is analytic.

Now the last thing where after that I am going to stop that if a function say a function fz.

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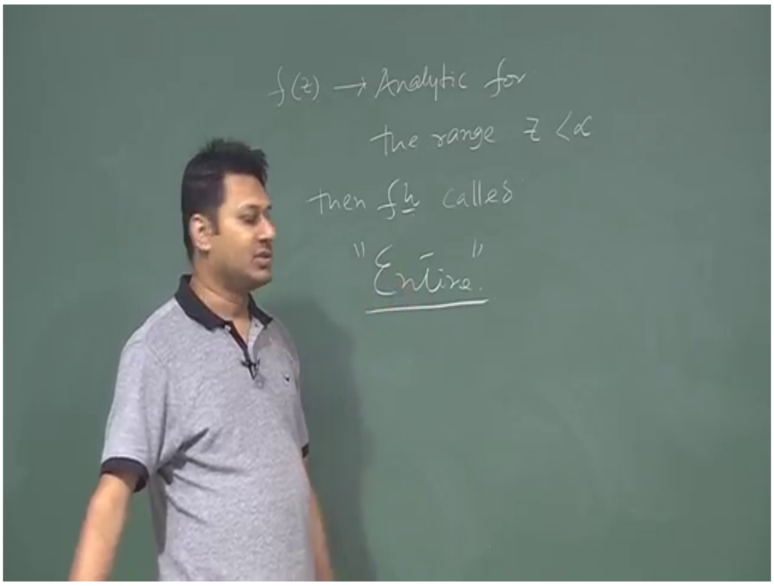

I say this is analytic in this region and also we have some derivative over the region that is the definition of analytic. So, now, if I say fz; if is fz is analytic for the range z less than infinity; that means, I want to find I say the function I find a function which is analytic; that means, it is it has some derivative with some z and also the neighborhood of z, but I am saying that the range of this z is less than infinity; that means, entire almost entire plane except z into infinity point.

Then this function has a special name called entire function; this is called the entire function what is the meaning of entire function entire function is this some kind of function which is analytic over a range of z which is less than infinity. As z tends to infinity the function may not have the analyticity, but apart from that it is analytic in the entire range in this case; this function is called the entire what are the examples of some entire functions say.

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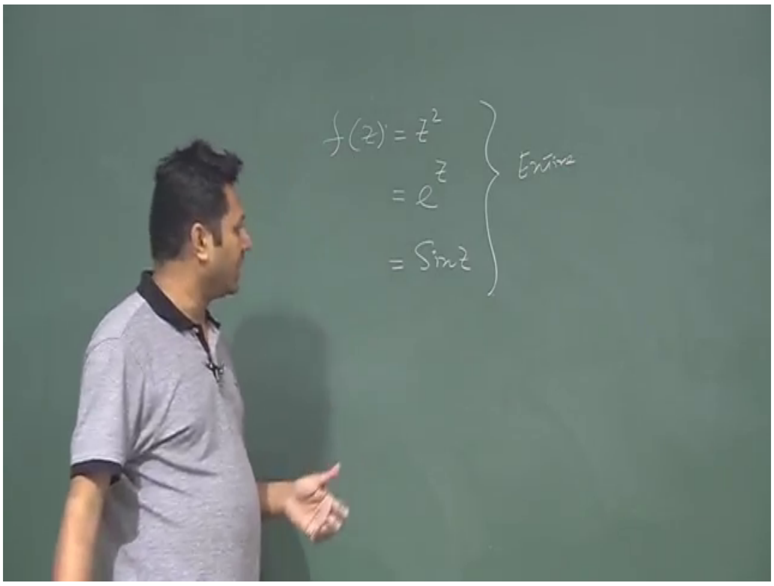

This is also very well behaved function say z square or any polynomial of z e to the power of z sin z cos z and so on.

So, all will be functions are in the set of entire function these are the entire function. So, entire functions; we will come to some different kind of functions in the maybe in the next class we will encounter different kind of functions which is which has some kind of singularity which is very very important in the context of complex analysis, but before going to that step, we need to know what is the meaning of entire function that the derivative.

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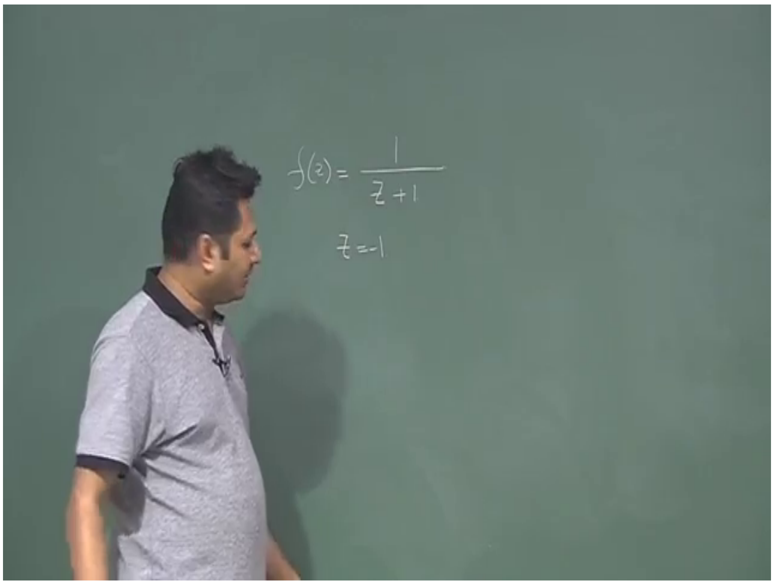

We have a derivative wherever you want you have the derivative except the point z tends to infinity z is infinity then, but you can readily understand that for example, z plus one this is also in function, but here the function the way the function is given is not an entire function because at z equal to minus 1; the function will blow up and as a result will not able to define this function at z equal to minus 1, but this point is not z intends to infinity this is minus one is a finite point.

So, this kind of function is not entire function it should have some kind of some kind of singularity, but that thing we will cover in our future classes. So, today I am going to stop here and today we learn a very important thing that the entire the function is called analytic if this function has a Cauchy Riemann they satisfy the Cauchy Riemann equation we show some kind of example and we find that this couchy riemann equation is satisfied under the condition that the function fz is not an explicit function of z star.

So, whenever you find a function f which is not a function of z star explicitly then you can readily say that this function may be an analytic function, but you need to check meticulously whether the Cauchy Riemann equation is satisfied or not next thing is that there are many well behave function we find today which should have the derivative in the entire range except the z tends to infinity point and this function is called the entire functions.

So, entire function as a well behave function as I mentioned and in the next class we will go with that we will find many important things for analytic function and also we will start learning how to tackle with this special function where we have some kind of singularity; that means, the function is blowing up at that particular point particular finite point of z. And these singularities play a very important role and how to tackle with this singularity. And all these things and more about the analytic function and how the analytic function behave; what is the speciality of the analytic function. We will learn in our next class.

With that let us conclude today's class. So, see you in the next class with all these things.

Thank you for your attention.