Mathematical Methods in Physics-I

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Lecture – 33
Set of Complex Number, Stereographic Projection

So, welcome student to the next class of complex analysis. So, in the last class, we mentioned something about defining the roots of a complex number, but today we will going to do something different. So far in the few first few couple of classes we discussed about the complex number and all these things, but mainly discuss about the complex analysis which is important and which is the goal of this course.

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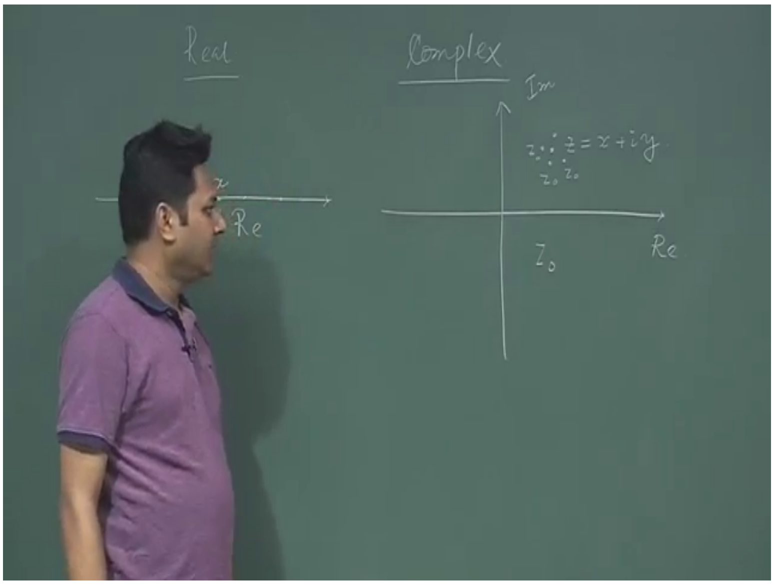

So, after having the idea of how we can deal with the complex number, how the roots can be find out, how should be the representation of a complex number in some coordinate system say Cartesian x plus iy. In polar, r e to the power i theta, now we are in a position to deal with this kind of effects this kind of I mean thing which is related to complex analysis.

For example, today before going to in detail what is complex analysis I like to mention something important. So, this is the real case and this is the complex case. In the real case, if you remember in the first class that if I want to define a point, it should be over the real axis; this is real axis. So, all the points whatever be the value should be on the real axis. On the other hand, in complex case, instead of having one line, we have a plane right. A general point x is represented by this point on the real axis in case of real analysis, but in complex analysis things are slightly different, because if you want to find out what is the point here which is defined by z. It is not related to one coordinate rather related to two coordinate x and y and represented by this so that means entire plane all the points into this plane are represented a complex number.

Now, since this is over the line it is one-dimensional things are easier in understanding for understanding in this real case, but for complex case things are not that easy compared to the real case. But we should understand what is going on here, so that is why after having the knowledge that here the points means there is a distribution. If I say here some points are there in the neighborhood of x that means, x minus x 0 mod of that should be some value epsilon that means, I want to find out a point very near to x. And if I write this that means x 0 is somewhere here, whose difference from x is some lowest number epsilon 0 or lowest number epsilon which is very close to the point is very close to x 0. This is the representation normally we have in case of finding a limiting condition or defining the limit and so on. But here what is the meaning of this statement, if I say for example, my z 0 is very close to z. Then what should I say because is the z is here is the z 0 is here, is the z 0 is here is the z 0 is here; all the points z 0 may be closer to these things.

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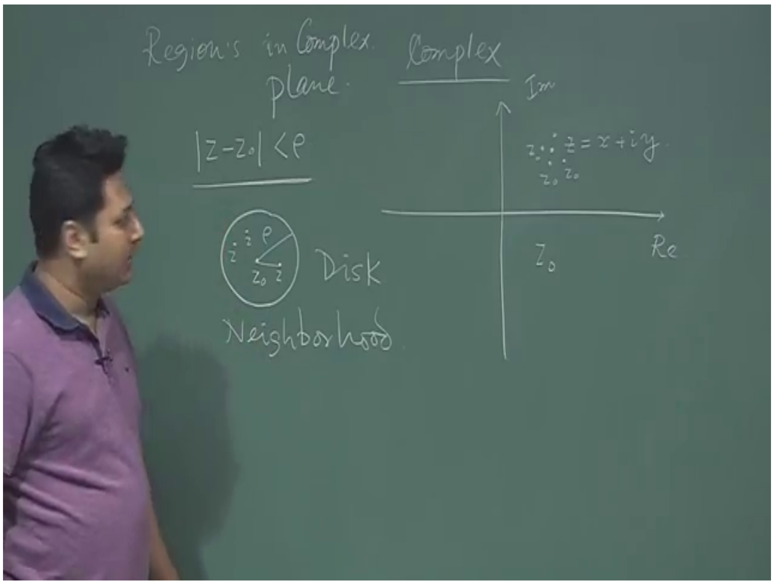

