Mathematical Methods in Physics-I

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Lecture - 30
Geometrical Interpretation Of Complex Number

Welcome back student to our complex analysis course. Which is in the mathematical methods, course is the one of the major part we are dealing with. So, in the last class I mentioned something interesting that z 1 greater than z 2.

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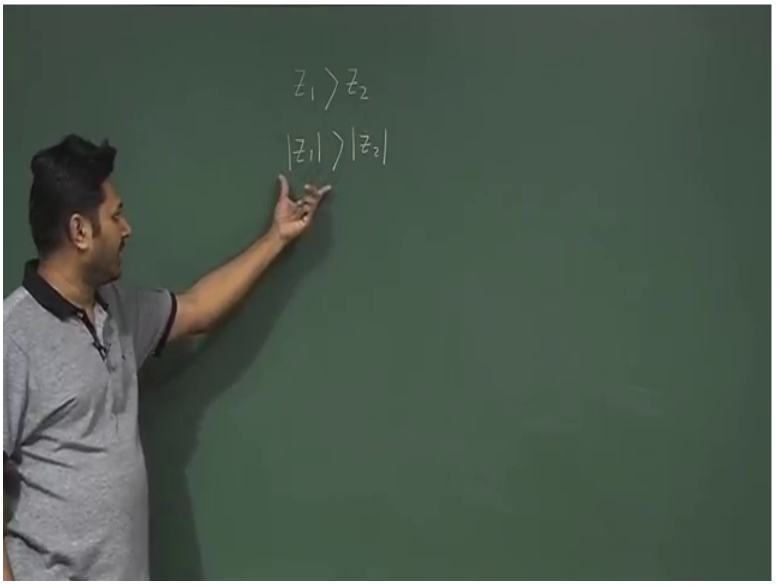

It is a meaningless sentence or meaningless expression, z 1 greater than z 2 with a mod sign is a meaningful expression. This is a meaningless because I am comparing 2 points here I am comparing 2 lengths which is justified.

So, now let me show you the geometric representation.

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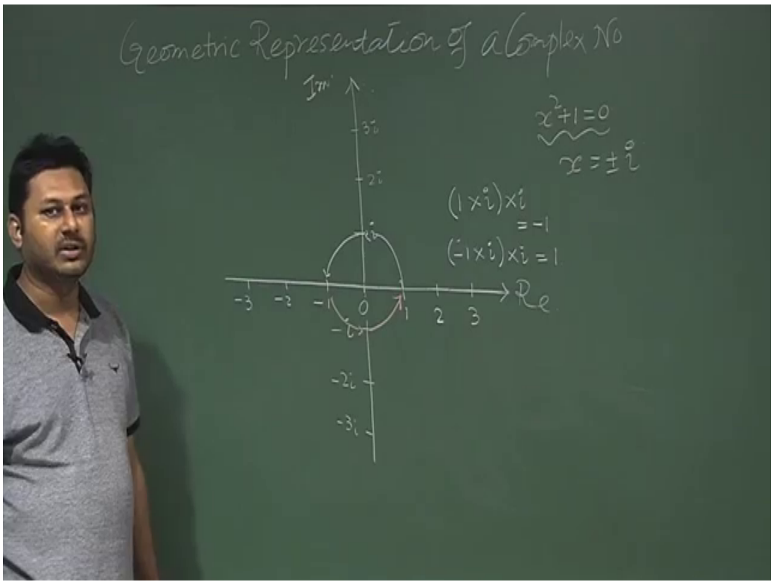

So, geometric representation of a complex number; so geometric representation of a complex number; what is the meaning of geometric representation? In the very first class I mention, that I have a line which I call the real axis over which all the real quantities are there.

This is 0 and this is positive this is negative all the quantities are there. Now I find x square plus 1 equal to 0. This kind of solution the solution of this things, do not have any real solution. So, the solution I will find something which I called imaginary.

So, now the imaginary numbers if the imaginary numbers are there, then we need to put this imaginary number somewhere here, but it should not be in the real axis it should be somewhere else. So, for that for them we need to find another axis which is perpendicular to the real axis, and this is your imaginary axis. If I say this is my one this is my 2, this is my 3. This is my minus 1, this is my minus 2 this is my minus 3 and so on then in this side I should write this is my i, this is my 2 i this is my 3 i and so on.

