Mathematical Methods in Physics-I
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Lecture – 34
Complex Function, Concept of Limit

So, in the last class we try to understand what is the meaning of stereographic projection.

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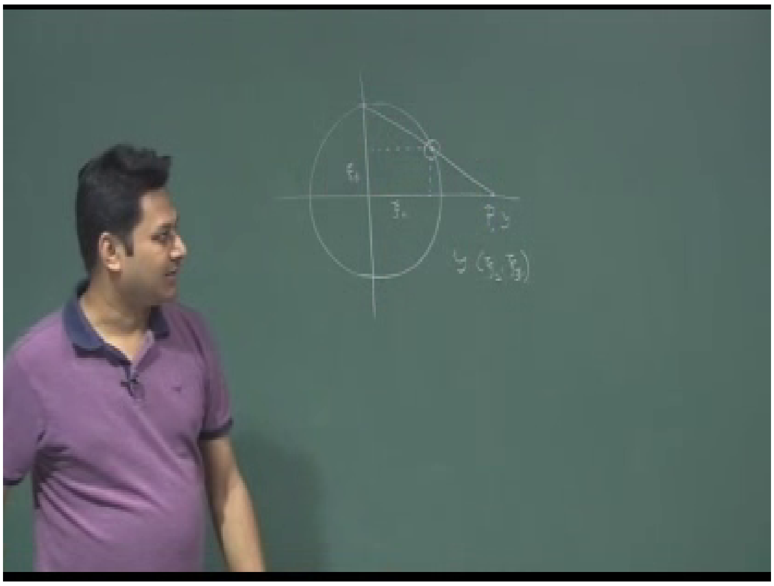

So, let me remind you once again that stereographic projection is something, this is the case. So, if I want to find a point here say this point is p, then for each point I should have a unique point here and we say the there is a relationship between this point, say if this is y and this is zi 2 zi 3. Then, I we calculate something that these things divided by these things is equal to these things divided by these things, in different way you can do that. I mean this is not a unique way to do the relationship, but the point is that you can find y as a function of zi 1 and zi 2 and when you do that; that means, for each zi 1 and zi 2 or zi 3, here zi 2 and zi 3 because I am not taking x.

So, for each value you will have a unique value of y. So, if y tends to infinity this point goes to here in the North Pole you can do that. I mean this is not a very hard thing to do today, we will start something important as I mentioned in the last class; which is complex function.

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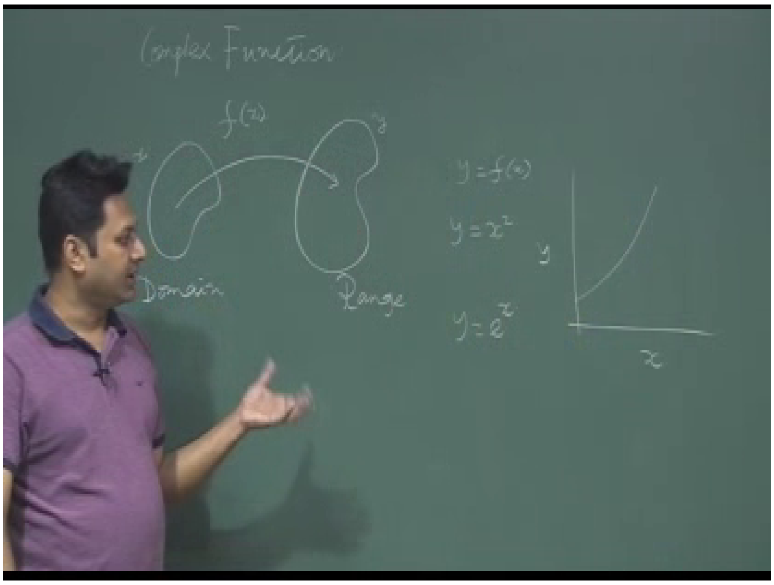

So, real function we know what is the real function. So, let me go back to this function analysis and all these things. So, I have function is something which is rule, we have a set of point here which I called the set of point x. I apply the rule and as a result I am getting a set of point and that the rule I called f x and this is y. So, y is f x is the rule and for each point I should have another point here, this is called the domain and this is called the range.

