Mathematical Methods in Physics-I
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Lecture - 35
Derivative Of Complex Function, Cauchy-Riemann Equation

Welcome back students to this numerical course, Numerical Analysis Complex Analysis course where this few numerical problems we have done in the last class regarding the limit and all these things.

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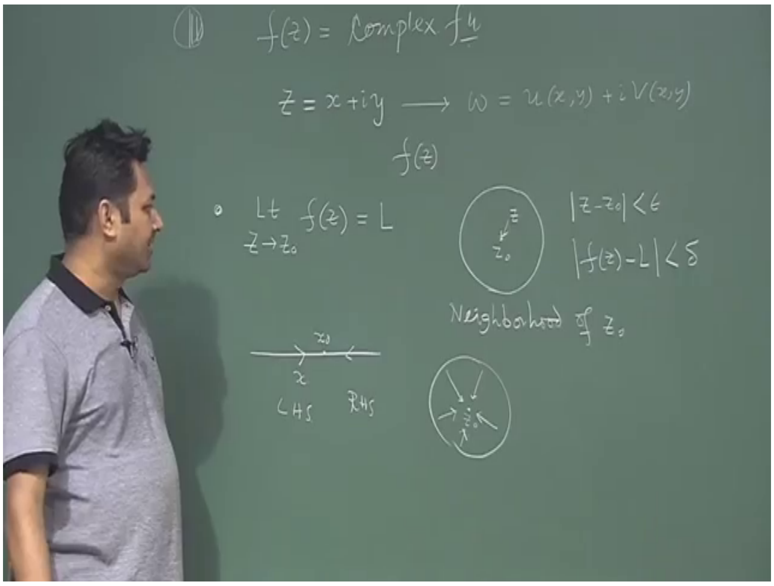

Before going to that numerical problem, we should start the thing from where we stopped. I mean f z this is a complex function and in last class, we mentioned that z is equal to x plus iy is the variable.

So, when this variable is changing, x and y should change and as a result we will have something which is f z which is a rule and in w plane, we have something this. So, this is the variable. This is not a single variable; this is a dual variable. Two variables are associated with that x and y and when x and y is related to a rule I called fz, then in the w plane we have another complex number ux u plus ivx, where u and v are function of x and y. We have shown example and few numerical problems in the last class. We will not be going to do any other problems right now.

So, next thing was when the function is defined, we defined something limit say limit x tends to 0 or x tends to some point z 0 function of z is l. So, essentially the meaning is if we have a point z 0 here, z is some point which try to come at this point z 0 with a condition that z minus z 0 is less than some quantity epsilon. Then, the function and this whatever the value I have here, the function try to tends to a value l with the condition that f z minus l mod of that thing should be less than delta.

So, this is called the disc or neighborhood of z 0 and z is approaching to this point here. We have something quite different that we have in case of real functions. What is the difference here in the real case? The approach of some points say x to x 0 is only two way, either it is left hand side or in right hand side, but in this case, in complex case what happened we have a region and one can approach to that point z 0 any path or any from any direction, but the condition is that the limit should exist if it is valid for all the cases.

In the last class, we showed that there are few functions where if I approach from one direction, I will have one limit and if I approach from other direction, I will have other limits. So, that means the limit does not exist. So, after having this, it is an important thing that we need to ensure that in from all approach, I will have the same value and then, only we can say the limit exist. Also, at the same point I should mention what the meaning of continuity is, where this z into z zero fz.