In the downwards I, also I have, minus i minus 2 i minus 3 i and so on. So, now, the things are clear, if I have one number one. If I multiply that with another quantity i essentially, I am having this point going to this point so; that means, a 90 degree rotation is there somewhere. Then this point again I multiply with i, I will have minus 1; that means, this point again have a 90 degree rotation and I am coming back to minus 1 point. Again I am multiplying this minus 1 minus 1 with i and when I multiply these things then I will have a minus i; that means, I am again rotating 90 degree. And come to this point let me use another colour this point again, I multiply this minus i this quantity with I will have one. So, I will return back my original position which is here.

So; that means, by multiplying a real number with some number which is which is complex, which is totally imaginary is eventually I am rotating this point from here to here. And then here to here and here to here I can see that. So, I have a real number and just multiply with i or 2 i or 5 I whatever I like, what we are doing we are putting this point or rotating this point somewhere here.

For example, tow is here if I multiply with 3 i then I have 6 i it is not a rotation it will be like this. So, the point is changing there is there is something which is changing the angle also. That means, angle is something which is important that for the timing we should note, but there is more on that. So, let me now represent a purely complex number, complex number is something which has a real and imaginary component together.

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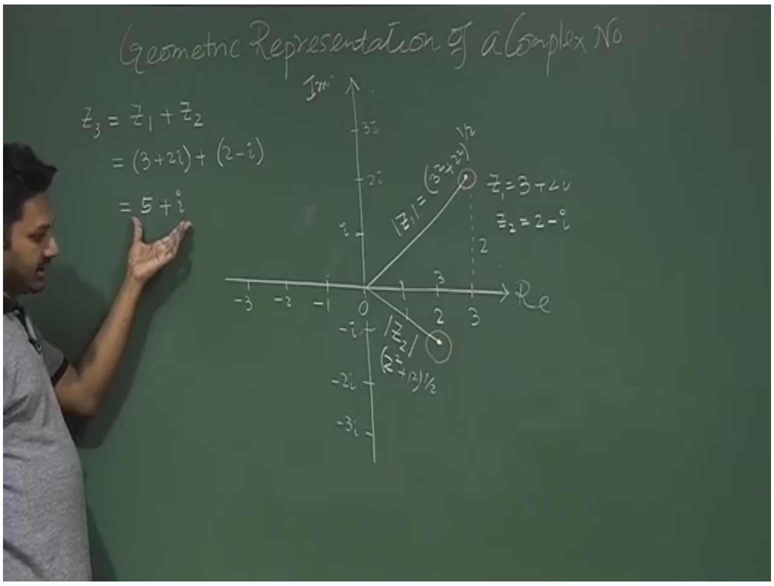

This is this is i, this is minus i this is 1. So, now, any number complex number z, which is say x plus iy where x and y both are real can be placed in this complex plane, this is called the complex plane real axis in along x and imaginary axis is along y. So, entire plane is complex plane, and all the points each individual points are the point in the complex plane. So, if I write say I want to find out what is the value of z is equal to 3 plus 2 i, x value is 3. I need to go 3 1 2 3 and y value is 2. So, the point is here. So, this is the point. So, z is essentially gives me a point. That is the thing last class we mentioned this is the thing which is giving you the point.

Now, I want to find out another point this is z 1 another point z 2, which is say 2 minus i. This is also another point where the point locates it is 2 12 and 1, in the down stair because it is minus I; so somewhere here. So, so the point is here. So, z 1 and z 2 are merely representing 2 points in the complex plane that is all it is not more than that. It is just 2 point which are representing 2 things in 2 point 2 in 2 points in the plane. Now if I want to say that z 1 is greater than z 2, that I mention in the last class that; obviously, meaningless because this point and this point are just merely point, but now I am if I say mod of z 1 is greater than mod of z 2 that is meaningful that is meaningful why.

Because mod of z 1 is nothing but the length of these things from 0, this is your mod of z 1. What about mod of z 2? Mod of z 2 is also the length of these things. So, how I calculate the length, we know that if it is if this value is 3, this value is 2. The length is nothing but 3 square plus 2 square root over of that. So, it is nothing but z 3 is nothing but 3 square plus 2 square whole to the power half, which is by the way is the norm of this quantity the mod of this quantity. So, mod of z 1, we know by formula it is root over of x 1 square plus y 1 square if it is represented by x 1 and y 1 then it is here it is nothing but 3 square plus 2 square to the over that so; that means, 9 plus 9 plus 5 whole to the power of half.