So, I will take 1 point from domain, I will apply the relationship or the rule over that and I am getting another point in another domain. So, another which is called co domain or range. So, example is say y is equal to x square, I take 1 x value here I squared that and I will get another point here I will take another x square that I am getting and if I do that then I will have a relationship between these. So, I will have y I will have x, for every x I can plot y here this is the rule. So, if I plot I will have a parabola. So, no problem with that if I have a different kind of function I will have different. So, if I have y e to the power x, I should have a different kind of function and I can if I plot it will be something like this and so on. So, the point is that I can always plot in real case I can always plot 1 variable with respect to another.

But what happened in case of complex function. So, let us first define what is the meaning of complex function?

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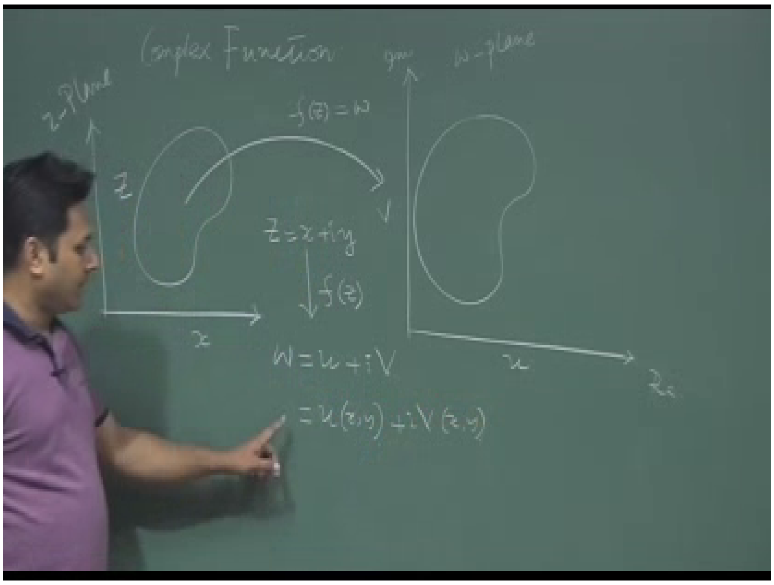

So, complex function is the same thing that, I have a set of point complex point say Z, this is a set of complex point in the complex plane. I have a set of point in a region as I mentioned in the previous class, I apply a rule of that point this is the point in the complex plane set of complex so; that means, this is called Z plane this is x and this is y. So, in the Z plane I have a distribution of points and over that particular point I apply the rule exactly the way I apply and as a result I am getting here something as a result I am getting here something.

Now, the question is whatever I am getting is also essentially be a complex number. So, I called it as W and I can also put these things here, but now my plane is changed this complex plane, if I write this complex plane as Z plane then this complex plane. I should not write the Z plane I should write is as W plane, in the W plane I will have a distribution of point because of the fact, I am applying 1 rule over that and different points that start generating, but since these points are again complex points. So, I should have a real and imaginary axis here like previous, but I should not write it as x and y rather I should write it as u and v. So, in sum I will have Z equal to x plus i y this is my variable.

I apply my function if Z over that this is the rule, I am applying over that as a result I am getting something W which is u plus iv, which is again a complex function and this complex quantity I should write in this way, now if you note that you and we should be function of x and y both. So, it is better to write u xy plus I v xy there are different kind of function you can imagine. So, this is the first I need to understand what is the rule here or what is the meaning of complex function, it is exactly like the real function. The real function I have a real value of variable and I apply some kind of rule over these real points or real variables and as a result I am getting something which is real and this real thing if governed by something which is called fx.

But here I am doing the same thing the logic is same, but instead of having x I should have Z as my variable, which contains 2 variable x and y. Now when I apply these things through a rule if Z I am getting a new thing which I call w, but interestingly this W is also complex. So, I should write u plus iv, where you should be a function of x y and v should be a function of x y as well.

Now, let me give you some straightforward example then things will be clear.

