Mathematical Methods in Physics-I
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Lecture – 28
Monomial Basis, Factorial Basis, Legendre Basis

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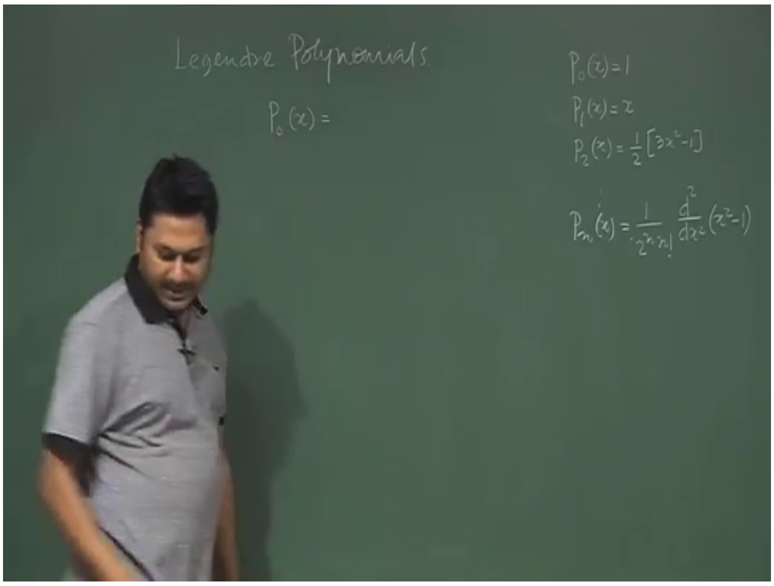

Welcome back students; so in our last class, if you remember we show some polynomial, which we call Legendre polynomials. The significance of these polynomials is, this is P 0 x was let me write it here P 0 x was 1, P 1 x was x, P 2 x was half 3 x square minus 1, there is something like this. There was a generating function that we also mention; if I want to find out what is the value of P n then it should be essentially 1 by factorial 2 to the power n factorial n then d 2 dx square x square minus 1 something like that. The thing is that these polynomials are forming a orthogonal set. So, how many octagonal set we find let me write it here once again.

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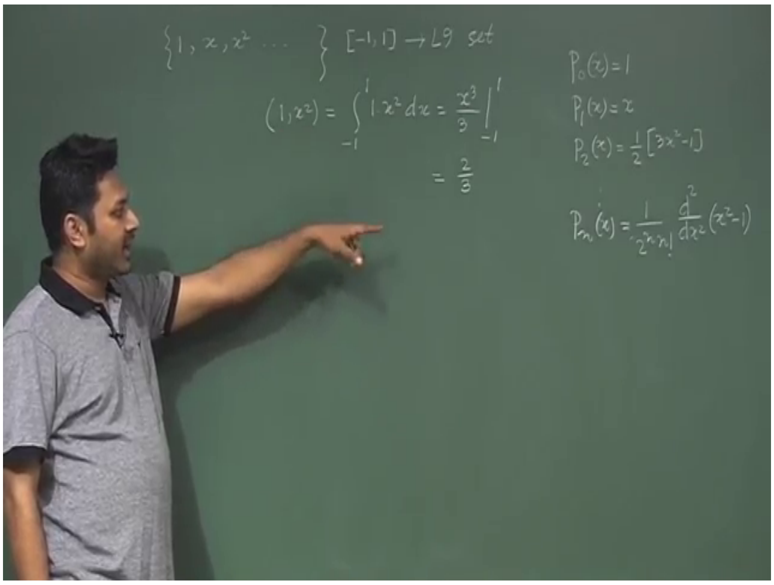

So, the first very first and fundamental was 1, x, x square this these are say in the range minus 1 to 1, these were linearly independent set of function, they are not orthogonal. If you calculate the orthogonality, you will find that for example, if I try to find of the inner product of 1 and x square, it should be something like minus 1 to 1, 1 into x square dx. It should be xs cube by 3 minus 1 to 1, it will come up as 2 divided by 3 which is not equal to 0. 1 is a function, and x square is a function, I just take these two function and then try to find out what is the inner product between these two. If the inner product is any of this, this, this linearly dependent set of function if the inner product is 0 then we can say they are orthogonal to each other, but here I find there is not equal to 0, they are not orthogonal, but then they are linearly independent.