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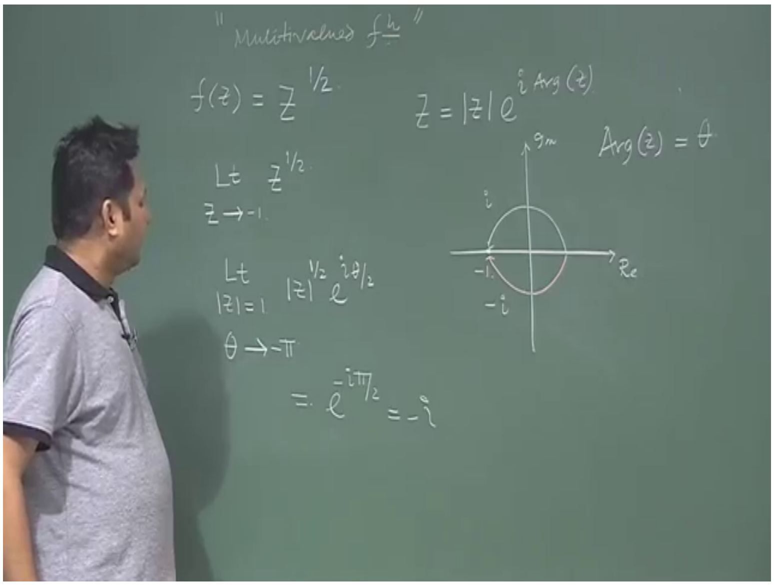

If this value is this, then we can say this function f z as a continuity. So, again the limit condition that it should have is the existence of the limit at that particular point that is the thing.

So, here I should mention one important thing and that is what about the multi-valued function. So, multi-valued function is not a very new thing because we have already discussed these things in our early class. So, let me give you the example say f z is z to the power half this, then what happen if I put z in this way mod of z and the corresponding phase term which is argument of z, I can write the entire z in this way.

So, mod of z is nothing but I just write z in terms of the polar coordinate which is r e to the power i theta. Here r is nothing but the mod of z. So, I just put mod of z here and e to the power i argument of z. Argument of z is nothing but the theta it has and big argument suggest that there is a limit of this theta. Now, this limit actually gives something quite different here. If I try to find out what is the value of the multi valued function at certain limit, for example if I want to find out this limit. Limit z tends to say minus 1 z to the power half.

What is the meaning of that? If I draw that in the complex plane actually which point I am targeting here, where is the limit, I want to find is minus 1 point. This is real; this is imaginary. I want to go this minus 1 point and when I try to find out the limit that means my approach should be anything from all the direction whatever I want to approach this because the condition is that z tends to minus 1. So, here I will do that since it is the minus 1. So, these things are nothing but z is 1 i. Now, divide this z tends to minus 1 this argument into two parts. One is the amplitude part i say z is 1, fine mod of z, but theta which is the argument tends to pi. That means, my one way to approach is something like this. Then, if this is the case, the limit is slightly modified here. The function z to the power half I should write in this form. So, then my function, let me write it here limit. This is equal to 1 and argument which is theta tends to pi and this point z to the power half. I write mod of z e to the power i theta by 2; mod z to the power half because here my z is defined in this z to the power half or root over of z can be defined in this way.

Now, if I do that, if I put this limit here, what essentially I am getting mod of z, I can put directly because I am reaching this point with the minus 1. So, that means mod of z should be 1. So, it is simply the limit is theta tends to pi e to the power i theta by 2 which is e to the power i pi by 2 which is e to the power i pi by 2 is say i because cos pi by 2 is 0 sin pi by 2 is 1 i. So, here it is i, ok.

So, one approach is something like that I am going from this to this. I am getting this. Let me try to do the same thing with another path or another approach . So, I will try to find the limit when I am going from this to this. Please note that here the direction is changed. So, in the previous case if I put theta tends to pi here, I should write theta tends to minus pi because I am going in the different direction.

So, now this limit is modified as mod of z is equal to 1 and theta is approaching minus pi because I am going to this point from the other direction, then z to the power half e to the power i theta by 2. If I put this limit here, you will readily find that it will be e to the power minus i pi by 2 which is minus of i. That means, from this approach the value of the limit is i and from this approach the value of limit is minus i.

So, that means whatever the function is given, if I try to find out the continuity at that particular point, we will find that the continuity is not there. There is a discontinuity.