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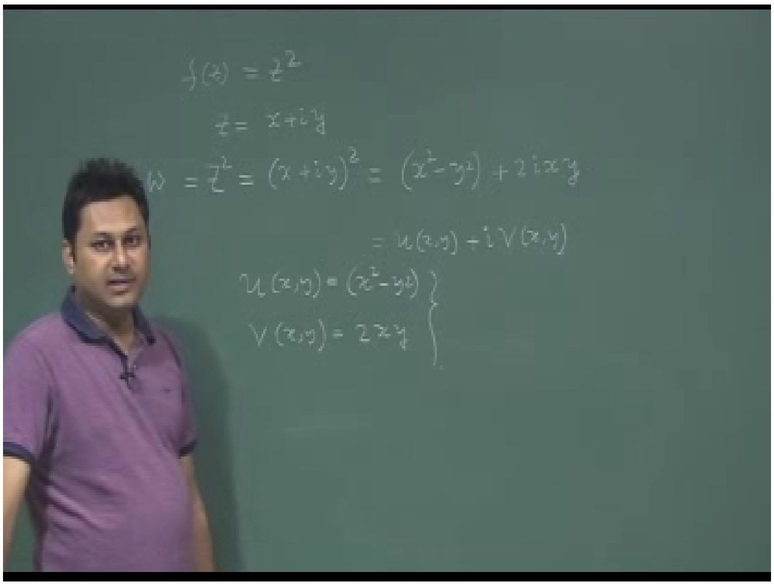

So, after having a very basic idea about how the complex functions working. So, say fx fz is say Z square, this so my rule is given. So, what is my Z x plus I y this is my Z what is my Z square.

By the way Z square is nothing, but W what is Z square is x plus iy square, which is x square minus y square plus 2 I xy that quantity and that quantity is nothing, but u function of xy plus I v function of xy because my W is u plus iv. So, u xy should be corresponding. So, when both the things are equal then, I can readily write u of xy is x square minus y square and v of xy is 2 of x y this is the relation.

Now, the problem is in the complex plane, the problem is you cannot draw anything because 2 variable are associated with Z, for these 2 variable again you have u and v u and v is associated also 2 variables. So, in the real case there was 1 variable that is associated with that and you can plot this 1 variable to another variable.

Here you have 2 variable associated with Z. So, when Z is changing essentially 2 variables are changing, when 2 variables are changing then it is difficult to in to the it is not possible to draw v and u simultaneously, what you can do that you can fix 1 value and then plot and then find out the relation. I will do 1 simple problem, but before doing that let us try to find out few other functional form also. So, this is base.

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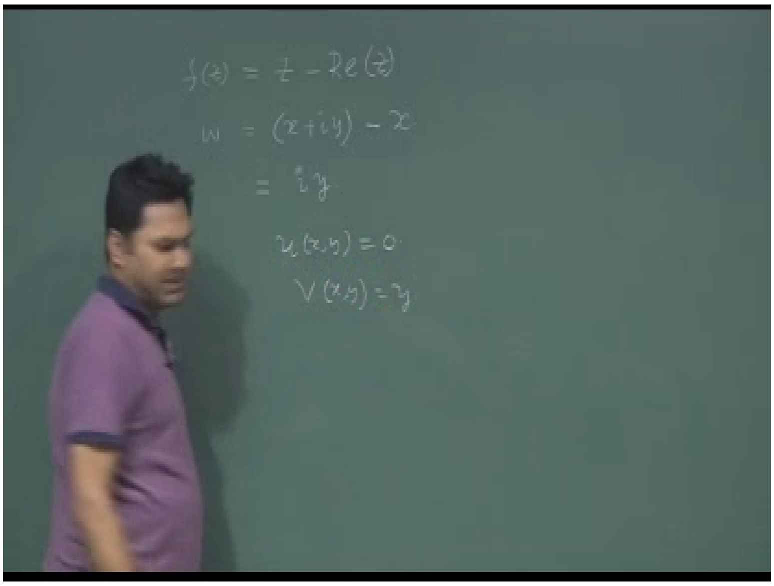

So, what is the function of say Z minus real part of Z this is also a function. So, this functions if I write in the xy form. So, W is x plus iy and real part of x means it is minus of x, real part of Z mean means it is real part only.

So, if I write it is just iy. So, iy so that means u of xy here 0 v of xy is y.