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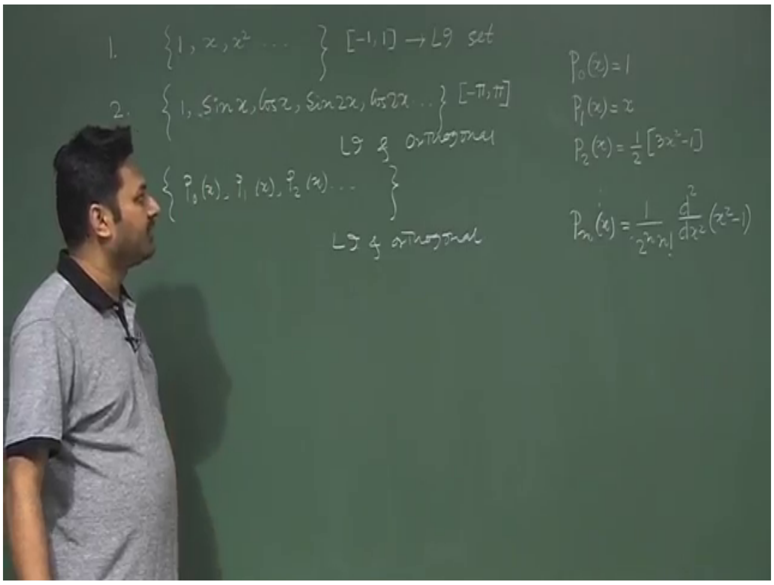

If you calculate the Wronskian, you can readily figure it out. So, this is one kind of linearly independent set and you can expand the function in terms of this. Another was one sin x cos x sin 2 x cos 2 x and so on in minus pi to pi, this limit, this range. So, this set of function are also linearly independent and there also linearly independent and orthogonal that means, if you take the integration over this limit you will find that they are going to vanish. So, also they are potentially form the basis. So, here is a set of function which can form a basis, here is a set of functions which can form a basis. If you remember the Fourier series is based on this basis. So, the Fourier series any function if I want to expand in terms of Fourier series, we will going to use these basis.

Next we find another last a which is nothing but the Legendre polynomials; potentially they are also form a set which are linearly independent and orthogonal. If they are linearly independent and orthogonal, there also can be used as a basis function. And if I expand this function in terms of in some coefficient, I can regenerate whatever the function I want, so that means, they are forming a basis function. So, there are three different kind of basis function that we have. Also if you remember let me erase that the middle one that we have already used in the Fourier series and we know how to deal with that.

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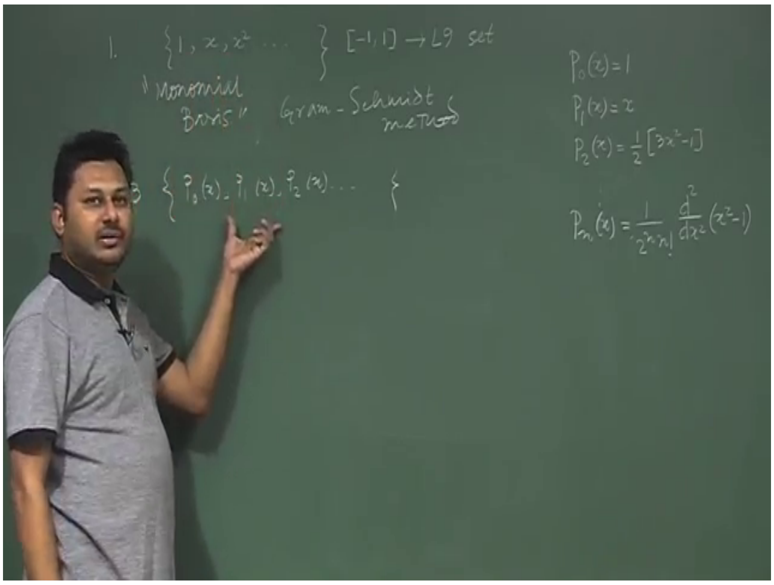

But from here to here in the last day, we calculate that if a set of linearly independent function is given to you, how to make this linear independent set of function a set of orthogonal function or orthonormal function. If I put the normalization constant here, then it should be normalized also. And we use the standard Gram-Schmidt method of orthonormalization. But please remember that when you make the Gram-Schmidt orthonormalization method then we put our inner product formalism as phi i, phi j is equal to integration of some limit say a to b phi i, phi j dx if both the function is dx.

