Mathematical Methods in Physics-I
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Lecture – 32
Roots of A Complex Number

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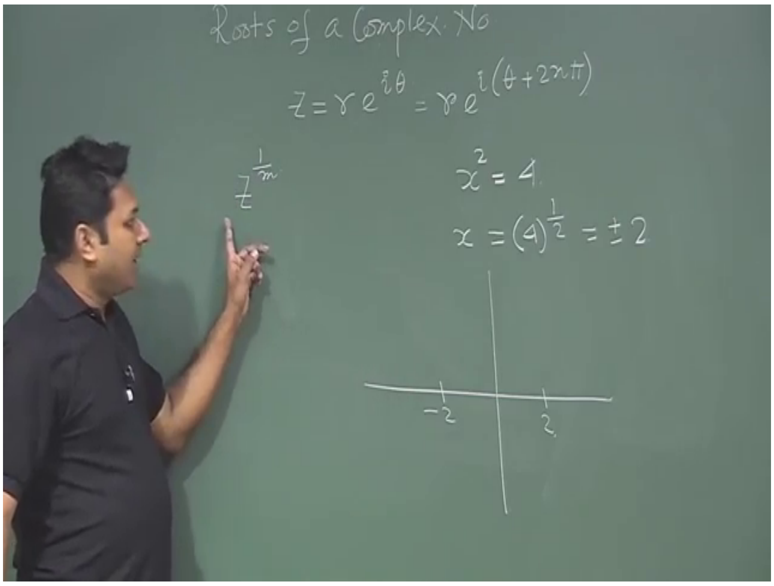

Welcome back students to the next class of the complex analysis. In the previous class, we learned how different points are placed in the complex plane; and we mentioned that by using the concept of principal value, we will evaluate the root of some complex numbers. So, let us start with this. So, roots of a complex number. So, if z is given to you like r e to the power i theta mind it this theta now I need to change because this theta is not the principal argument rather in general it should be e to the power of i theta plus 2 n pi.

Now, I need to find out what is the value of z to the power 1 by m. Before that let us go back to how to find say x square is equal to 4. So, what is the value of x 4 to the power half, which is plus minus 2, so that means, I will have here 2 and here minus 2, two point and this is the root of a real number. You can have any number and when you make a square root of that then you have two numbers; if you have a cube root of that you suppose to have three numbers; if the roots are real then three real roots you will get. But the important thing is that the plus minus sign gives you two roots here. Here I am doing the similar thing, I am doing the similar thing, but instead of having a real number I am having a complex number in my hand which is z, z 1 to the power n.

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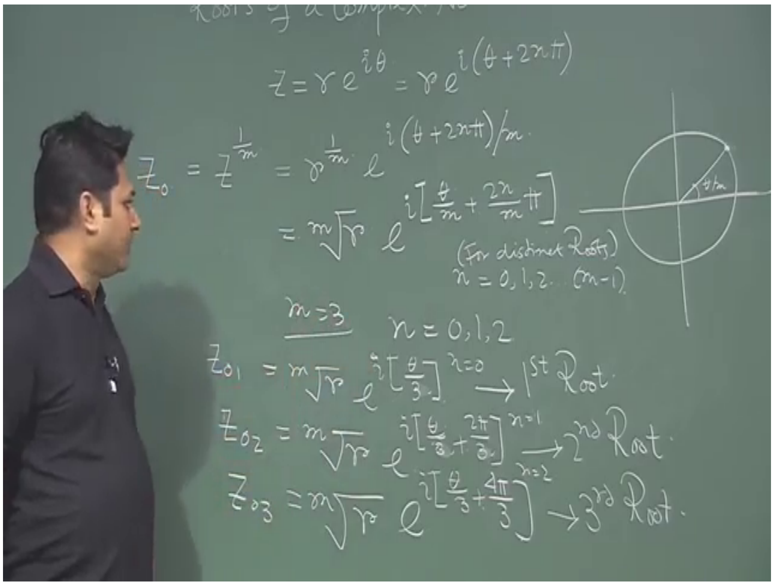

If I do that in the right hand side, I should write r whole to the power 1 by m e to the power i theta plus 2 n pi divided by m right this quantity. I called it as my z 0, because z zeroes are the root of whatever mth root I am try to find out. So, this is the mth root. So, if I write it will be m root over of r e to the power of i theta by m plus 2 of n of m pi. Now, this value of n is 1, 2, 3, 4, 5, 6, 7, 8 and so on, but here in order to find out the root what should be the limit of n. If you look carefully you will find that when n is equal to m, I will have a value which is this plus 2 pi.