So, I need to define few things which is important in understanding what is the meaning of a neighborhood. So, it is called say regions in complex plane. So, region I am talking about here what is the meaning of region. So, this is my complex plane, I already described now I try to find out the point z which is very close to z 0, so that means, in the similar notation I should write this should be less than equal to rho. What is the meaning of this statement here, this statement suggests that I have a circle where z 0 is sitting in the center of this circle; rho is my radius. all the point set of point says z, z all the point z is satisfying inside the circle, all the points z is satisfying this expression. If you look carefully that if I take one point z here, if I say this is my z point z minus z 0 mod that means, the length of this quantity is always less than the rho that means the radius of this circle.

So, inside this circle whatever the point I have can be represented by this expression. We called this as disk, this things is behave like a disk. And these things are called neighborhood if I say my z point is close to z 0 with the condition this, we can say that all the points sitting inside this disk and these points are z points are neighborhood of z 0 that is the thing. So, there are many regions we can define in this way. So, let me do one by one, because it is important that how do you define the region and all these things, I will do all these case and show.

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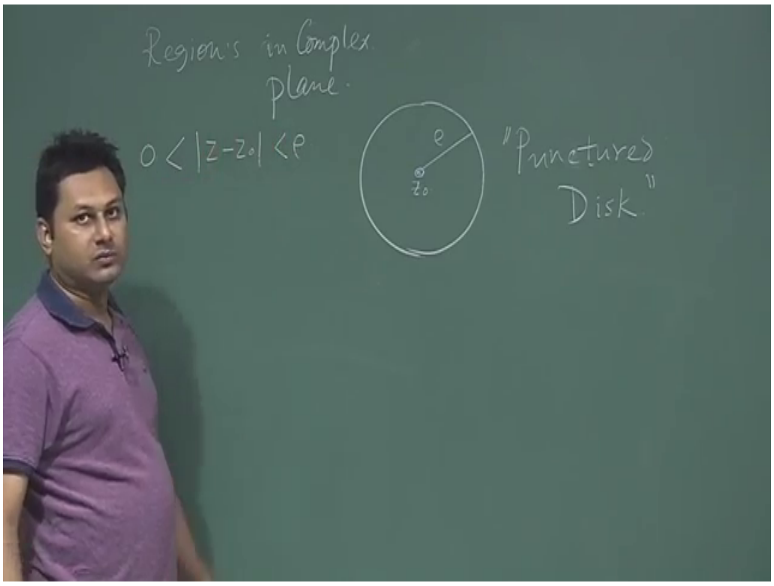

So, I understand what is the meaning of neighborhood, neighborhood is something by the way if I write this condition, I just changed this condition that means, z minus z 0, the z point is less than z minus z 0 is less than rho, but greater than 0. That means, my disk this is my disk, where z 0 is sitting in the center, this is my rho. I am taking all the region except this point except z 0 point, z can take any point in this disk having a radius rho except the point z 0. That means, there is a hole or there is a point here, which is not equal to z, z should not be equal to z 0. In that case this disk is called punctured disk. It seems that something is punctured here, because this point I am not going to take this point that is why it is called punctured disk. So, if I exclude the z 0 point then I have a punctured disk if I do not exclude this point I will have something, which is in the neighborhood of z 0, this is the thing.