If you remember another thing that this definition we put by making the weight function if I say w is my weight function is 1 or we define that in terms of r, r is 1. So, (Refer Time: 06:54) reference I mean if I try to write it more general way critical way then it should be something like phi j and r dx, r was the density function or the weight function whatever you say. But if I put r equal to 1, then my inner product definition will come to this and because of that I come across this linearly independent set of function, if you remember. But someone can change these thing by putting some value of r, some function of r, whatever. If they put some value of r then readily you can understand that this definition will going to change inner product definition will going to change because of this weight function or the density function r, and you are start getting a different kind of or set of functions which not be essentially the Legendre polynomials. It will be some different kind of polynomial, but still we are in a position to find a set of orthogonal function which is not in this form, it may be different it may be (Refer Time: 08:04) polynomial, they are different kind of polynomial Legendre polynomial.

This polynomial are also behave like a linearly independent set of functions, but here we put a specific example which is the Legendre polynomial. And we find that it can be derive directly from this basis to another basis by using the Gram-Schmidt method. By the way at this point I should mention that there is a name of this basis we normally call is monomial basis. So, this from monomial basis I can generate my set of functions which are Legendre polynomials essentially.

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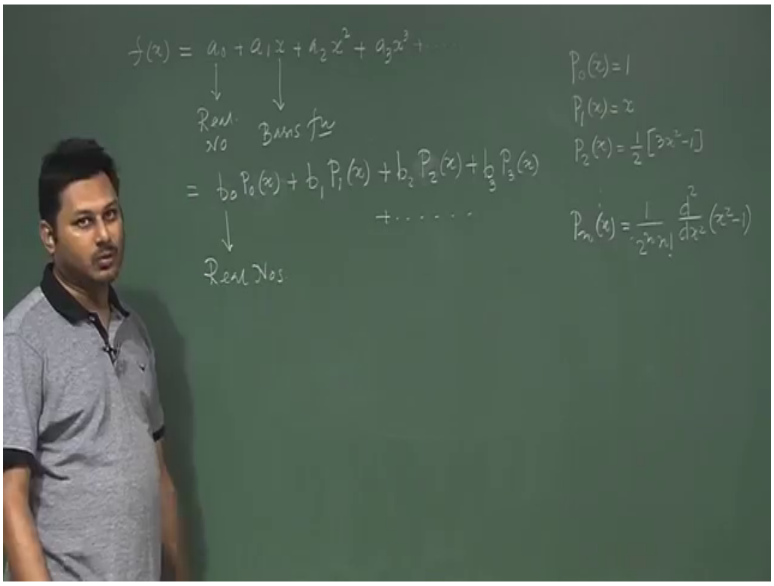

Now, a as I mentioned a function let me try to do something some example for example, some problem for example. So, a function I try to make it general. So, say a 0 plus a 1 x plus a 2 x plus a 2 x square plus say a 3 x cube the something like this. So, this function is defined in terms of monomial basis, why, because this is a coefficient this is the coefficients, some real numbers or coefficients; and these are the basis function. Now, the question is, I can write this thing in terms of the other basis also.

For example, the same function, it is exactly like a vector if you let me write it here the vector form if a vector v is given to you can write this vector say v i, e i in this basis. And also you can write this is with another basis which is prime basis, this is the natural basis normally we put this notation for natural basis. This is another kind of basis which call the prime basis, but the same function I can write it here. And we know that if I write it in this form then there is a relationship between these and these. Here also we are try to do the similar kind of thing that means, I am writing a function in terms of monomial basis and it is possible to write this as like this and so on, I can extend the term up to my liking. So that means, a function is represented in terms of this basis which is one x and then we can expand the same function in another basis which is this.