In the similar way, we have this quantity this length this length is nothing but this mod of these things which is this length and this length is 2 2 square plus 1 square whole to the power of half root over of 5. So, here I have something, and here I have something now I am saying that mod of z 1 is greater than z 2 because it is giving you the length of these 2 2 points.

So, now I understand; what is the meaning of these things. Now few interesting things of that I have 2 points these and these. So, it seems to be a vector like thing. So, now, I want to find out z 3 which is the addition of these 2 quantity z 1 plus z 2.

So, I know the addition formula z 1 is 3 plus 2 i and z 2 is 2 minus i. So; that means, it is something 6 3 plus 2 sorry 5 and then plus 1; that means, 5 plus I. So, if I put this 5 plus i, it will be somewhere some point in here; and how we are grading these things. So, it is basically the vector rule we are we are having. So, vector will suggest that if I add these

2 quantity like one vector plus another vector, I will generate another vector. So, let me do that like this. So, I need to extend these to find out where the point should be locate it is 4 it is 5.

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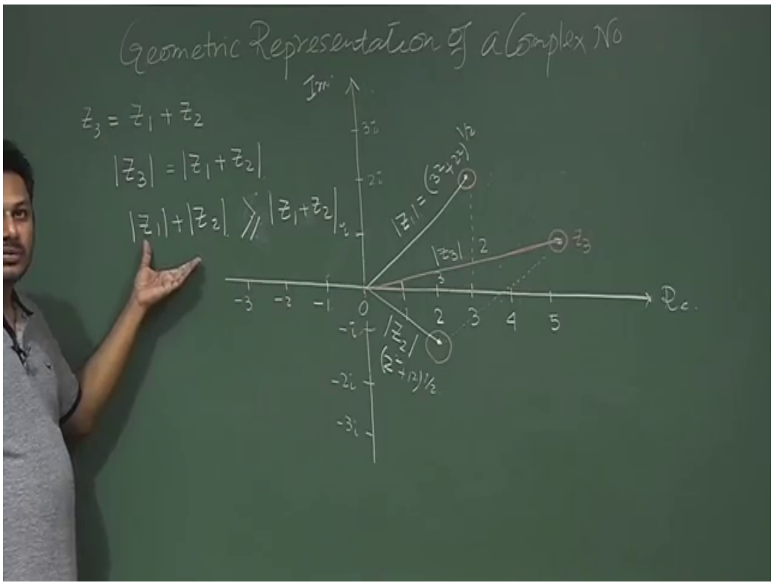

So, if I make a parallelogram to this quantity my scaling is not correct, but if it is correct then it should be somewhere here and if I add these 2 things. This is a vector parallelogram. So, 5 plus i is somewhere here 5 and this is I. So, my addition of 2 things give me another point which is this; this is my z 3, but z 3 as I mentioned this is the addition of z 1 and z 2 z 1 is this z 2 is this, if I consider these 2 as a vector then this is nothing but the vector addition of these 2 quantity it is nothing but the vector addition of this quantity. So, this quantity red line is basically gives you this point and this red line is basically gives you mod of z 3 the length of these things.

So, addition of 2 quantity in this plane addition of 2 complex quantity in this complex plane is nothing but if I consider these as a vector this as a vector and if I add these 2 vectors separately. I will have a resultant vector it is something like the resultant vector which is given by this z 3 and from 0 to z 3, if I if I join the length of this line is basically the norm or the mod value of this quantity. So, from this we can now say one important thing if z 1 and z 2 is this if I want to find out what is the relationship between mod of z 1 and these things. So, mod of z 1 is equal to mod of z 1 plus mod of z 2. This now we know that this is the third angle third this is one this length and this length are same. So, from here we can have one important identity mod of z 1 plus mod of z 2 will be always less equal to sorry always a greater equal to mod of z 1 plus z 2 mod of z 1 plus z 2 is nothing but z 3 which is the sum of these 2. So, these 2 are if I make a mod of these things, it will be always be less than whatever and this is nothing but the triangular law.