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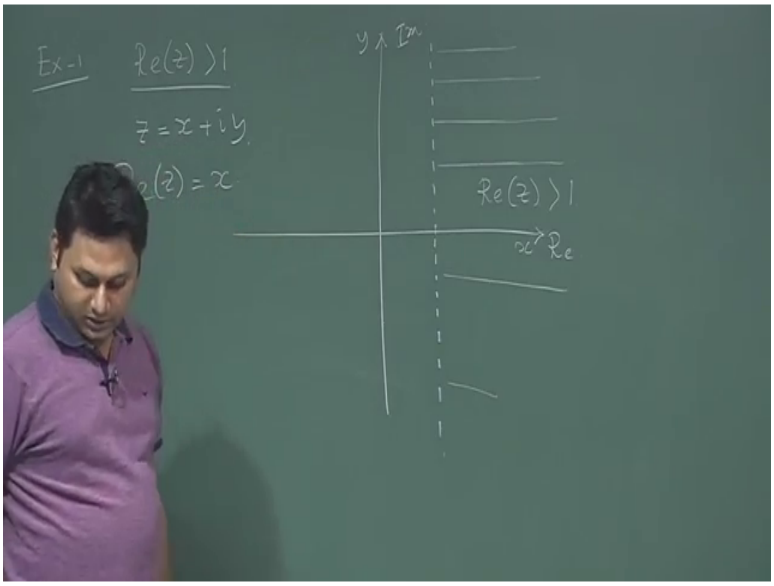

After having the knowledge of this disk and all these things as I mentioned region in complex plane, so let me erase this because I need to draw many regions here with the example. So, let me start with the example one what is the region of say real z is greater than 1. If you are asked to draw this region in the complex plane what should be the region. It is quite easy; this is my complex plane. Real z means, what z is x plus iy. So, real z is equal to x as simple as that so that means, this is my real axis, this is my imaginary axis, I call it y, I call it x and then I say real of x is greater than one; that means, x is greater than 1. So, x is greater than 1 means I have this point 1. And if I make a dotted line here this dotted line suggests that this region this region is x less than 1 and this region is x greater than 1. So, I am talking about all the x here which is greater than 1. So, my region will be here. So, this will be my region, this will be my region, this will be the region real z is greater than 1.

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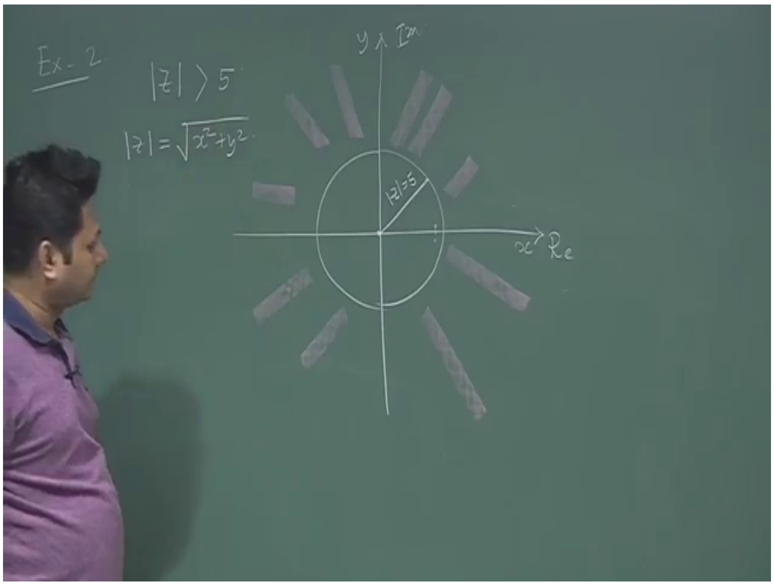

What are the other example of the regions. So, let me erase this I will going to use the same coordinates. So, I am not going to erase that. Example 2 say I need to erase this also. Example 2 say mod of z is greater than 5, mod of z is greater than 5, what is the region, what is the meaning of mod z. Mod of z means whatever the point the I mean you can do that meticulously mod of the z is nothing but root over of x square plus y square. So, root over of x square plus y square is greater than 5, so that means, I should have a circle here with a radius 5. So, this is my circle. Whatever the value of z, you have here mod of z is equal to 5, this is my circle. Whatever the value of z here over the circle is z mod of z is equal to 1. Whatever the point inside is z less than 5, but my region is mod of z greater than 5.

So, all the point outside this circle these, these, these, these, these all the points outside this circle will be the region mod of z greater than 5. Let me erase that. So, I believe now you understand what is the meaning of the region and when I am saying some region you can readily understand that in complex plane how this region are placed, how this region it is quite easy and let us take few other examples also.