Now, the question is now if you this is also some real numbers, now the question is if this is the real number, then this real number should be same as this. Real number may not be same as this, real number may not be same; as a and b are somehow related, but they are not equal. If they are equal then this has to be same, we have also proved that in our previous some previous class. In one particular basis, the function or a vector can be expressed uniquely for that particular basis; uniquely means this value should be unique for that it not going to change every time for fixed basis. But if I change the basis, this coefficient will going to change, so that means, now I need to write some different notation. Here if I write a 0, I need to write b 0, b 1 and so on.

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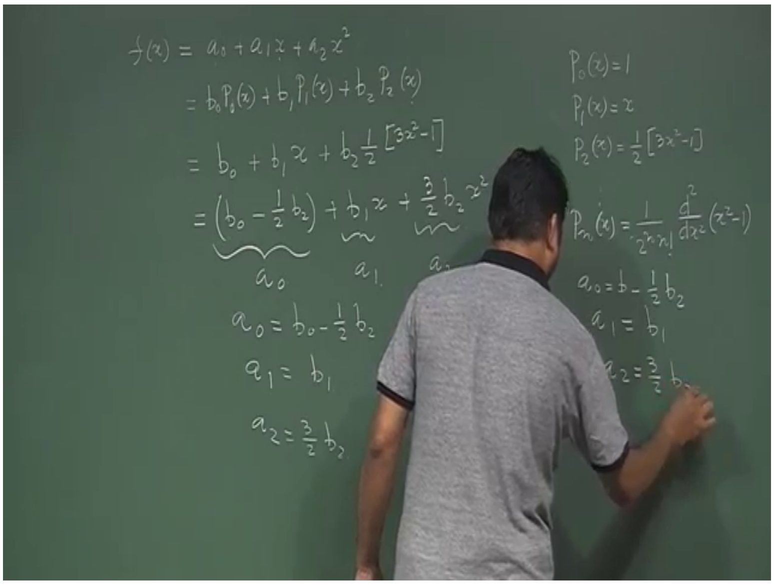

Now, I mean you can readily find out what is the relation between these two. So, now, I write explicitly and let us take for up to this say some function is given like this and you are asked to write it and try to find out. You are asked to find out what is the relationship between a 0, a 1, and a 2 with this quantity. How to do that, quite simple, in fact, you can

I believe all of you have readily I mean find what should be the readily in your mind the scheme is there. So, how to do that?

So, I will just expand this P 0. So, here P 0 one is 1, so it is b 0, b 1 is x no problem with that. b 2 is P 2, P 2 is defined as this if I expand the same function in terms of my given basis which is the (Refer Time: 13:59) polynomial I will have this things. Now, if I slightly rearrange these things I should write because there is a b 2 half multiplied by this plus b 1 x plus 3 by 2 b 2 x square. Now, you can readily find that this quantity, this quantity and this quantity is written in the form of monomial basis. This is of order one, this is x and this is x square; exactly the way it is given, so that means, essentially this is equal to a 0, this is equal to a 1, and this is equal to a 2.

So, there is a straight way there is a relationship between this quantity. So, a 0 will be b 0 minus half b 2; a 1 will be b 1; and a 2 will be 3 by 2 b 2. a 1 you know, a 2 you know which is b 1 and b 2, 1 b 1 and b 2 is given to you 0; a 0 is known to you. So, b 0 you can find out because now you know what is your b 2 and you replace these and you can readily figure out what is the relationship between the coefficient of monomial basis and the coefficient of your Legendre polynomial basis.