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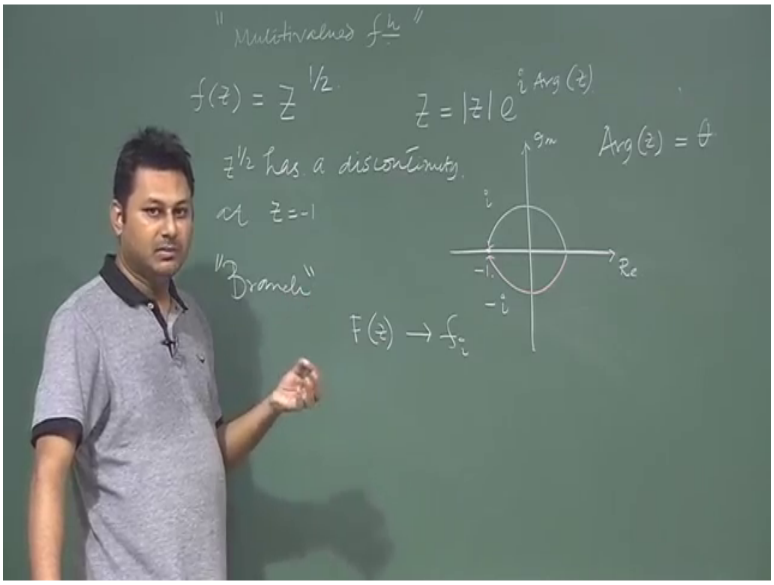

So, that means the function z to the power half has a discontinuity at z equal to minus 1. Now, from here one important thing is emerged in complex analysis which is called the branch. I will try to cover this in the future class, but not today. In order to understand this branch, you need to know few other things associated with that.

So, before that I want you to know the basics of the derivative and all these things, then I will go to this particular thing called branch. Branch is nothing but a multi-valued function should have different set of values. For example, f z is a multi-valued function for which you have a set of value fi and this fi is continuous to some limit and then, again after other limit, you will again continuous and after other limit, it is again continuous by having the same value. So, then we can say that fi is a branch of these things and for different branch, we will have different values.

Then, if we cut this discontinuity point, we will have something called branch cut. So, this is entirely a subject where you need to spend more time. So, I will do that in the later class when these things will be more familiar to you, but since I try to make you understand; what is the meaning of continuity and all these things, so at that point I am calculating what is the continuity of these things at that particular point with our knowledge. So far and we find that there is something called multi-valued function and for this multi-valued function, we have this kind of problem of this continuity.

In that case, since it is a multi-valued, you have a set of values which is same that other set of values in other branches. So, these are there at different branches which are at some point, it is discontinuity. There is a discontinuity between two branches, but you will have a same set of values. So, after having a very rough knowledge about this continuity limit, all these things, now we are said to define more important thing which is the derivative of the complex function.

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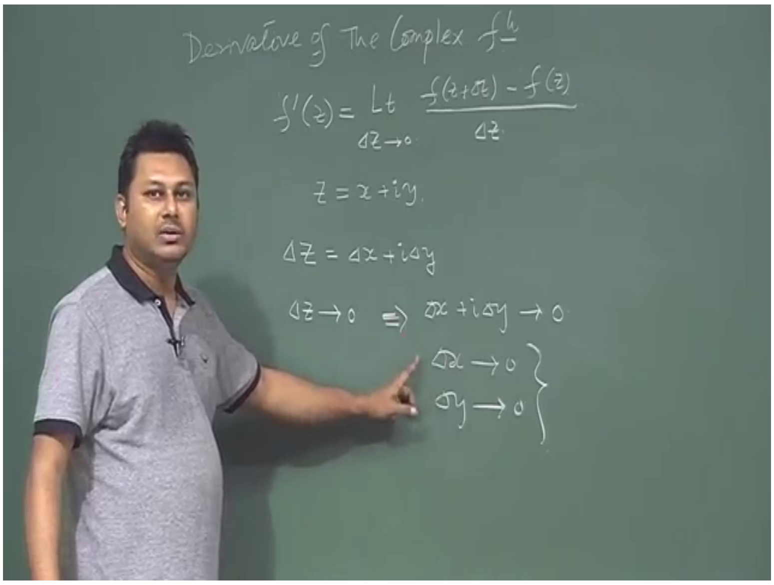

So, the derivative is defined as usual. I mean it is nothing as I say what is the meaning of derivative f prime z is essentially the same thing that we have for real case delta z is a small increment and the function f z minus function, this divided by delta z. This is the standard definition that we have for derivative. In real case here we are following the same thing. So, f z is nothing but the derivative of a function fz. f z prime is defined as the derivative where it is a small increment of the variable z. Delta z is a small change in the variable. This is the original value.