So, you know the triangular law to 2 sides if add up 2 sides is always greater than the third side in a triangle. So, here we are using these tools, and this is a very well known law in n complex analysis in the similar way. In the similar way I can let me erase this because I believe you now understand the entire concept of these 2.

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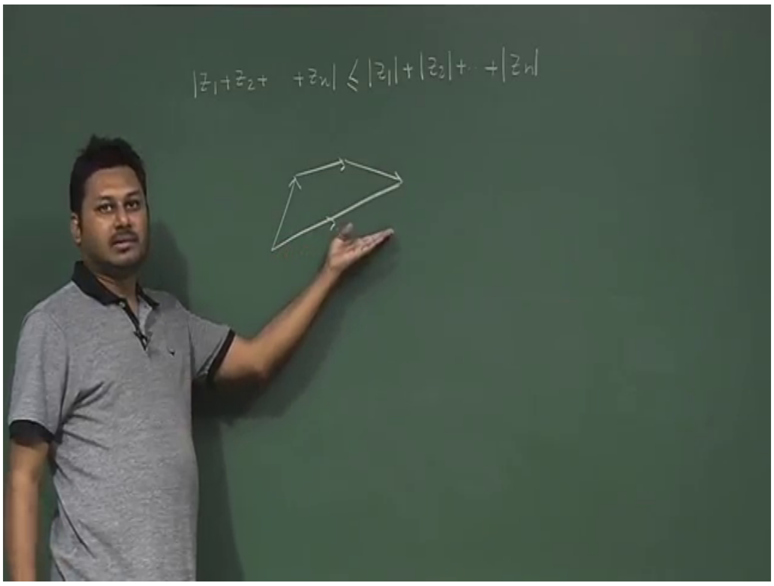

With this we can also write a general form z 1 plus z 2 plus zn is always less equal to mod of z 1 plus mod of z 2 plus mod of zn. In general, in the same way if I trying to add this is my one vector, this is my another vector this is my another vector and if I add these to vector result in vector. So, this plus this plus this always be greater than this now the next thing is that. So, far we are dealing with the complex numbers and these complex numbers are represented in Cartesian coordinate.

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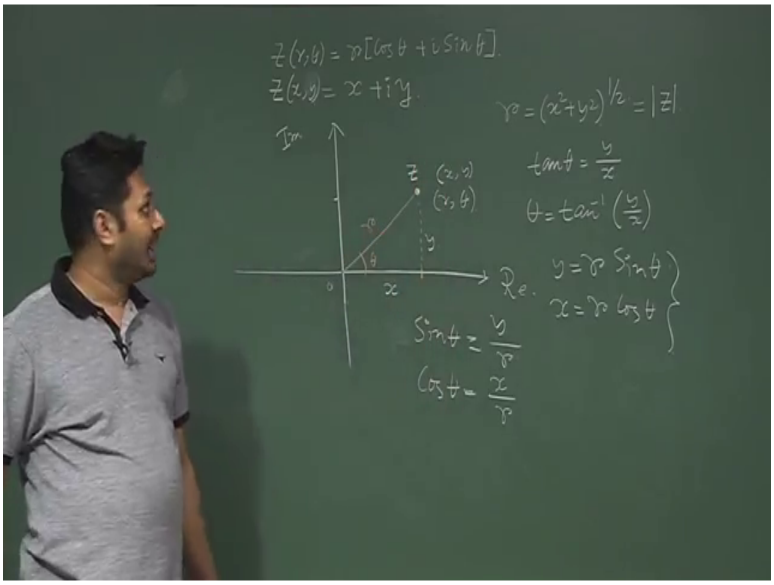

So, z is represented by x plus I one in Cartesian coordinate, I am representing that and also we find that this is my real axis this is my imaginary axis and zi represent like x plus iy 2.

So, some point here this is my x this is my y; so somewhere here. So, this point is my xy which I represented like x plus iy and this is my Cartesian coordinate. Now we can also write the impacting in different coordinate system which is polar. So, we call it a polar form what I need to define polar depicts to define this point, which thing I need first thing is the coordinate which is here x and here y.

So, once the coordinate is giving to you in the Cartesian form you readily understand, which point you are talking about and this point is nothing but the complex number. Now what I try to do that I try to define the same point in different cord coordinate system and here my coordinate system is polar. So, I if I have this angle theta and this r here then I can also write these things what should be the coordinate of this I should write it as r theta.