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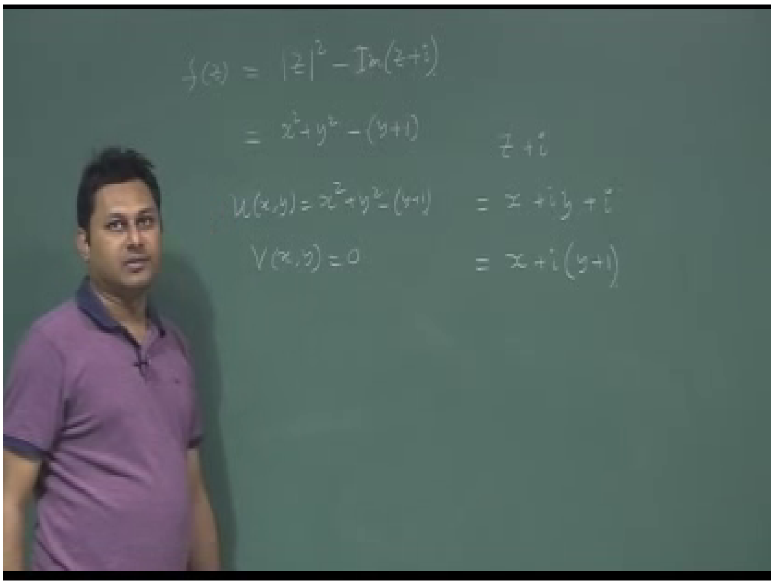

There are different kinds of function; you can consider for example, mod of Z square minus real say imaginary part of Z plus I, this is also a function mod of Z square means mod of Z square, means x square plus y square minus imaginary part of this quantity. What is the imaginary part of this quantity? So, Z plus I is equal to x plus iy plus i, so x plus I y plus 1. So, real part is this and when I say the imaginary part; that means this quantity. So, this quantity is nothing, but y plus 1.

So, entire function now become a real function, the way I define this it is maybe a real fit is not necessarily that function of Z, is always gives you a real or always gives you a imaginary; it may be complex and real thing is nothing, but a special case of complex function complex quantity. So that means, here I should write u xy is x square plus y square minus y plus 1 and v xy is 0, because there is no imaginary part associated with this. So, after having these things let us go back to that problem, how to define a function.

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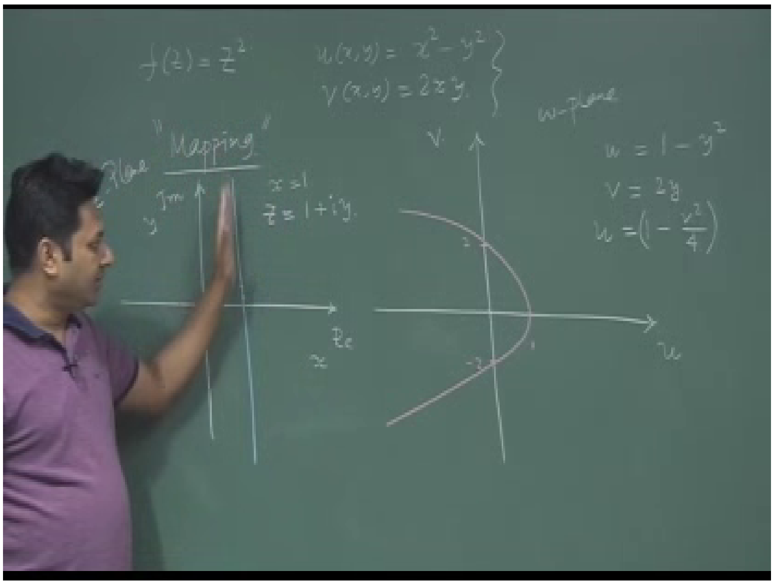

So, let us say this is my function f Z is equal to Z square, now since we are not able to. So, it was I do that. So, my u x y was x square minus y square and v xy was 2 xy, what I am trying to do that I try to find out for different Z how this u and v is related. So, since it is directly it is not possible to draw the variation, what we can do is called the mapping.