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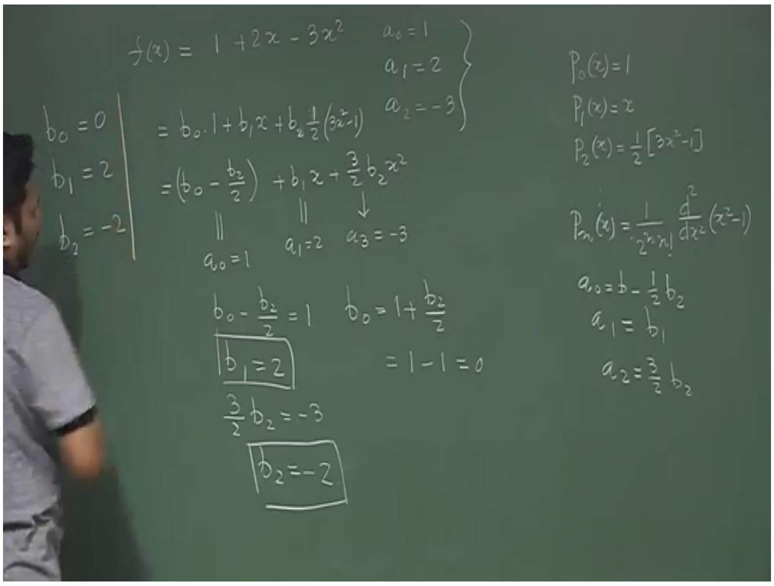

So, let us do a very simple problem. So, let me write somewhere the relation I will going to use that or I can directly do that quickly. So, say 1 plus 2 x minus 3 x square a polynomial is given to me. Now, we are asked to find out, so what is my a 0 here 1, what is my a 1 here 2, what is my a 2 where minus 3. If these values are given then I will like to write it as b 0 into 1 plus b 1 into x plus b 2 into half 3 x square minus 1. I can write it in terms of my Legendre polynomials. Then quickly b 0 minus b 2 by 2 is here plus b 1 is here plus 3 by 2 b 2 x square exactly the same way I did just few minutes ago. So, this is this value is how much this value is a 0 which is equal to 1. This value I readily figure out this is a 1 which is 2; and this value is a 3 which is minus 3 so that means, I have an equation b 0 minus b 2 by 2 is equal to 1; b 1 is 2 readily I find; and 3 by 2 b 2 is equal to minus of 3. So, b 2 from here i figure out it should be minus 2.

So, I know what is my b 2 which is minus 2, I know what is my b 1 which is 2, I need to find out b 0. So, b 0 is 1 plus b 2 by 2; b 2 I already figure out here which seems to be let me check once again if I am doing something wrong here or not. b 2 is minus 2; if it is minus 2, then b 0 comes out to be 0, maybe let us I will check that. So, it seems to be b 2 is minus 2, if here put minus 2, this 1 minus 1 - 0. So, b 0 is come out to be 0. So, let me write somewhere here what value I am getting and then put that and check whether it is coming or not. So, my b 0 is 0, b 1 is 2 and b 2 is minus 2.

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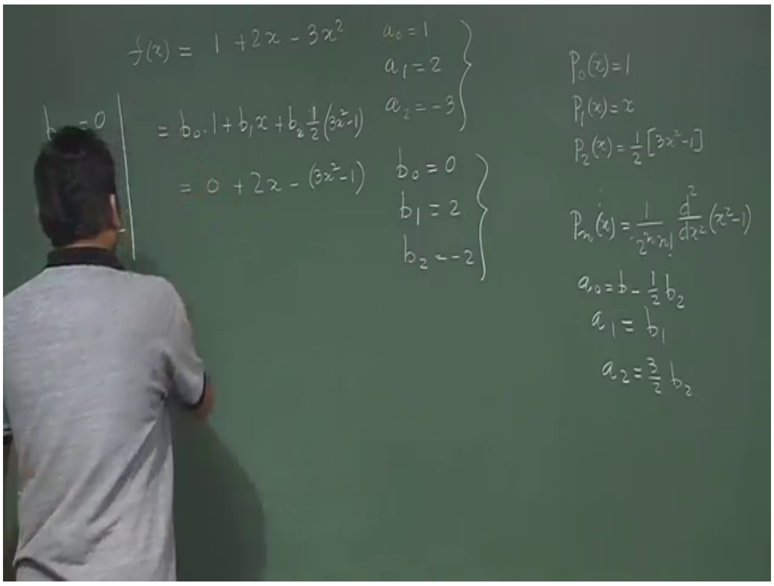

Now if I put this here then you can readily find out if I put this value, then it will be 0 plus P 1 is 2, it is 2 x minus 3 x square minus 1. So, this is this quantity this quantity as same. So, if this is the coefficient for monomial basis the coefficient for Legendre basis Legendre polynomial basis is this. It is same thing like writing one vector and changing that vector to another basis same vector that the coefficient is going to change. Mind it here it is 1, 2, 3, now it is 0 come to 0, 2, minus 2. Now, at this point, I will like to mention something quite interesting that if a polynomial is given to you let me erase this with this also I do not need to write this or either this, let erase.