So that means, whatever the value I will have as a root. So, how many roots you will find for m. If m is 1 single value then because of this n is changing, you will have in fact, infinite number of points with the value of n. I will show you the example then this then this things will be clear, but I would like to appreciate at that point that when the value of m is equal to n, you will have n and m same. You will have pi theta divided by m plus 2 pi, so that means, whatever the theta by m say this is by theta by m and when I say this is plus 2 pi. So, I will return back to that point and so on if I so. That means, in order to have the distinct root my n should have some kind of restriction here because I try to find out what is my distinct root. So, here the value of n will be this, values of n rather n will be 0, 1, 2, to n minus 1.

For this n is fixed say m is 3 as an example, so how many n value you will have n will be 0, 1 and 2, not 3 because if n is 3 then n and m will become same. So, I need to go with m minus 1. So, whatever the m I have three I need to have 0, 1 and 2. So, for each n 0, 1 and 2, I will have one individual number here. For example, z 0 1, I will have how much m root over of r e to the pi power of i theta by m that will be my first value. And essentially this is my first root, because in the real case I find that if I have x square is equal to 4 I should have two roots here if I have to the power 1 by m; that means, I should have my m number of root. As an example, I put m equal to 3, so that is why I should have three roots. So, that is the thing I try to figure out how these three root are emerging.

What will be my second root, it will be theta by m plus n 0, I put here this is for n equal to 0. Second one for how much n equal to 1, when n equal to 1, I will have 2 pi divided by m, m is fixed here. So, m is 3, so rather putting theta I should put theta by 3, theta by 3, this one. So, now what will mean by z 0 3 third root, this is my second root. What will be my third root my third root will be m root over of r. So, mth root of r this term will not going to change this is a fixed term. What is changing is the phase part which is important here e to the power of i theta 3 plus 4 pi of 3, because you put n equal to 2 here, so it will be 4 pi and divided by 3. So, first root, second root, third root, so three root I will have.

Now if I go to n equal to 3, then you will find that is theta by 2 6 pi. So, 3 will cut to 6 pi it will be 2 pi. So, theta by 2 plus 2 pi these things and the first root are same point, this is my third root. So, the general expression of finding a root is nothing but this mind it here I am using m and n, m is the given value which order of root I need to find. And n is an integer n finding the root for distinct root, I should write here for distinct root the range of n should be up to m minus 1, so that I will have for 3, I will have three distinct root. As I mentioned if I go to the next root and this next root will be converted to the first one if I put n is equal to 2, n is equal to 3 here, here I put n equal to 0, 1; and here n equal to 2 not n equal to 3, I stopped here. After that if I go then I will return back to my first one. So, let me give you few examples then things will be clear.

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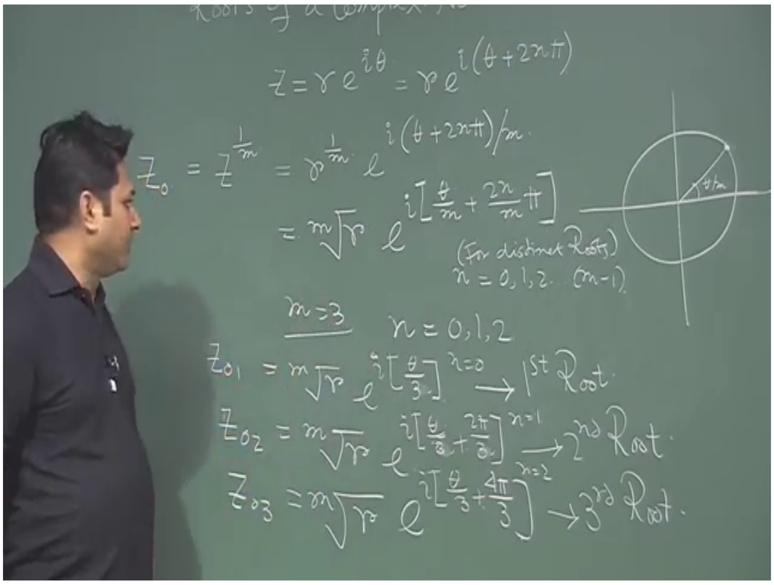