It is called the complex mapping. So, what I will do in this mapping procedure, what I will do that my Z variable is changing. So, in Z plane I will vary my point, real and imaginary which is x and y this is my Z plane and try to find out that if the Z point is varying here how in the W plane things are changing. So, here this is u and this is v and this is W plane and related with the function which is defined here, f Z is equal to Z square. Now in order to do that as I mentioned I need to change my variable to a single variable. So, that is why I say x is one. So, I fix my x and y can vary.

So; that means, in the Z plane the variation of Z with the limit with the restriction y equal to 1 should be this line. So, my Z is varying over this line in my Z plane. So, my Z point is varying over this line, because y equal to x equal to 1 is my point, x equal to 1 is the restriction and x equal to 1 is nothing. But this points this line and over this line my Z is changing so; that means, whatever the y value I can think of so my Z is essentially 1 plus iy. So, there is no restriction of y. So, y can change like this, if my Z is changing in this way what will be the form here in u nv that is the question. So, the problem is Z is changing like this way I want to find out how it will look like here, in W plane it is called the mapping. So, I am I want to map this particular variation of Z with the restriction x equal to 1, so that I can do that.

So, here we have a relation. So, I can just put this x equal to 1 here and try to find out what is the relationship between u and v that is all, that is the recipe of doing these things. So, I have x equal to 1 in my hand, here what I will do that I will put this condition here, when I put this condition here I will have a relationship between u and v and then I will put this relationship. So, you will be if I put x equal to 1, 1 minus y square and my v is if I put. So, it will be 2 of y. So, I should have a relationship between u and v and the relationship is u is equal to 1 minus y square if I say it will be v square by 4.

This is the relationship between u and v, after having a relationship between x and after putting a value x equal to 1, I have a unique relationship between these 2. If I now plot this point these things, how it look like that is that that is the answer. So, it seems that when v is equal to 0, when u is equal to 0 v is how much, v is v square by 4 equal to 1 v square is equal to 4. So that means, v is equal to plus minus 2. So, at this point the value of v is here and here, which is 2 and 2 this is point 2 this is minus 2 rights and then when v is equal to 0 then u is equal to 1. So, somewhere here this is 1 and it seems to be a parabola. So, simply it will be something like this. So, this is the mapping of this particular straight line for which I should have a restriction x is equal to 1, now when I have a restriction x equal to 1. I should have a straight line for that; that means, this is my variable.

Now, my Z is not vary over the entire plane, rather it is varying just 1 single line and 1 when we have a 1 single line variation, then I can map this to the W plane and try to find out with this relation when u and v are there. How u and v is related and I find there is the relation so this is one example of mapping. After having the relationship of mapping. So, let us now try to find out something interesting, which I already mention actually is the limit, I am not sure whether I will able to complete this topic today, but let me first introduce what is the thing, so concept of limit. So limit is a well minus known concept in a real case.

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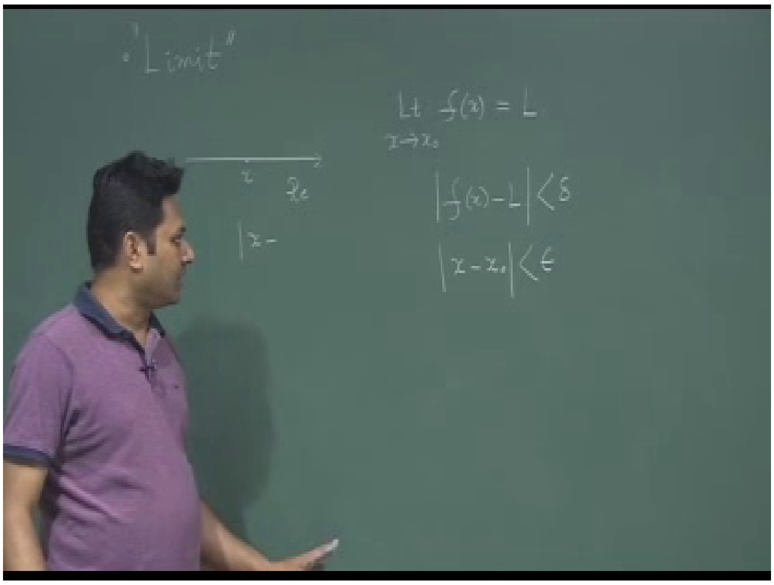

When you have in the real case say this is my real axis, 1 point is x and because of when I have limit; that means, say limit x tends to x 0 fx is say L. The meaning of this line is that x minus fx minus L, will always be less than equal to some quantity delta. If my x approaches to x 0 in such a way, so for every x which is following this relationship; that means, x minus x 0 is less than a small quantity delta so that means, for that x if I put this x here then this value will be close to some other value L with the fact that, if I take a mod of these things that should be less than delta. So, as if I go to closer and closer.

