Mathematical Methods in Physics–I
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Lecture – 27
Polynomial Space, Legendre polynomial

Last class we started something called Legendre polynomial and at least I wrote this term, but I did not mention anything about that. So, let me do that in today's class.

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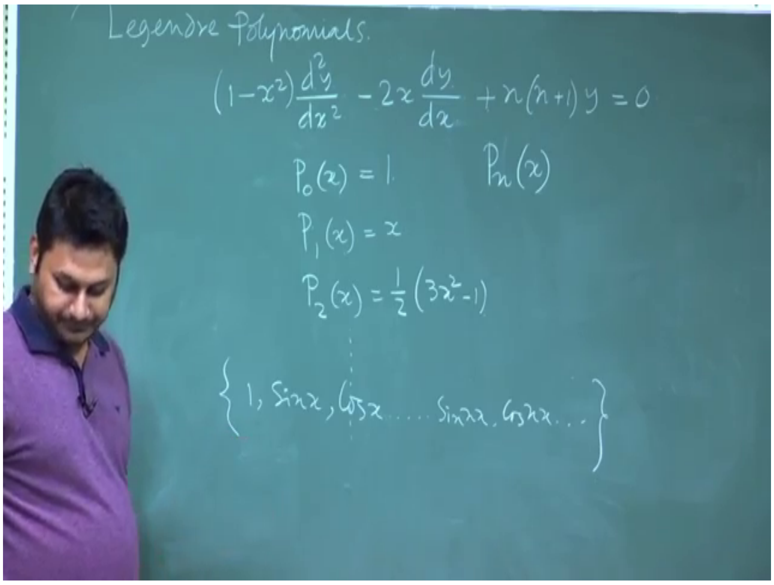

So, if I have a differential equation in the form then we have a series solution of these things, a series solution the solutions if I write it should be something like this P 0 is 1, P 1 x is x, P 2 x is half of 3 x square minus 1 and so on.

Depending on the value of these things I should have a general solution series solution we called it Pn x. If you note that these things there are other terms also are forming polynomials through polynomials this is one polynomial this is another polynomial this is a degree 1 polynomial this is a degree 0 polynomial this is a degree 2 polynomial and so on.

So, these solutions individual solutions they are forming the polynomial are linearly independent to each other and they are also orthogonal since they are orthogonal to each other and linearly independent to each other obviously, they are orthogonal they should be linearly independent. So, now, with using this polynomials one can express or expand some function like the Fourier series.

If you remember in the Fourier series the basis function was 1, sin x, cos x, sin nx, cos nx and so on these were our basis sets which forming a basis and I can expand a function in terms of them. Here instead of having a sinusoidal function I have some polynomials they can form the basis, but before that we should also learn about this how to get this and all these things.

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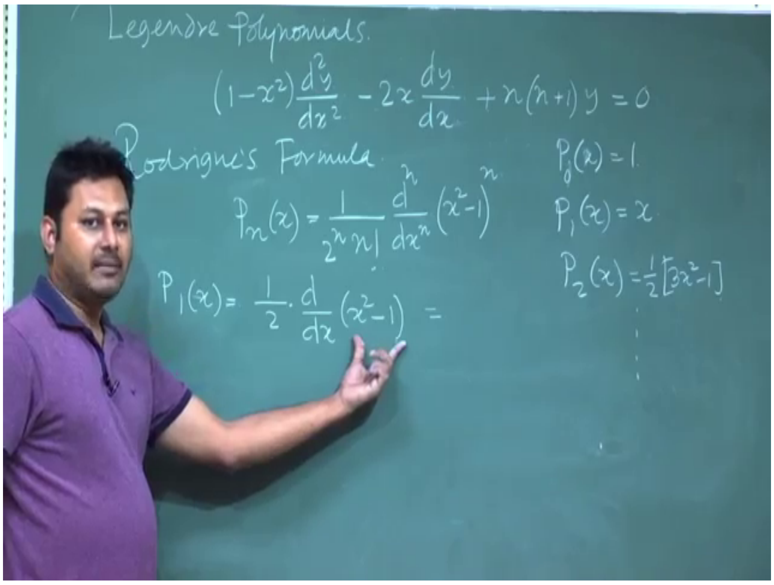

So, let me write somewhere here in the right hand side few values of few polynomials say P 0 x is 1, P 1 x is x, P 2 x is half of 3 x square minus 1 and so on in order to remember how these polynomials are there we have something called Rodrigue’s formula Rodrigue’s formula.

What is the formula? This formula suggests how to find out this value Pnx and the formula is 2 to the power n 1 by 2 to the power n n factorials in its derivative of the quantity x square minus 1 whole to the power n. If I evaluate I will going to get these things. So, let me do that elaborately so that you can have the idea of these polynomials.