Let me take a, so let us find out what is my z here is i. What is the root, what is the value of this root over of i, what is the value of that. So, I know z is equal to i that means, it is one e to the power of i, this i suggest this value is pi by 2 plus 2 n pi this is my argument of z this is the principle value added by these things, so this total thing is argument of this. Now, z to the power half is one this no problem with that this remain 1 e to the power i pi by 4 plus 2 n pi divided by 2 making to the power half. So, the entire value which is here I should have put to the power half here.

Now, what is my restriction, I say if my m is given, here m is 2, the n value would be m 0, 1, to 2, m minus 1. So, here m is so 2, so I will have only two values, n is 0 and 1. So, if I do my first root z 0 1 this is my first root, it should be 1 e to the power of i pi by 4 which is how much cos pi by 4 plus i sin pi by 4. Cos pi by 4 is seems to be 1 by root 2 plus i 1 by root 2, it seems to be something like this. In order to ensure this is the first root; in order to ensure if you square that if I square these things 0 1 square, so this value is 1 by root 2 plus i square. So, this is half plus 2 i 1 by 2 minus half, half-half will cancel out this will give me i, so that means, this quantity is something which is the square root of this. So, first root, I calculate the first root.

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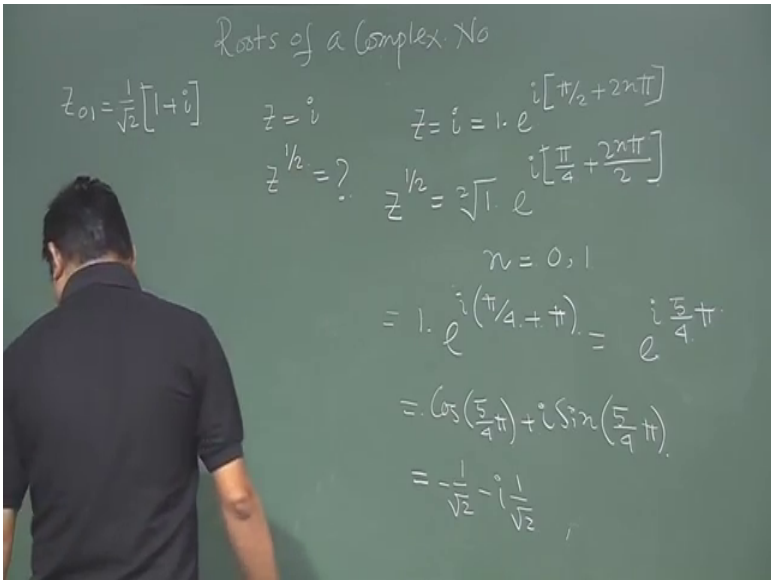

And the first root comes out to be let me write somewhere here. So, z 0 1, this is my first root; and this first root is something like half 1 plus i. What about the second root? The second root for second root n value should be equal to 1. So, if I do that, then it should be 1 multiplied by e to the power i pi by 4 plus 2, 2 will cancel out if I put n, so there should be a pi, this quantity. This is e to the power of i 5 by 4 pi. So, now, if I write this as cos 5 by 4 pi plus i sin 5 by 4 pi and 5 by 4 pi if you calculate from that you can calculate because it is with plus 1. So, we know that this value is how much cos a plus b, so it will be something like cos a cos b. So, it seems to be minus of 1 by root 2 and sin a cos b. So, minus of this is seems to be something like this. Minus of 1 by 2 these things so entire thing will be having a negative sign; obviously, because e to the power things is multiplied by e to the power i pi. So, only a negative sign will be appearing here. And if you calculate you will be having that if we squared that you will be having the same i here.