So, I want to find out how the function is changed with respect of the change of the variable. When the change of variable tends to be 0 for very small change, how the function is changing? Well, here again the same problem arise that we have in case of finding limit and all these things. So, my z here is x plus iy delta z should be delta x plus i delta y. This is a variable z which is defined by x plus iy as usual, where x and y both are real quantity, delta z is a change of the variable. I want to change the variable z which is defined by this where the small change of x and small change of y is associated with this change, fine. No problem with that.

Now, if I say delta z tends to 0, that means this entire quantity tends to 0 which suggests that delta x plus i delta y tends to 0. When delta x plus i delta y tends to 0, that essentially means that delta x tends to 0 and delta y tends to 0. So, now the same question repeatedly we are encountering that if I say delta z tends to 0 or delta z or z tends to some particular point, what should be my approach here? Here we are having a similar kind of problem. Delta x and delta y both are going to be 0. So, I want to find out how or there are how many ways they can go to 0. So, let us just do a simple thing.

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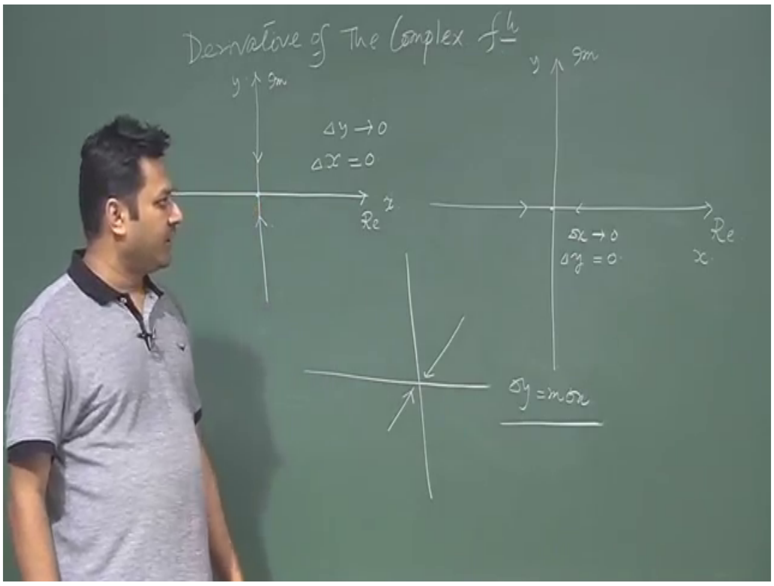

So, this is the real and imaginary axis. If I want to go to this zero point, for example if I want to go to this zero point, one approach last day we actually did it, but still one approach is that I go, this is y and this is my x. One approach might be I can go from this to this. That means, delta y tends to 0 and x is equal to 0. That means, I put x equal to 0. That means, I am not changing this point. So, these points along the values are already fixed.

Now, I am saying that the other portion that means y is going to zero along this path, so that the point ultimately reaches here some point. In the second case also, I can have my figure is I need to modify slightly. So, this is my real axis, this is my imaginary, this is y, this is x. Other thing is that I can approach like delta x tends to 0 and delta y is 0. Here I should put delta x. So, then the change of y, I put 0 and I try to find out what happened when delta x tends to 0.

So, I am approaching along real axis to go to this particular point by changing x, but I am saying that the change of y is already 0. So, delta y is not going to this point. So, delta y is already 0. By the way, there are many other ways as I mentioned. There are infinite ways to approach this point. For example, in general I can also come to any arbitrary direction from here to here, where they can follow this simple relation. Here also delta x tends to 0 gives you delta y tends to 0, but here the thing is I am going to some particular direction depending on the value of m.