So, first let us do P 0 is a 0th order derivative. So, I will and here is 0 here is 0, so 1 radial it is there what about the P 1 P 1 if I do P 1 x which I wrote here x. So, I will going to verify that. So, P 1 means n is equal to 1 this is 1. So, 1 by 2 the derivative will be first order x square minus 1 n is 1. So, here I should put 1 if I make a derivative of this quantity I will have 2 x and these 2 will cancel out. So, I will have only x.

So, these things is verified I will have these things from Rodrigues formula. What about the next term let me erase these things because I will going to I need this space actually.

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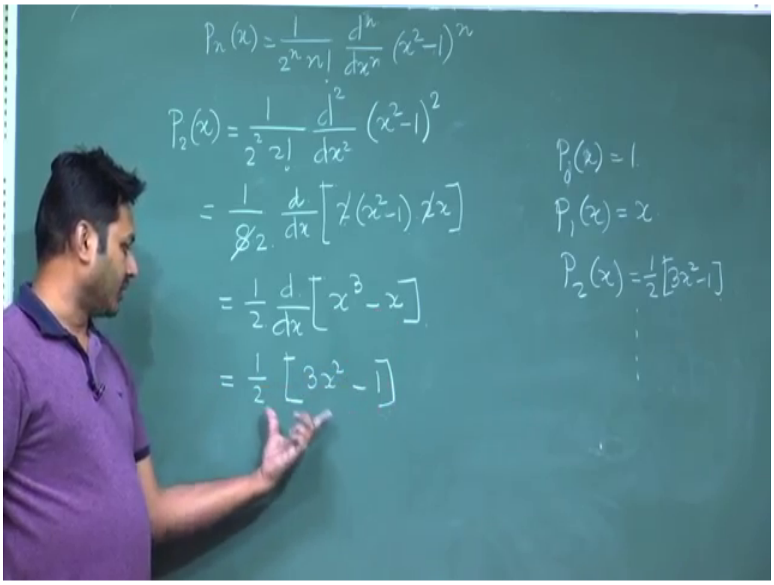

So, the Rodrigue’s formula I once again write Pn x is equal to 1 by 2 to the power n n factorial d n dx n x square minus 1 whole to the power n this formula I have already written.

So, I am also going to use this for say P 2 P 2 x, how to write P 2 x? 1 2 to the power 2 and factorial 2 which is the initial term, then d 2 dx square I will have x square minus 1 square of this quantity this if I have this then I will have in the denominator 2 2 square 2 to the power 2 is 4 and multiplied by 2, so it will be 8.

And now the derivative d dx when I make a first derivative of these things I will have 1 2 term here 2 of x square minus 1 and for this x square again a 2 of x term here. So, this 2 this 2 will cancel out and I will have here 1 2. So, it will be half it will be half d dx of x cube minus x something like this.

So, it will be half if I do that it will be 3 x square minus 1. If I make a derivative of this quantity it will be 3 x square and for x it is 1, so half 3 x square minus 1 which is written here, but I did not reading if f 3 P 3. So, I need to calculate that also let us do that it will be just an exercise.

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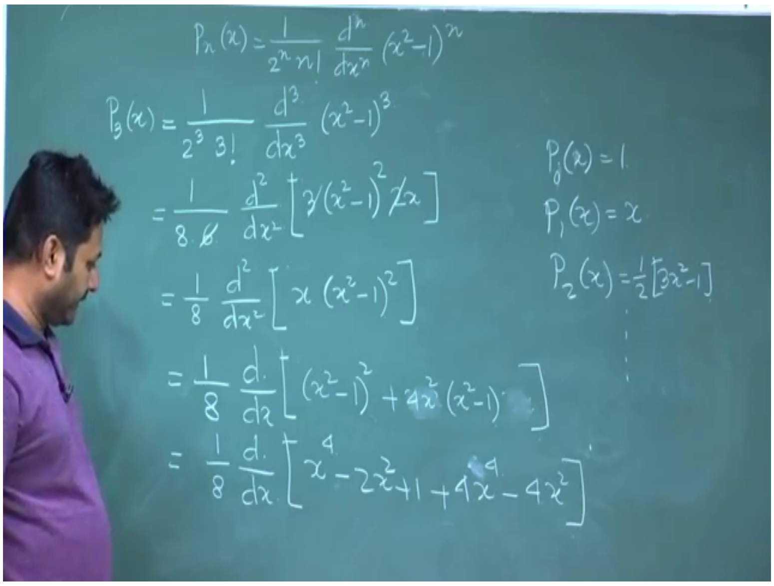

So, P 3 x I like to find out. So, it should be 1 2 to the power of 3 factorial 3 initial term and then d 3 dx 3 x square minus 1 whole cube this.