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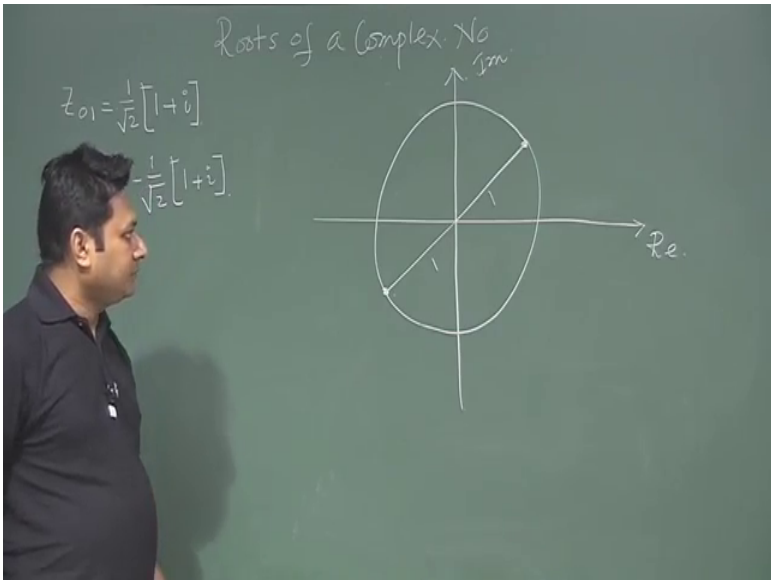

So, the second root what I am having seems to be is minus of 1 by root 2 1 plus i. So, I have two roots. So, now, if I put this two roots in my complex plane, so how it look like. This is my real axis; this is my imaginary axis. So, I need to find out what is the location of this z 0 1 root 1 and z 0 2, this is root 2 in the complex plane. And I find that if this is 1 by root 2 and this is 1 by root 2. So, this value one point is here and another point is somewhere opposite. So, both are minus, so with a magnitude of 1.

Essentially I will show that in the next example we will show that. It seems that these roots whatever the roots I am getting here if it is 2, I am getting the two points which are located here and here, but on the circle over the circle of radius one. So, this is one this is one. So, over the radius one over the circle these roots are placed. Now, if I do the same thing for some other values, for example, so let me find out the roots for some then look how the roots are placed in the complex plane that is interesting.

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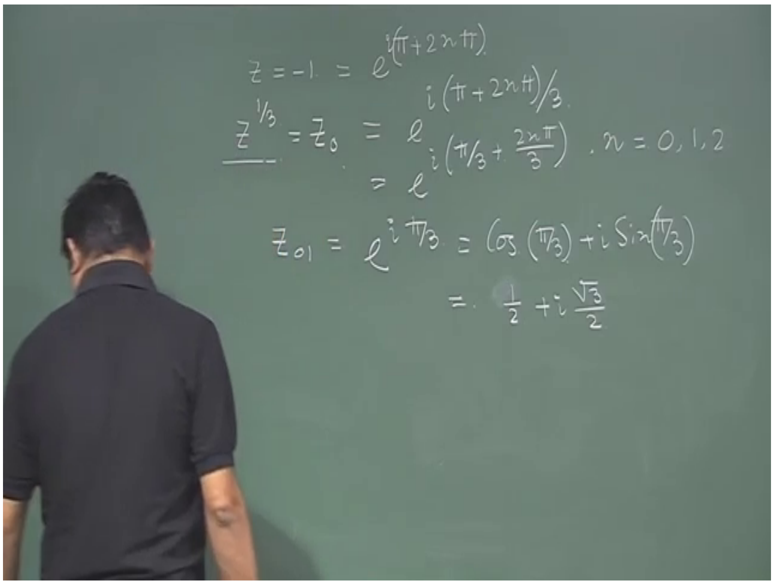

So, let us take another example. Say z is equal to minus 1, I need to find out z to the power 1 by 3 this I want to find out. So, minus 1 is in the r theta coordinate I need to write. So, it will be e to the power of i pi minus 1 is e to the power of i pi, but I will should not stop here because I need to write the total thing which is this quantity e to the power of i pi 2 n pi. So, z to the power of 1 by 3 it is z 0, if I say this quantity is e to the power of i pi plus 2 n pi divided by 3 or e to the power i pi by 3 plus 2 n pi divided by three this is the quantity I have.