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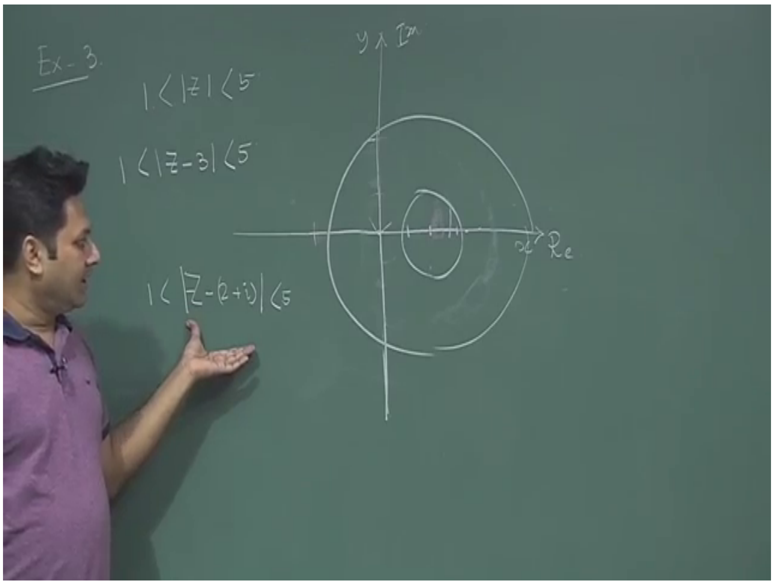

Say z minus z 0 in general is by the way mod of z less than 5, if the example was given like this, which means whatever the z I am talking about is less than 5 that means, all the region inside the circle will be the region I am talking about. So, next example, example 3 say z minus z 0 is say less than rho the previous example the previous exactly the previous example is a dicks thing it is nothing but I have a point z 0 here, I just shift my coordinate system, I have a circle over here this is my rho. So, all the z here inside this circle is following these things. So, I will talking about this region that is all. So, I am not going to discuss this anymore.

What about this point mod of z less than 5 greater than 1, this is this is interesting region, why because two conditions are imposing over z so that means, I should have again a circle of radius 5, z point is less than 5, so all the point inside. But another thing is there, that means, mod of z is greater than 1. So, I should have another circle here with the radius one. So, the bigger one is of radius 5, the lower one is of radius 1. I am talking about all the zs that are in this region this annular region. So, this is the region I am talking about. So, this is the annular region where all the points are placed with the condition that this. So, this is my annular region which is defined by this expression fine.

More example you can take I mean just change the coordinates. So, z minus 3 less than 5 less than 1, the same thing what do you need to do that you need to find out what is the point 3, here the point 3 is here somewhere not here. So, let me erase this and draw this once again. So, this is just the coordinate shifting and if I shift the coordinate the circle and everything will be same only thing is that. So, 1, 2, 3. So, here is the point and encircling this point I should have one circle here with 5 and one circle here for 1 and this is the region I am talking about, this is the entire annular region I am talking about with this. I can also do that with z minus 2 plus i mod of this quantity is less than 5 less than 1 greater than 1 so that means, this point will going to shift here and I will do the same thing and so on. So, I believe now you have an idea how to define a region in a complex plane. So, it will be useful because in the future class in the future problems or in the future theories this concept is required. Now, I know what is the meaning of region.

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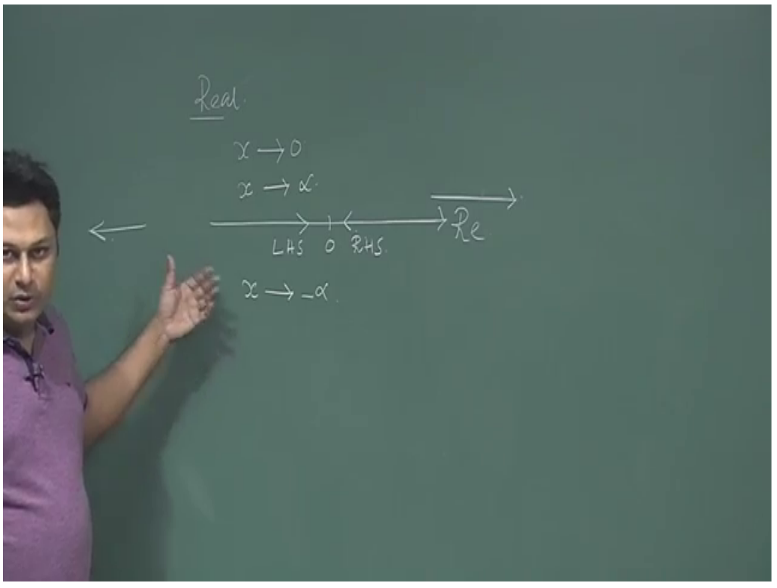