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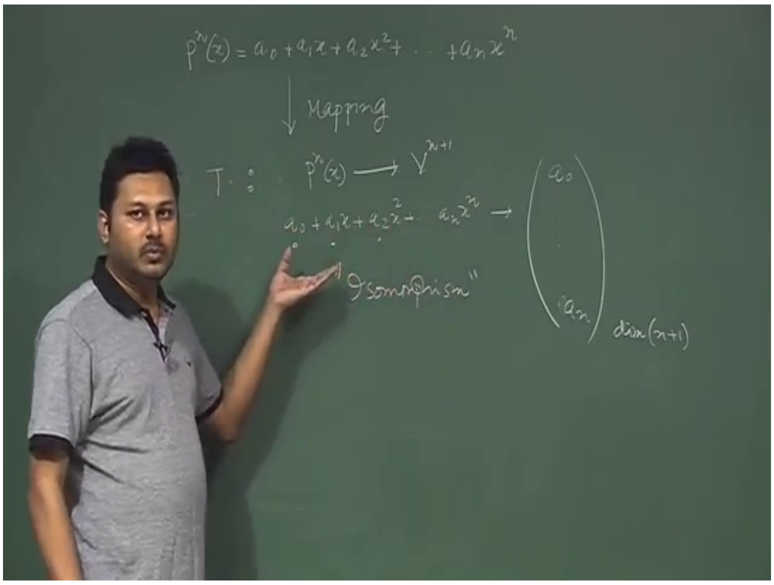

For example, a polynomial is given to you nth order polynomial, I write it as P n, which is a 0 plus a 1 x plus a 2 x square plus a n x to the power n, this is a polynomial given to you. Once the polynomial is given to you, you can readily have a mapping. What kind of mapping that I will this is the T, which is the mapping, T will do what P n x polynomial it will transfer T is some kind of mapping which transfer let me write it in this way, which transfer nth order polynomial to a vector of order n plus 1. How if I have a polynomial like this a 0 plus a 1 x plus a 2 x square plus a n x to the power n then this transformation or mapping suggest as that form that polynomial I can make a vector which contain only this coefficient, this coefficient, this coefficient, this coefficient.

So, if I write this coefficient, I will have a vector of dimension n plus 1. So, from a nth order polynomial I can make a vector, this vector is nothing but the coefficient is give the coefficient that is given to you I write this coefficient in a column vector. And when I write this column vector I can have direct mapping from this polynomial to a vector. This sometimes call is isomorphism is a technical term that for a given any given polynomial I can write this polynomial in terms of a vector with this mapping.

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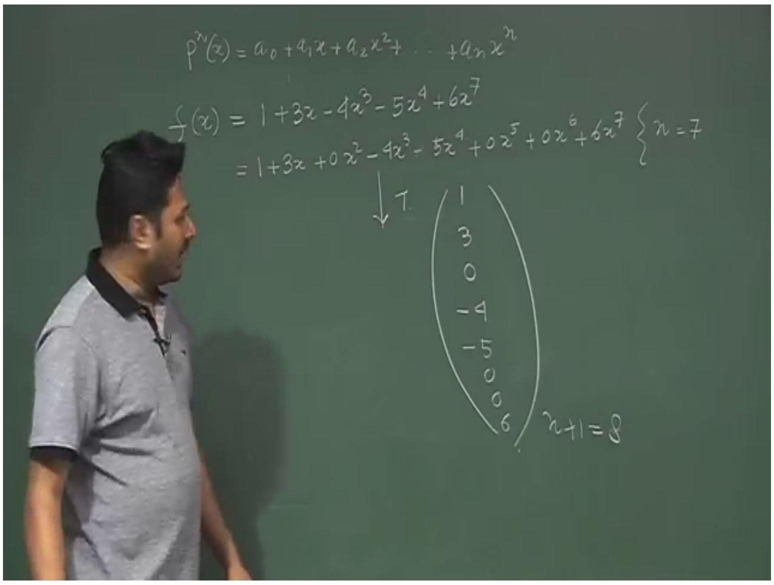