So, m is nothing but the slope and I can reach this point whatever the process or whatever the path I like, but we will stick with these two very specific case to figure out very interesting things. So, let me do that first.

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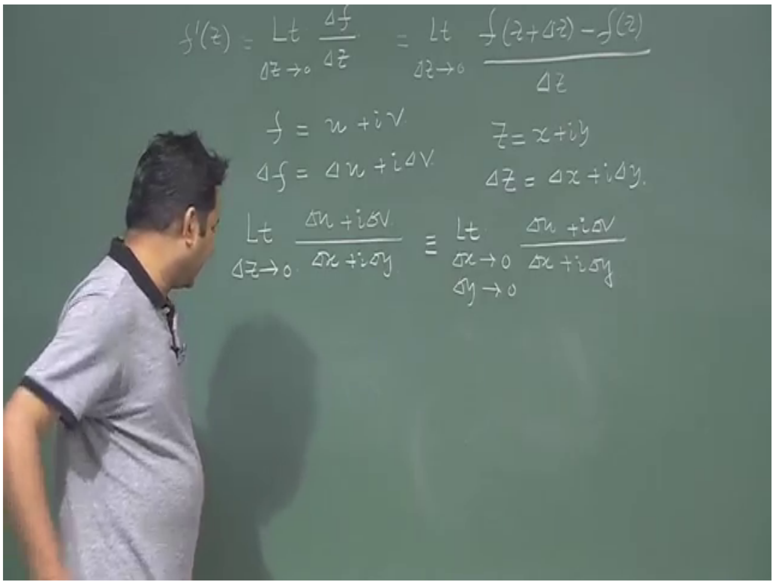

So, I try to find out this quantity which by definition is change of delta f with respect to z. Let me write it once again. It is limit delta z tends to 0 delta f divided by delta z. It is the same thing that I have written previously. So, it is limit delta z tends to zero function of z plus delta z minus function of z divided by delta z function of z plus delta z minus function of z. I just put delta f. The change of that thing now f is equal to u plus iv f of x is equal to u plus iv. So, delta of f is equal to delta of u plus i delta of v. That is the additional thing because in the previous case I mean in the previous class, what happened that I defined how the function is changing if you remember f z if I say this is a z square.

So, z square itself can be represented as x square minus y square plus 2 of i xy or u of x of y plus i v of x of y, where u of x of y is this quantity and v of x of y is this quantity. So, now by definition I will have limit delta z tends to 0, delta f is delta u plus i delta v divided by delta z, z is x plus iy delta of z is delta of x plus i delta of y. Already I use this notation.

So, delta of x plus i delta of y, this condition is equivalent to the limit individual delta x tends to 0 delta y tends to 0 delta u plus i delta v divided by delta of x plus i delta of y. After having these, now the question is what should we do? How to go this together? As I mentioned there are several ways to go there. So, the path is according to the choice. I mean if I choose some specific path, I should go with that, but every time I need to check that every time the value, whatever the value I am having is the same thing. So, keeping that in mind what I will do that I will take the two basic paths that I just showed before doing this. That means, one path is let me write it here. Path 1 maybe I can erase this and show that here path 1.

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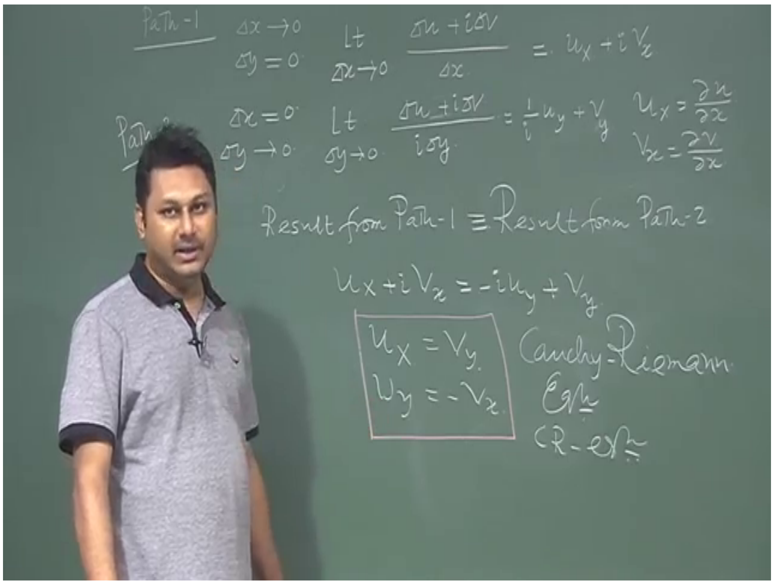