This quantity f for every x which is following this epsilon which following this relation, I should have a additional relation between these 2 things. So, that I will have a f minus l is something less than delta. So, this is the delta epsilon the well minus known delta epsilon, relationship between when we define the limit of that. So, we know these things for real case say in our previous courses, it has been taught very rigorously. So, I am not going to be detail in that. So, I believe all of you are aware of this just this is a recap how the limit is defined.

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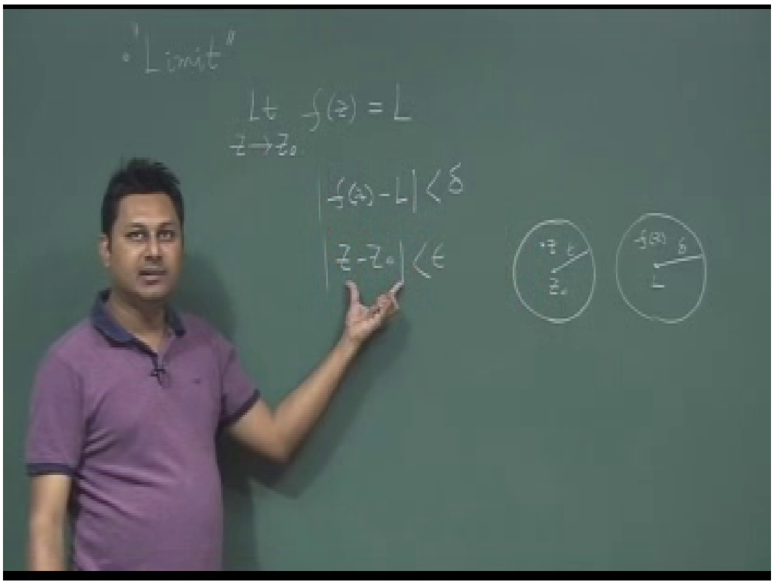

So, now, I am going to do the same thing for complex. So, my lit complex case Z is go to Z 0 f of Z goes to some value say l.

In this case also I should have a relationship that fz minus L should be less than delta, when Z minus Z 0 is less than some epsilon. So, I have. So, Z minus Z 0 you can understand that this is basically a circle in the neighbourhood of Z 0, I will have a point here Z I am approaching to this point Z 0, with the condition that it should be less than these things. Here in W plane when this when I put this function here. So, I will have this in W plane. So, W plane also I have a point L here and my f Z or W is such that they will also follow the similar kind of things. So, something likes this. So, I am approaching this point Z is approaching minus Z 0, with the limitation that it should be less than some quantity epsilon. As a result of these things in that another plane, I should have this quantity which is also approaching to L, with the limitation that this minus this mod of this minus this should be less than delta this is 1 thing, but important thing is approach, let me draw that in bigger way.

So, this is my Z 0 point when I am saying Z tends to Z 0. So, it can approach from this point to this point, it can approach this point to this point.

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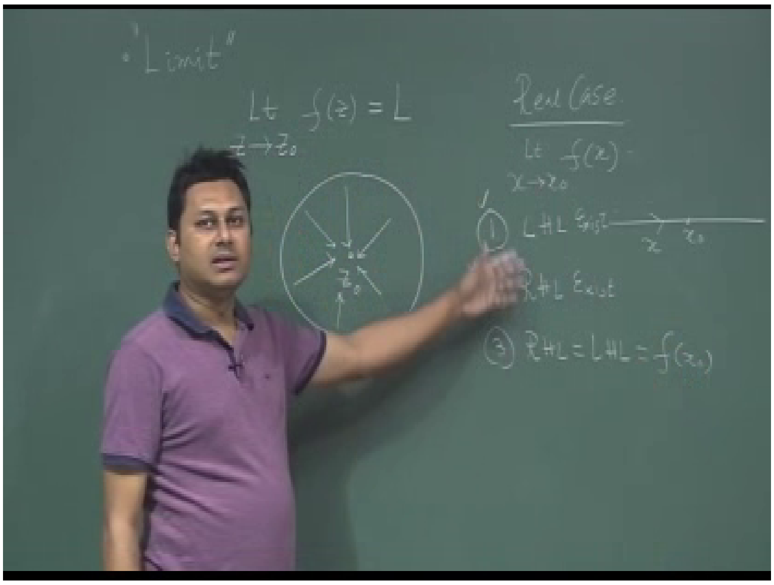