For example, the example is straightforward and quite simple even I think you have already understand what is the meaning of these things. So, say a polynomial is given like 1 plus 3 x minus 4 x cube minus 5 x 4 plus 6 x to the power 7 something like this. So, you can find that there a few terms few order that is missing. So, I can write this as 1 plus 3 x plus 0 into x square minus 4 into x cube minus 5 x to the power 4 plus 0 x to the power 5 plus 0 x to the power 6 plus 6 x to the power 7. So, the order of this polynomial is 7, n value is 7. If the order is 7, then I can form with the mapping T mapping a vector of this form, 1, 3, 0, minus 4,, minus 5, 0, 0, 6, this will be my corresponding vector quite simple. And if you find what is the dimension 1, 2, 3, 4, 5, 6, 7, 8, so it should be n plus 1 dimension, which is 8.

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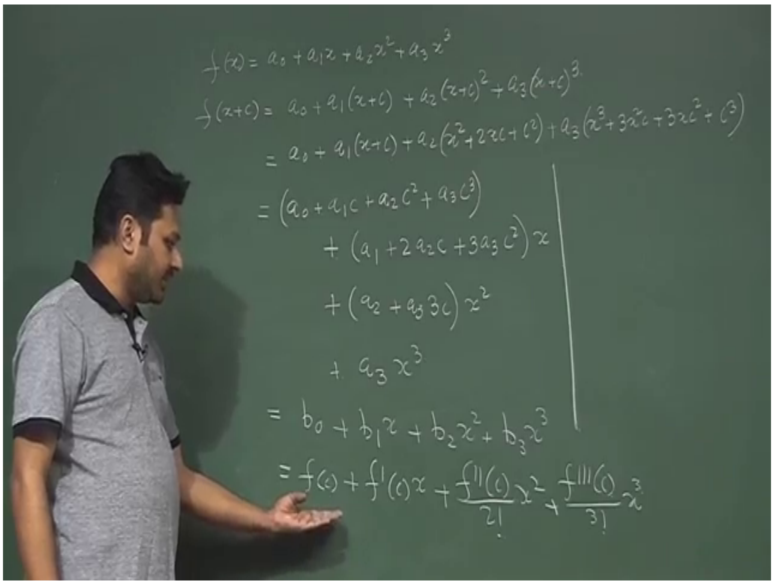

Now, another important thing in this polynomial, let me do that. For example, I have a polynomial f x is equal to a 0 plus a 1 x plus a 2 x square plus a 3 x cube. If I have this polynomial then if I translate that that means from x point I should add something here, which is called translation. Then in the right hand side, something will change and I have some expression like this. Once I have some expression like this, then if I rearrange these things.

So, let me write it once again a 1 x plus c then a 2 x square plus 2 x c plus c square plus a 3 x cube plus three x square c plus 3 x c square plus c cube. Now, if I rearrange the term then I will have a 0 plus a 1 c plus a 2 c square plus a 3 c cube, first term I am rearranging these things plus second term a 1 all the coefficient with respect to x plus 2 a 2 c then plus 3 a 3 c square x. The next one is all the coefficient with square a 2 there is a square plus a 3 into 3 c, this quantity into square, hope I am not missing anything and finally, the cube finally, the cube this is a 3 just a 3.

So, even though I am changing these things, so this is a new coefficient. So, if I write this in terms of new coefficient, it will be b 0 plus b 1 x plus b 2 x square, I should write x cube here plus b 3 x cube. This was the basis, this is another the same thing I am writing, but with a translation. Now, these things b 0, b 1, b 2, b 3, if you find if I check carefully then what is my b 0, b 0 is nothing but f of c. If I put this quantity in place of x if I put c then I will have f c. What is my b 1, b 1 is the derivative if I make a derivative and then if I put c on that then I will have these things.