So, first it is 8 into 6 2 to the power 3 is 8 factorial 3 is 6 let me put that double derivative I want to reduce 1 derivative, if I do that it will be 3 multiplied by x square minus 1 square of these things and for 2 x I will have for x square I have 1 2 x here this quantity. So, here again 2 and 3 will cancel out with the 6. So, I will have 1 by 8 d 2 dx square x multiplied by x square minus 1 square of that. Again I want to if I want I need to reduce 1 order here. So, d dx if I reduce to 1 order then I need to make a derivative of this quantity.

So, let us do that in this way the first derivative and the second derivative, x square minus 1 first derivative multiplied by second 1 plus second one multiplied by the first derivative. So, it is 2 x square minus 1 2 x square minus 1, and then for 2 x I will have another for x square I have another 2 x I believe this fine 1 by 8 d dx of if I expand it should be x to the power 4 minus 2 x square plus 1 this is 1 term and rest term is plus 2 to 4 x square. So, 4 2 into 2 is a 4. Let me write in this way. So, it will be multiplied by 4 x square. So, it will be 4 x to the power of 4, 4 x to the power of 4 minus 4 x square is comes out to be something like this.

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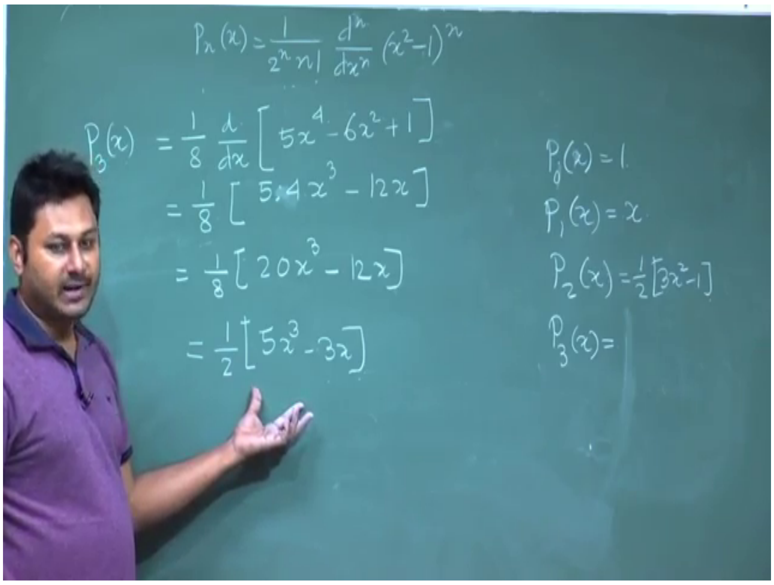

Now, if I simplify this it will be 1 by 8 I try to calculate P 3 x it is 1 by 8 d of dx of this quantity which is now which is now something like this quantity.

So, it will be 5 x to the power of 4 and then this quantity which is minus of 6 x square hope I am doing correctly and 1 plus 1 term is here this one term will not going to be effective because the derivative is there. So, now, if I make this a derivative it will be 5 x to the power 5 into 4 x cube minus 12 x it will be something like this.

So, it is this is 20. So, let me write it this is 20 x cube minus 12 x of this. So, we have already got the polynomial, but next I need to reduce a bit. So, if I take 4 common of that, so I will have half 5 x cube minus 3 x still I have a polynomial of higher order. And this is the form of P 3. So, if I write here somewhere P 3 which I find from Rodrigues formula this up to this I remember. So, I write it, but after P 3 and P 4 it is very difficult to remember and you have already find that how difficult not difficult, but the process is lengthy because you need to calculate the derivative of different order for where P 3 you need to calculate the derivative for third order, if you go to 4th order and fifth order sixth order it will be larger and larger.

So, half 5 x cube minus 3 x square 3 x sorry is my form of the Legendre polynomial of order 3. Now after having the Legendre polynomial from Rodrigue’s formula next we need to find out whether they are really these things are really orthogonal to each other or not. So, how do we find out the inner product by taking the inner product? But before taking the inner product one thing I should mention earlier, but now I am mentioning this function is limited to minus 1 to 1.