It can approach this point to this point, it can approach this point to this point and So, on there are infinite number of way to approach, this point as I mentioned in my earlier classes that there are infinite number of way to approach in real case what happen. So, I want to find out the limit x tends to x 0, fx is equal to I want to find this limit. So, what is the value of limit, first what we choose left hand limit; that means, when I approach from this is my point x 0, when I am from approaching from this side to this side it is there, first I check second thing left hand limit exist checked right hand limit exist, check if these 2 limits are same. So, right hand limit is equal to left hand limit is equal to the value of function value of the function at that point. If it is true then I we can say that their limit is there. So, limit exists, but here since the approach is not left end and right and all the directions, so we are in some trouble.

So, we are not directly we cannot say whether it is approaching to this limit or not whether the limit is existing or not. So, let us take a very straightforward simple example before, the end of this class next class again we will start from this concept.

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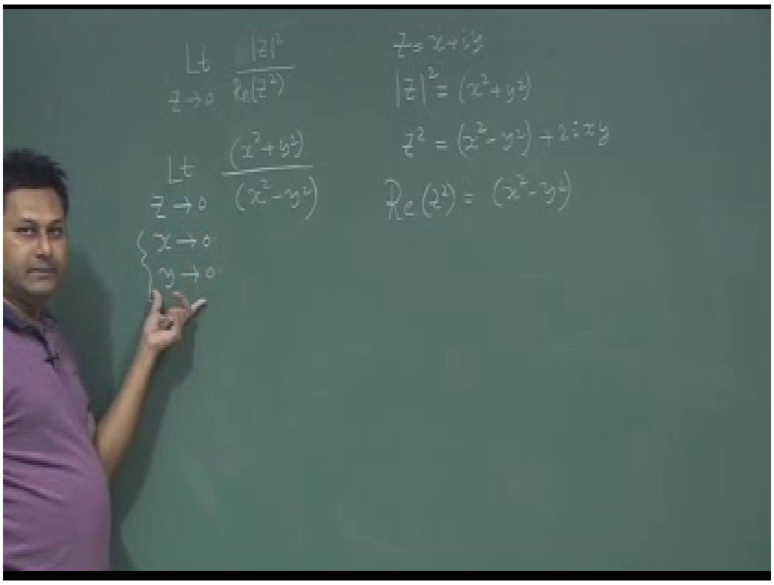

So, let us try to find out limit Z tends to 0, mod of Z square divided by Z square, mod of Z square divided by real part of Z square say something like this.

I would have to find out the limit at Z equal to 0, just give one example then I will show more things. So, what is the case here, Z mod of Z square is how much x square plus y square, real part of Z square. What is Z square mind it Z is x plus iy. What is Z square Z square is x square minus y square plus 2 I xy, when I say real of Z square it is x square minus y square this quantity. So, if I put here eventually what I am saying that Z tends to 0 which essentially means x tends to 0 and y tends to 0. This quantity is x square plus y square I just put everything in terms of x and y and this quantity real of x square is x square minus y square I have this.

So, this is my condition this is my limit I need to find out. So, limit Z tends to 0 is essentially means this right. So, now I will do one by one and check.