Next thing is important because in the next case we need to find out what is the meaning when I say z tends to 0 or z tends to infinity. So, why it is important let me do that in the real case when I say x tends to 0, this is my real plane or real line rather not real plane real line. This is my 0 point x tends to 0 essentially means I am going from either this side or this side there are two options through which you can approach to this zero point either left hand side approach or right hand side approach. This is the two approach through which you can come to this particular point 0. In the other way, x tends to infinity means you are going to one direction and that is this go infinity. X tends to minus infinity you are going to that direction which is infinity right. So, you have a proper direction through which you can understand that when I am saying x tends to infinity that means, I am going to which particular direction, but in complex case this is slightly tricky.

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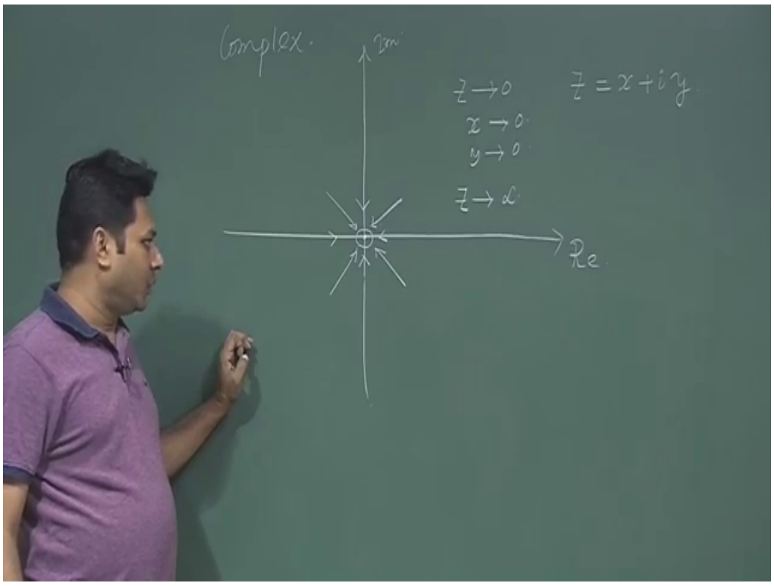

In the complex case, we have a plane here. Again you should remember that here we are dealing with the plane. Here we have real axis, here we have imaginary axis. If I say z is tends to 0 that means, I am going to the point z equal to 0 means what x 0 and y 0 individually because z is equal to x plus iy. When I am saying z tends to 0 that essentially means that x has to be 0 and y has to be 0; that means, they tends to 0; that means, I am try to reach this particular point where both x and y equal to 0, this is the point where z equal to 0. So, this approach is not any particular direction because I can approach from these two this, I can approach from these two these and I can approach from any arbitrary direction I want, this is something which is entirely different from the real case. In the real case, my approach for 0 is for two different directions, but here you should have infinite way to come to a particular point which is 0 here for example.

Now, if I say z tends to infinity, the similar problem appears, so which direction I should go when I say z tends to infinity, what should be the direction where I should go. So, in order to reduce this ambiguity, so something we normally do.

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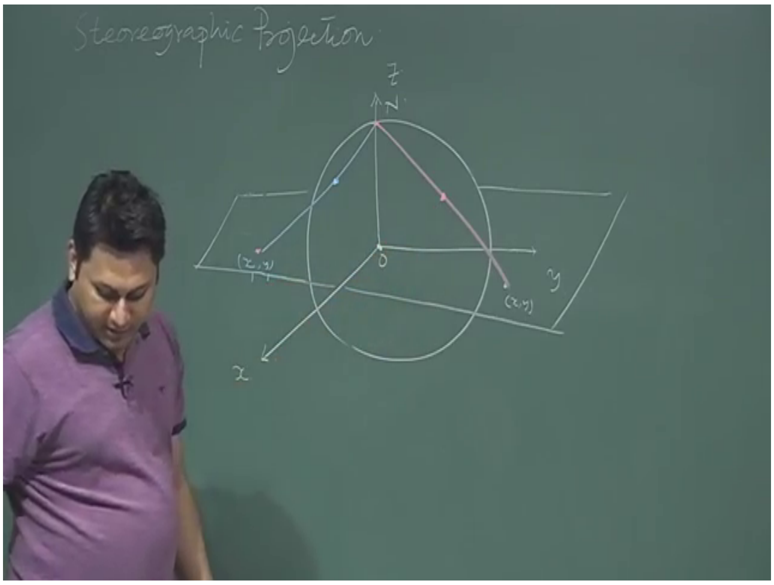