Now, for n equal to 3, we should have three different three different root three distinct root. So, if I do that then n restriction of n here will be up to 0, 1 and 2. So, now, like previous one, z 0 1 is e to the power of i pi by 3, this is my first root I must say. So, it is cos pi by 3 plus i sin pi by 3 cos pi by 3 that means, cos 60 degree; cos 60 degree is seems to be root over of cos 60 degree is half and plus i root 3 by 2 seems to be something like this. So, first root I calculate and I have this one. If I write here my z 0 1 is half plus i root 3 by 2.

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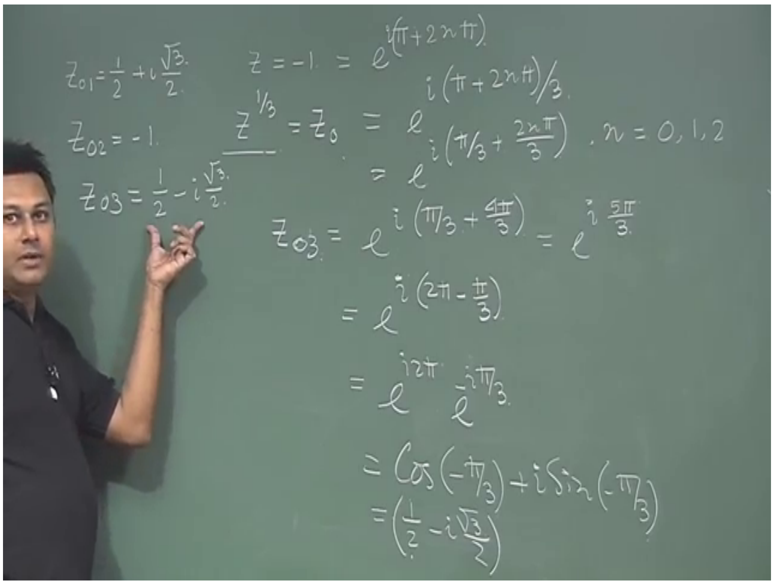

After having the first one, let me go back to the second one, which is the process is exactly the same that we have done in the previous case just instead of having one root, now I have should have three root because here n value is three. So, e to the power of i pi by 3, this is the second root. So, second root the value of n should be one here. So, 2 pi by 3, this value is how much e to the power of i if I just add these two things, you can find that it is e to the power i pi, e to the power i pi means this is minus 1. So, minus 1 again another root we know that if I take minus 1 multiplied by minus 1 multiplied by minus 1 you will return back to minus 1 that means, for minus one minus one is seems to be another root. So, z 0 2 is minus 1.

What about the last root z 0 3 it is e to the power of i pi by 3 plus i now put n equal to 2, it will be 4 pi by 3. So, it is nothing but e to the power of i 5 pi by 3. So, 5 pi by 3 can be represented can be written as say 2 pi minus pi by 3, to make the thing easier because 2 pi minus 3 pi by 3, so this quantity I can write as e to the power i 2 pi minus pi by 3. These things and these things are the same; I just rearranged the term slightly, so that I can have, so e to the power i 2 pi multiplied by e to the power i with a negative sign pi by 3.

So, if I do that then you can see that this quantity e to the power i pi gives you 1, and this quantity e to the power minus i pi by 3 is the old value I mean that means it will be cosine of minus pi by 3 plus i sin minus pi by 3. You can write in this way either or write cos pi by 3 minus of i sin pi by 3. So, if I do then I will have here half minus i root 3 by 2, I will have one value like this, this is my third root. So, let me write here my third root 0 3 is half minus pi root 3 by 2, this quantity. So, now, I know how to calculate these things.