This is a different coordinate polar coordinate in the polar coordinate; I also write these things now if I write r theta readily you can understand. There is a relationship between r and theta and x and y and we know that very well. So, r is nothing but x square plus y square whole to the power half it has to be because r is the length of this number. So, if this is my z which is x and y. So, r is nothing but the mod of z. So, which is equal to mod of z what about theta also we know because I know, this is x I know this is y. So, if x and y is known to me then what I can do I can do tan theta is y by x. So, theta is tan inverse of y by x. So, I have r in form of x and y I have theta in form of x and y and if I have r and theta in my hand, I can readily write this into another coordinate system r and theta.

But apart from that I can write these things in a different way not in the different way, but let me now replace this x and y in terms of r. And theta my goal is to write z in terms of here z when I write z, I should write it here z is a function of x and y right. I need to write z which is a function of r and theta how to write that. So, now, my x has to be replaced and y has to be replaced in terms of r and theta in terms of r and theta. So, from here I know that sin theta is how much, this divided by y divided by r cos theta how I define cos theta x divided by r.

So, my y is essentially r of sin theta and my x is essentially r of cos theta. So, now, I can write my x and y in terms of r and theta. And an rn theta once I do that then I can readily write this as r cos theta plus i sin theta. So, I define the same point, I define the same point in terms of xy which is coordinate system, but apart from that I can also write this in a different way which is cos theta plus i sin theta after having that, let me now erase this hopefully you understand, what is the geometry a beautiful thing comes to our hand. Now, I defined a complex number in a complex plane with x and y and an r and theta which is defined in this way.

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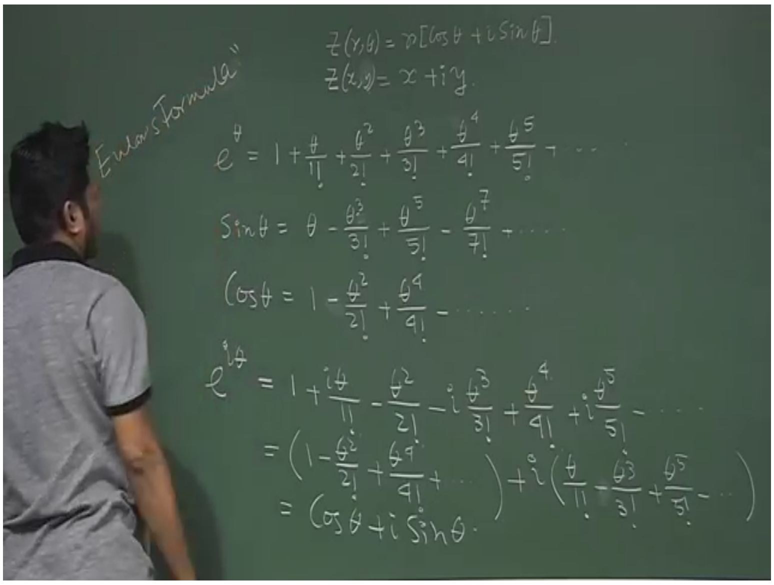

So, let me do one thing. So, what is e to the power theta expansion of e to the power theta is 1 plus theta by factorial 1 plus theta by factorial 2 plus, theta square theta cube by factorial 3 plus theta to the power 4 by factorial 4 plus theta to the power 5 by factorial 5 and so on. This is the expression of e to the power theta; what is the expression of sin theta which is theta minus theta cube by factorial 3 plus theta 5 by factorial 5 minus theta 7 by factorial 7 and so on; only the odd numbers with changing sin cos theta is one cos theta is 1 minus theta square by factorial 2 plus theta to the power 4 by factorial 4 minus so on even term even power with changing sign.

Now once I have sin theta and cos theta in our hand then readily, I can put one thing that if I put e to the power i theta instead of theta then this expression is slightly change and it will be theta e to the power i theta is 1 plus i theta by factorial 1 minus, theta square by factorial 2. Because theta square there is a minus because in place of theta I am putting i theta and then I make a square of that. So, minus sin will b because of that I then minus of i of theta cube by factorial 3. This term will be plus of theta to the power 4 factorial 4, this term will be plus i theta to the power 5 by factorial 5 and so on.