So, let us do that very briefly which is called stereographic projection. The concept is not very hard to understand. It is straightforward concept. So, what I will do that for example, this is my coordinate system x, this is y, and this is z; x, y is my complex plane for example, and z is the direction which is perpendicular to the complex plane. And now I draw a unit sphere over that encircling this particular coordinate system where the plane the plane is something this is the plane and this plane the circle is in exactly in the middle, so that the plane is fall in this way.

Now, the thing is that if you understand properly if this is say N this is say O, N stands for say north pole. If you want to find out a point here x and y or any point if you want to define, you can define a very unique way. So, the way is for example, this is my point which is x y coordinate. And if I draw a line from into that particular point, it should cut somewhere here, I am drawing a particular line here starting from the north pole and join the point which I try to find out. So, x y is the point where I need to so which point I need to find out. So, I joined this. When I join that, this line must cut the surface of this sphere in some point here.

In the similar way if I have some point here again x y some x 1, y 2 say different points, so that is why I put a different name. Again, if I draw this from here to here again it will cut some point on this sphere. So, this is this is a safe sphere which is encircling the entire thing I am drawing a line over the sphere and it cut this sphere over some point and then go to this point. So, this cutting point is unique for this unique point. So, for all the planes all the points over the planes, I will have a unique point over the sphere so that means, this point can be projected by all the points over the sphere. This is the infinite number of point in the x y plane; and for each point I will have a unique point here over the sphere.

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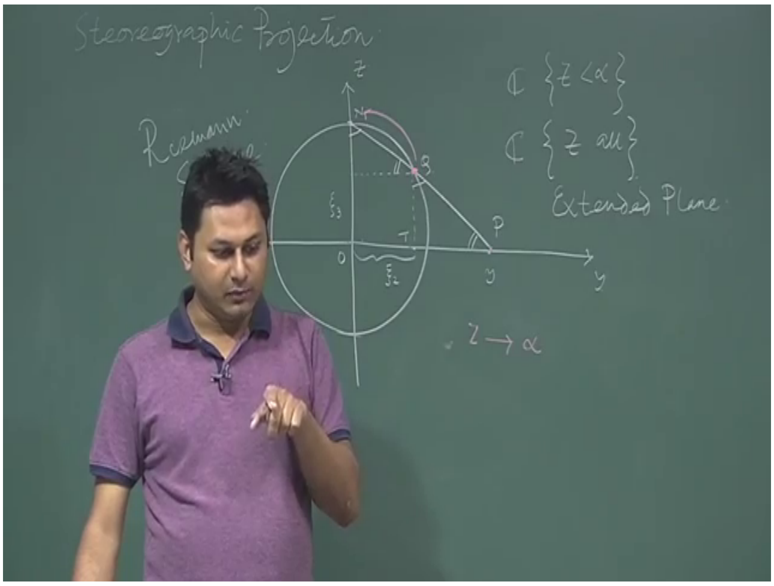

So, since the drawing is 3D maybe it should be maybe it is a difficult for you to realize the thing. So, let me write it draw it in a 2D case. So, it may be easier for you to understand. So, this is the coordinate system I say this is my z and say x equal to 0 so that means, this is my y, and the sphere is like this with unit radius. And here somewhere I have a point say x as I say y is equal to 0, so I will have one point over this I reduce the dimension, so that it should be under so that means, I make a slice of this sphere. So, that I will have a circle and this is my north pole N, this is my point O and I am joining this point here. If I join this point here and you can readily find that here there is a cutting point this point there is a cutting point and this is my y.

So, now you can readily understand there is a one to one correspondence to this point and this point if I change this, this cutting point will going to change, if it is cutting here this cutting point will change. So, there is a one to one corresponding you can readily understand what is the relationship between because since it is the mix sphere I can have the coordinate all the points coordinate over all the points of this sphere say psi 1, psi 2 and psi 3, this is a coordinate over this sphere. And since the sphere has a unit radius, so I have a additional expression that psi 1 square plus psi 2 square plus psi 3 square is equal to 1. Psi 1, psi 2, psi 3 is nothing but the polar coordinate of this thing if you do the polar coordinate then psi 1 will be sine theta cos phi; psi 2 will be cos theta sine phi and sine theta cos phi and then psi 3 will be just cos phi. Just do it and you will find that if you make this square it will be 1. Since r is 1, I should not put r, I should just put 1.