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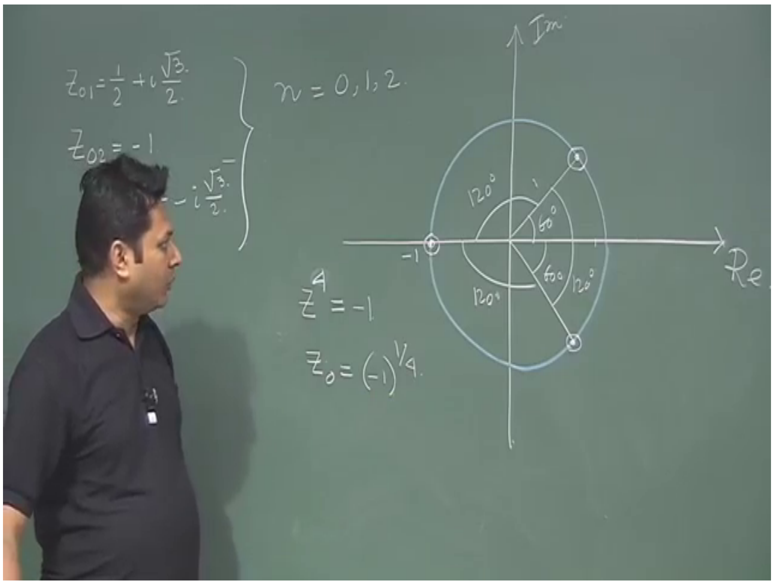

And if I plot that if I put this value here somewhere in the real and imaginary the complex plane rather this is real and this is imaginary. And now start putting the point this is half, minus 3 by 2. So, half somewhere here, and 3 by 2 somewhere here. So, the radius is 1. So, let us put first all the cases the amplitude is 1. So, this is 1, 1 by 2 square plus 3 by 2 square that means, 1 by 4 plus 3 by 4, this is 1, this is 1 this is also 1. So that means, if I draw a circle of radius one like this all the roots at least this three root should fall over this circle. First point is somewhere here, second point is minus 1 that means it should be here because this circle is radius of radius 1, this is 1, this is 1. So, this has to be this point is minus 1. And what about this point, this point is somewhere here.

And you can see that this points are placed very symmetrically, because it is what is the angle between this I need to calculate the angle between this so that means, slightly change because it should be 60 degree my drawing is not good. So, because the first point this is 60 degree, this is 60 degree. So, this angle between this two is 120 degree; the angle between this is this is also 120 degree, and the angle between this and these two is also 120 degree. So, three roots are placed over a circle which is of radius 1 they are apart from 120 degree, so that they can fall this. So, this three distinct root are coming because of the fact I put n is equal to 0, 1 and 2 in my argument right.

So, now, students please note that if I put n equal to 3 here, then first root is here second root is here third root is here. If I put n equal to 4, n equal to 3 - in this case then I return back to my original root here, which is not a distinct root. So, in order to make the distinct root, I need to follow these things and this is the result I am getting. Also you can find out different other values, for example, if z to the power 3 is equal to say minus one what should be the value of z; that means, eventually I need to find out this quantity z 0 is minus 1 whole to the power 1 by 4 again. So, the same problem instead of having three root I need to find out what is the 4 root. And if you do that if you try to find what are the root then again you need to calculate in this way. And you find that you have four root placed very equally spaced four root over a circle with unit radius with the separation between them is having a separation between them which is related to this angle.

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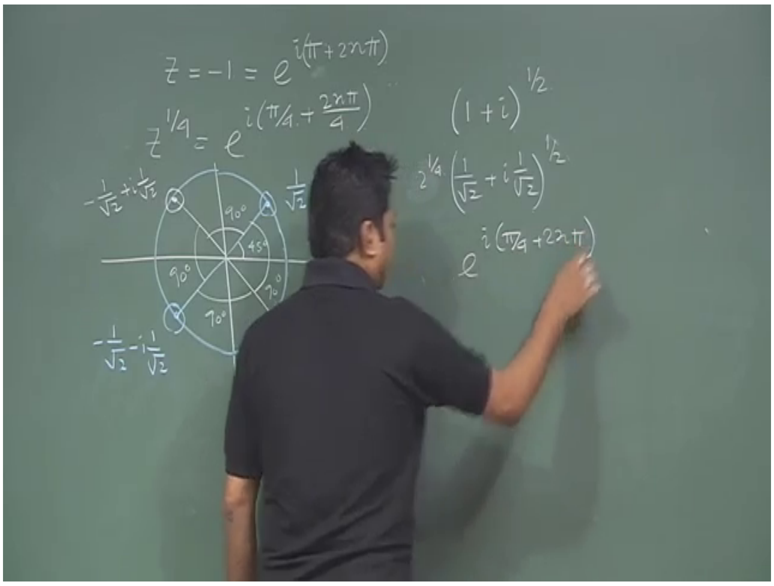