So, path 1 is what I say delta x tends to 0, but delta y is 0. I showed that. So, if I put this here, then things will be slightly simpler. So, then my limit will be only over delta x because I am saying that delta x tends to 0, but delta y is 0.

So, then delta u plus i delta v divided by since delta y is 0 here, there will be no delta y at all. So, I will have only delta x in my hand. These things essentially gives you u over x plus i v of x, where u of x and v of x are nothing but the partial derivative of u with respect to x because here delta u delta x plus i delta u. So, if I write another step here, it should be let me write it. If you have any confusion over your mind under the limit delta x tends to 0, I can write this as del u del x and del v del x partial derivative because u is a function of xy, but I am taking y as a delta, y as 0 and deriving it with respect to x.

So, finally I have ux plus iv y where ux is du dx and vx is del v del x by definition. So, I will have one path and one path gives me this quantity. I can now choose another path say path 2 and now, I will say that my x delta x is equal to 0, but delta y tends to 0. If I do the same thing, my limit will be simpler. It will be delta y tends to 0, it will be delta u plus i delta v divided by i delta y from here delta x equal to 0. So, this term will be vanished and the only thing that we have is this.

Now, if I simplify it will be like 1 by i u of y plus v of x 1 by y u of y y i. I will cut out. So, I will have v of x v of, sorry v of y where v of y is a partial derivative with respect to y. Now, the thing is that if this is path 1, if the derivative exists, then I must say that the value regarding from whatever I have from path 1 and the nth result, what we find from path 2 should be same. So, the result from path 1 is same as the result at least because I am taking two paths in principle. It should be all the path, but for the two fundamental path rated.

Let us try to find out what this is what is the condition results; results from path 1 is equal to result from path 2. That means, this quantity and this quantity are same. So, let me just simply write the equation ux plus ivx has to be equal to minus of i uy plus vy. This minus comes because this i is here. I will just multiply 1 i and then, it come minus i. So, now I know that if two complex quantity are same, then the real part and imaginary part are same. So, if I write this individually, then it will be ux is equal to vy and uy is equal to minus of vx. This is a very very important equation. This equation is called Cauchy Riemann equation.

This equation is a condition. This is a necessary condition to have a derivative, but not the sufficient condition. This is necessary condition, but the name of this equation is Cauchy Riemann equation in short CR equation.

So, today I like to stop here, but today we did a very important thing in complex analysis. We find what is the meaning of a derivative in complex analysis and in complex analysis derivative must exist if this Cauchy Riemann equation is satisfied.

So, now this is the necessary condition to have these two things same in order to have the derivative, but important thing is that here whatever we have done is not a complete picture, but we just approach from one side and just check that for these two paths, at least for these two path things are same or not, but in the future class we find that this is a very important and very useful relationship between u and v in order to have a function which has some derivative in some region and if a function have a derivative in some region, a particular points is z and a neighborhood of z, then this function has a special name in complex and complex analysis, a terminology. It is called the analytic function. So, in order to have a analytic function, this condition mass must satisfy to each other and with that I will stop here.

In the next class, we will go with this and try to find out how the analytic function behave and what is the exact meaning of analytic function, what is the definition and all these things. With this let me stop this class here. See you in the next class and the next class is important because we will define; what is the meaning of analytic function which is very important in the complex analysis course. With that see you in the next class.