So, what is the coordinate of this point. So, coordinate of this point is say this is my from here to here this is psi 2, and here to here this is psi 3; and this O N is equal to 1, the length is one . So, now if I want to find out what is the value of y in terms of x in terms of psi 1 and psi 2, I can readily do that. So, if I say if I consider this say this point is say P and this point is say Q and this point is say T. So, from this triangle I should have a relationship, so it seems to be this angle and this angle are same. So, the ratio should be same. So, PO divided by PT is equal to this angle and this angles are same.

So, for this big case, it will be N O divided by 1 minus N O divided by N O minus z. So, let me check once again if I doing correctly or not is my triangle. So, two triangles are there. So, two triangles are there triangles NPO and triangles say QTP. So, this divided by this is this divided by this. So, PO divided by PT is equal to NO divided by this divided by this is equal to this divided by this, so NO divided by QT. PO is how much PO is y. So, let me and PT is seems to be 1 minus psi 2, NO is 1, and QT is nothing but psi 3.

So, if this is the case then I can have y is equal to 1 minus psi 2 by psi 3. In a similar way, if I do that for x I will have x is equal to 1 minus here it is psi 2 where it will be psi 1 by psi 3 and so on. The point is that if I have one point here, I can find out what is the value of this in terms of other coordinates so that means, this and this psi 1, psi 3 and y they are associated with one relation unique relation. So, for each value of y I should have one value of psi 1, psi 2 which is over the sphere and for 3D I should have entire points if I go to this I should have another component x and so on. So, that means, for each point I should have one unique point here that is the thing. It is just to show how to find out these things it is not a very critical thing to understand, but the point is if I go if I say this point is go to infinity, so that means, I slide this point to infinity then what should be the fate of this point.

If I go to infinity, you can readily understand that if P point is really very far away then this point will be shifted to the north pole, this point to be shifted to the north pole because this slope will gradually increase and this cutting point will go up and it will be equal to the north pole. Now, what happened if here I should have a point, so what happened so that means in this case also if goes to infinity in this direction this cutting point again merge to the north pole. That means, whenever the z is tends to infinity I should have a point which is uniquely go to the north pole of this particular sphere which we call Riemann sphere.

So, let me put the name of this sphere this is called Riemann sphere. And what is the beauty of these things what is the beauty of this projection the beauty is that even though I am going to infinity in my preferred direction, for example, here I am going to infinity in y direction, here I am going to in infinity in other direction, here I am going to infinity in different direction, so all over the plane I can go to anyway infinity, but this projection suggest that every time the point goes to N. So, that means, here with this projection what I am doing that I can say the directionality of the infinity. So, I can uniquely direct what is the meaning what is the meaning of infinity in the plane, so that means, what I am doing here I just this is my complex plane I just fold this complex plane. And when the point is infinity the folding means all the infinite point go to the north pole of this particular thing, so this is one thing.

And second thing is that so when I am saying I am talking about all my complex point where z is less than infinity with this condition I am excluding the infinite point that means, I am excluding this north pole in stereographic projection over the Riemann sphere. but if I say c where z is all that means, all value of z is possible that means, even I can go z tends to infinity and I can calculate then we can say that this is extended, this is called the extended plane. So, I am working in the extended plane where my z point can go to in x y surface, it can go to infinity.

Well, here I like to stop. So, today we will learn a few important things that what is the meaning of the region in complex plane because complex plane is not a straight line is a plane. When I am talking about a particular point over plane and say this point is approaching to a particular point then that means, there is a infinite way that point goes to that particular point not only that when I say z tends to infinity that means, there are infinite direction through which this can be possible. But the stereographic projection gives you the idea that you can fold this complex plane in such a way which we call the Riemann sphere that if I go to infinity that means, eventually I am going to the north poles; that means, a single point

So, in order to diminish these idea of going to infinity in infinite way, I can put my idea in such a way by putting this idea of stereographic projection that my unique when the point goes to infinity that means, eventually it is going to the north pole. So, I will going to stop here. And in the next class, we will start a very important thing which is complex function. What is the meaning of complex function we know the real function, but now we are in a in a position to explain what is the meaning of complex function and complex mapping. So, till then it is ok.

So, see you in the next class.