So, if I quickly try to find out, so minus 1 z is equal to minus 1 is equal to e to the power of i pi just to show how to calculate these things. So, I need to find out 1 by z, so e to the power of this I should write again with 2 n pi I should not forget these things. So, e to the power of i pi by 4 plus 2 n pi by 4. What should be the value of n here, n should be 0, n should be 1, n should be 2, and n should be 3. For this case, what will be the first value first value is e to the power of i pi by 4. What is this value, pi by 4 1 over 2 plus i 1 over root 2, it seems to be something like this. pi by 4 means 45 degree 45 degree meanings means this if you take a to the power 4 of this quantity, you will have minus 1 return back. If you have a square then this square plus this square is 0, you have two and then let me check once again whether I am doing the correct thing or not once again.

So, it is e to the power these things, so this. So, this quantity yeah this quantity if I took a square of this quantity just to check. So, it will be half plus i by 2 minus half. So, half, half will cancel out, it will be i by 2. And again if I make a no, it is simply i, it is simply i; and then if I make a square of that it will be minus also correct. So, my first root is placed somewhere.

So, if I draw now the important thing I like to mention here again I am doing the same problem, but now my root is 4. So, how many points you are going to have here, this is the circle of unit radius. And you will have the first point here at 45 degree you will have the first point because if you remember the root was i plus i root 2. So, this and this you will have the first. If you calculate then you will find that the next root will be fall will be here. The value will be without doing anything I can write the value 1 by root 2 plus i root 2.

Next value will fall here with both the thing minus and finally, you will have the point here, which is 1 by root 2 minus of i 1 by root 2. So, this will be the four roots of minus 1. So, in the similar way, you can find out what is the three roots of 1, you know which is one omega, omega square and you are well aware of these things. But the important thing is that these things are placed over a circle of radius 1, here in case r is 1 that is why it is 1, but in general it will be over the radius of r to the power 1 by m if m is given. And this angle is measure this is 45 degree. So, the angle between these two root is 90 degree, this angle is 90 degree, this is 90 degree and this is 90 degree, so that they can form a circle over the circle they can put.

So, with this note let me conclude that. So, today we will learn a very important thing that applying this all this power thing a principle value on all these things, we try to figure out how you can find a root. For example, this is the simple example, but let me quickly give you homework. So, what should be the value of say what should be the value of 1 plus i whole to the power of half, this what you will do that you will just multiply this. And if you take it, it will be 1 by 2 to the power 1 by 4 I guess and oh if you multiply, so it will be 2 to the power 1 by 4 and then you can write it as cos and sin with some. So, it will be e to the power of i pi by 4 plus 2 n pi. And then you proceed because now your r is slightly change instead of 1, now it is 2 the power 1 by 4 and then you proceed with these things with the similar procedure and you will find the roots of these things. Whatever the value is given say 1 by 2 is given if I can change it to 1 by 3, but whatever the value is given this is the process is quite general. And if you use this general process, you can find out what is the root of a given complex number.

So, at that point, I should end my class today. So, in the next class we start now i know how to find out the root and all these things, but important thing is that how the roots are placed over a in circle in the complex plane. So, we will learn more about the complex plane and more importantly we will learn the functions.

So, in the next class which is very important where we start the concept of complex function we know the real functions. So, what is the complex function how it can form, so from where it is coming and what is the consequence of having the complex functions because there are two variables are associated with that this kind of things we will learn in detail. With the idea of how this in the plane in the complex plane, how the points is defined, angular region a circle how these points will be defined these things we also learned before going to understanding what is the meaning of complex function. So, let us stop here. In the next class we will start from the concept of complex function.

With this, thank you and see you in the next class.