So, if I make a derivative this quantity is not be there, it will be a 1, a 1 then it will be 2 a 2 c 2 a 2 c and it will be 3 c 3 a 3 c square 3 a 3 c square. So, this quantity is the derivative of this quantity. What will be my b 2, b 2 is the double derivative of c with respect to 2 divided by factorial 2. If I make a double derivative this quantity will not be there this quantity will not be there here will be 2 a 2, I have a 2 here, here of a 3. So, if I make a double derivative it will be 3 into 2 - 6 a 3 x, so I will have 3. So, if I make it divided by 2, then it will be 3, so I will have this quantity. And finally, I want you to check please check if it is really coming or not it will be something like this. I believe you now start remembering what is this so called Taylor series. So, it is nothing but the Taylor series expansion with this.

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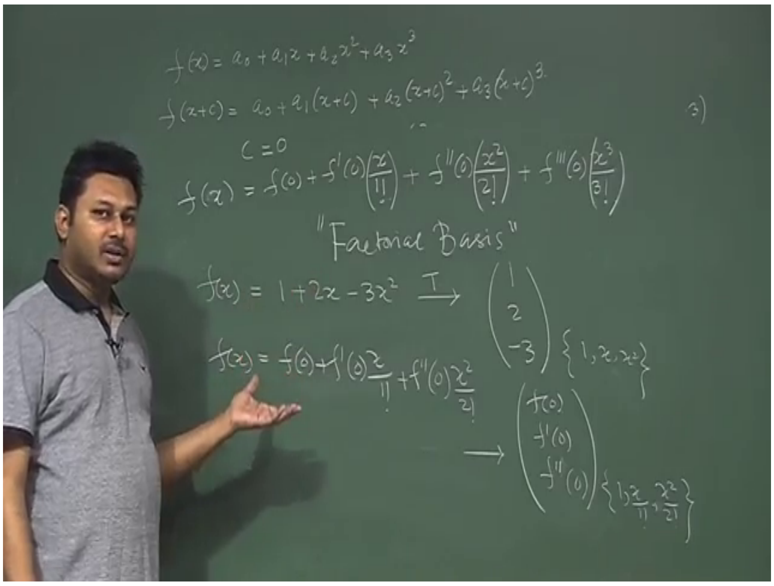

Another important thing is that another important thing is that if I put c equal to 0, then f x c equal to 0, I can also write in this form I put a factorial one here to make it more convenient, factorial 1 is 1, so I should not bother about this. And finally, these and these, if you check these and these, there is a different basis I can form. You can see this basis is slightly different from the previous one, this is a monomial basis I am writing the same function, but this basis is slightly changed, this basis is called factorial basis.

So, I can write the same function in terms of monomial basis I can write the same function in terms of factorial basis. So, now, if a function is giving to you then you can understand how to change is from monomial basis two factorial basis a quick example I will not do this problem entirely, it will be used as a homework, but very simple. So, f x for example, 1 plus 2 x minus 3 x square. So, what is the if I make it T mapping over that what will be the vector form the vector form is 1, 2 and minus 3 straight way.

Now, the question will be if I write this f x in terms of factorial basis, it should be read in the form like f 0 plus f prime 0 x factorial 1 and plus f double prime 0 x square factorial 2, so this is what is the basis here 1, x, x square this is monomial basis. Now, you can write this function in this way make a transformation just put this value here f 0, f prime 0, and f double prime 0, you will have a different value this will be for the basis 1, x, x square with these basis. So, I can write this is monomial basis as well as my factorial basis. Since, the function is given to you we can readily understand what will be your f 0, what will be your f prime 0, and what will be your f double prime, this is double prime by the way double prime 0.

So, then you can whatever the problem is given to you, you can write this function with the transformation with this basis or these bases according to your choice. Ok students, so this is the last part of the linear vector space. So, I will going to stop here. And in the next class, maybe I will going to start a new topic which is the complex analyze is a very important one. So, with that let us conclude here. Hope you enjoy this linear vector space are the different aspects of linear vector space. And now you have an better idea of how to do dealing with these things and from where it is coming everything is related to the function and vector is related to this linear vector space concept inner product and all these things which are very important concept. So, now, I think your understanding will be more better when you deal with these kind of problems. So, with this hope let us conclude here. So, see you in the next class.

Thank you.