Now if I look very carefully to this term one these thing these things all the terms which are associated with even powers if I take this term aside, I will have one series and what about this theta and this theta which is associated with the at the odd powers. So, what I will do that, I will take I common of that if I take I common then I have theta factorial 1 plus minus theta cube factorial 3 plus theta to the power 5 factorial 5 and so on.

So, it is like a miracle we have one series which is cos and one series, which is sin just multiplied by I. So, this is cos theta plus i sin theta. So, e to the power i theta is nothing but cos theta plus i sin theta, which I show you meticulously which is called the Euler’s formula which is a famous formula. And if I use this Euler formula, Euler’s formula then the point can be represented in a more compact way in polar coordinate.

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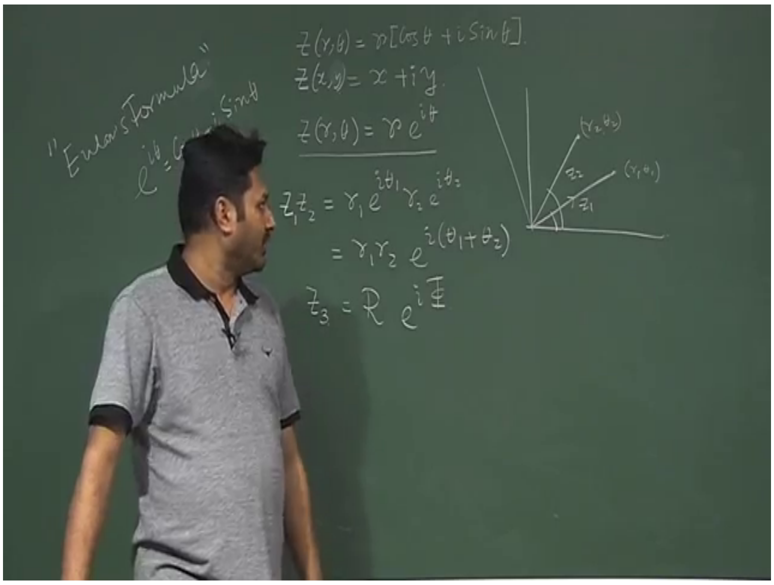

So, this quantity if I write in polar coordinate again with r theta it is r cos theta plus i sin theta. We know that e to the power i theta is cos theta plus i sin theta I will just replace this cos theta plus i sin theta to e to the power I theta. So, r e to the power i theta is the representation of a complex number in the polar form and very compact form very useful form. And this is something which gives you any interesting ideas and we will go to this for example, if I add, if I add if I multiply 2 quantity z 1 and z 2.

So, basically it is e to the power r e to the power i theta 1 multiplied by r 2 e to the power i theta 2 if I do that then I can write it as r 1 r 2 e to the power of i theta 1 plus theta 2 essentially what is the meaning of that r 1 and r 2, if I write big r and theta one and theta 2 if I write big phi. So, my new after multiplication, I have a new number new complex number I call it z 3. So, z 3 is now represented in polar form, but the interesting thing is that here. We have one angle here we have another angle. So, 2 quantity 2 points are like this, this is my z 1 this is my z 2 this point is r 1 theta one and this point is r 2 theta 2.

When I multiply these 2 things, these 2 quantities then we have another quantity whose amplitude is r 1 multiplied by r 2 some something bigger than that. And not only that the angle is this angle plus this angle. So, I will have something here at this point after making the multiplication of these 2 quantities. So, apart from that there are many interesting things that we will find in the next class, where we where we do many things with this form we need to find out for example, the square root of some complex number how to find out the square root of this complex number using this, this particular form.

We will learn that, but the concept is very nice that you can represent a complex point in complex a complex number in complex plane in terms of x and y which is convenient, but also you can do the same thing by putting the point in these things and you will find more information and you can extract more information with this form. And that is also equally important and gives you much more interesting aspects of complex number and you will learn that in detail in the next class.

So, with that let us stop this class today here. In the next class we start with this geometric representation and this polar representation and find out many interesting things which is related to complex analysis. And will be helpful for you to understand the complex analysis or complex points in a more convenient way. So, with that let us stop here.

Thank you for your attention. So, see you in the next class.