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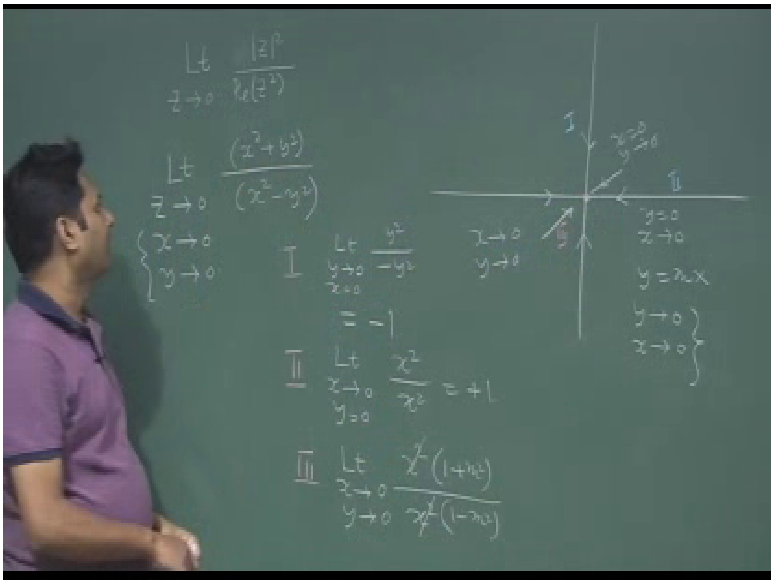

So, before that let me draw 1 thing here this is my 0 point, when I say x tends to 0 y tends to 0. I will have several option in my hand 1 option is this x, x is equal to 0 and y approaches to 0. I can go with this la this line I put x equal to 0 directly. So, that my calculation is simpler and I approach y tends to 0, this is my one path say this is one approach. One another approach you can readily understand maybe this I can approach from this side.

This side what is the condition y is equal to 0 and x approaches to 0, this is approach 2 and I can put any arbitrary direction I can go to 0 from any arbitrary direction. So, say I can approach to this direction y, where y and x are having a relationship so; that means, y is equal to m x that is a relationship; that means, y is approaching to 0 and x is approaching to 0 both are approaching to 0 in this case 3, I put it as a special case 3 both are approaching to 0, but they are maintaining a relationship y equal to m x m is a slope. So that means, through this straight line where so m is a slope they are approaching to this point.

So, there are 3 different cases where you can apply. So, because I need to check whether the limit exist or not in order to check, you need to ensure that whatever the direction you want whatever the direction, from which you can go approaches to 0 you will be able to reach this point. So, if I do then you can really understand for case 1, for case 1 what happened I put x equal to 0 y tends to 0; that means, x equal to 0 y tends to 0 gives you what value limit y tends to 0, I will have y square minus y square y square, y square you will cancelled out I will have a limit minus 1. So, I can have in fact, a value of the limit case 2, in case 2 y is 0 x is tends to 0. So, if I put y is 0 then limit x tends to 0 here y is 0 here x is 0 here y is 0.

So, y is 0 means this is going to vanish, I will have x square by x square which is plus 1. readily I find that in 2 cases in first case, the limit gives me minus 1 in the second case the limit gives me plus 1. That means, they are not equal so that means, the limit does not exist from here I can conclude, but interestingly in case 3 you will find something again. So, here both limit x tends to 0 y tends to 0 and if I put that there, then what happened the relationship. If I put then if I put y is equal to m x square then it will be x square plus 1 plus m square and in the down stair it will be x square 1 minus m square. So, x square x square will cancelled out.

I will have a value of the limit which depends on m, that means it depends the limit entirely depends on in which direction you are approaching to the point 0 ; that means, this limit does not exist the answer is the limit does not exist. So, here I like to stop because in the next part we will be go to more detail about the limit continuity and the derivative which is most important part here. So, before going to the derivative of complex function, what is the derivative meaning of derivative of the complex function we need to know, what is the meaning of the limit of a complex function and you find that here the major difference between the real case and imaginary case real case and complex case is that in the complex case, the approach of a particular point can be in from any direction and since it is in any direction it is difficult to find out the limit.

But there are few things through which you can find out whether the limit is exist or not in the next class again we will start from this concept of limit and find out do some kind of problems related to limit, find whether the limit is exist or not and all these things and then go to something called the derivative of a complex function, which is much more important here and then try to find out something quite important in the complex analysis which is called the analytic function. What is the definition of analytic function? What is the meaning of analytic function? Maybe in the next class we try to cover this, with this let us stop the class here. So, see you in the next class.

Thanks for your attention.