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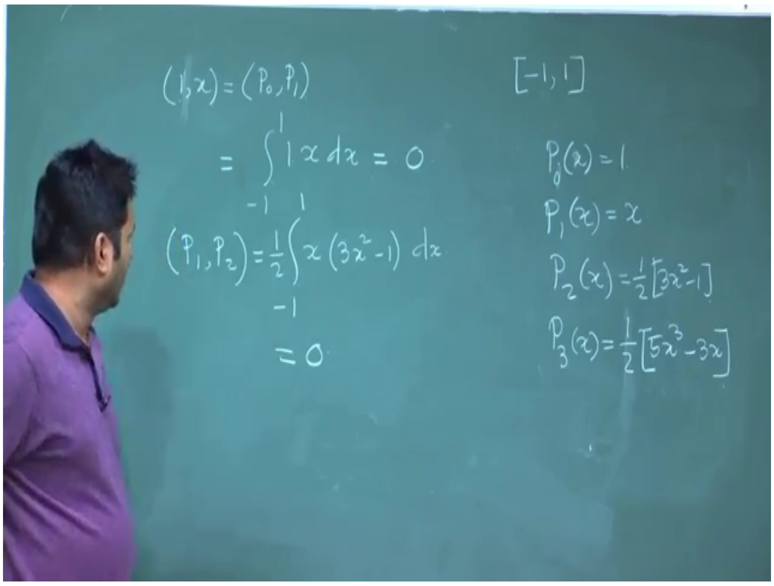

So, there should be some limit, there should be some range through which these functions are related or are defined. So, this is the limit for that because I required this because I need to calculate the inner product. So, now, if I calculate the inner product, let us find out. So, I am checking randomly. So, inner product between 1 and x or if I write in terms of P 0 and P 1, I say P 0 and P 1 are orthogonal. So, if I take the inner product of these 2 things I should have 0. So, really I am getting 0. So, let us do that minus 1 to 1 P 0 is 1 P 1 is x dx which is 0 because this is the odd function and limit is minus 1 to 1.

Let us check say for P 1 and P 2, P 1 and P 2 randomly I am checking. So, it is minus 1 to 1 that is my P 1 x half I take outside x multiplied by 3 x square minus 1 dx. Now, when I multiply this quantity the first term is 3 x cube and the second term is x. So, if I divide it there will be 2 integration 1 is 3 x cube and another is x. So, first one is also odd second one is also. So, this quantity is also going to 0.

So, now, you can you can test with other terms also I mean say P 2 multiplied by P 3 you will get that again it will give you something which is 0 so that means, they are orthogonal to each other. Now, if they are orthogonal; that means, they are potentially form a basis, but the important thing is that how to find out the normalization factor.

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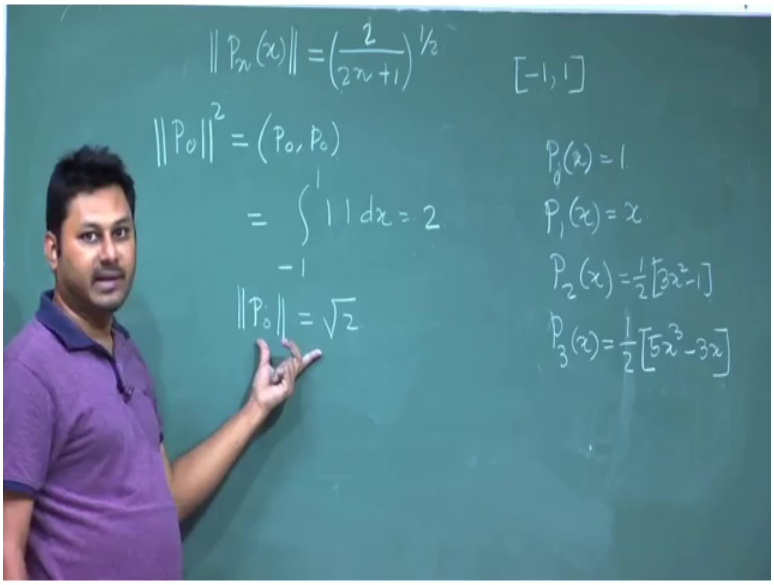

So, normalization also we can find out. So, before doing that I should write a general form also for normalization this is a general form of normalization. So, norm of the nth order of Legendre polynomial is 1 divided by 2 n plus 1 whole to the power of half this is the relation.

So, let us check whether really it is following or not, really it is following or not. So, first P 0 I want to find out the norm square of P 0 which is nothing, but inner product of P 0 and P 0 if I do that then I will have minus 1 to 1, then the function 1 into 1 dx which is how much; which is nothing but which is nothing, but 2 this formula I am making 1 mistake it should be 2 divided by 2 n plus 1. So far I remember it is 2 divided by 2 n plus 1 I will check that I will check that with this.

So, since it is x. So, x will be minus 1 to 1. So, that will be 2. So, what will be my P 0 this things it will be root over of 2 let us check with the formula whatever the formula I was reading initially I wrote 1 divided by 2 n plus 1, but I find it is not matching, but if I write 2 divided by 2 n plus 1 is it n equal to 0 then this quantity is not there. So, it is 2 with a root. So, I will have the same thing 2 with a root for 1 maybe it is easier. So, let us do that for x the next one.

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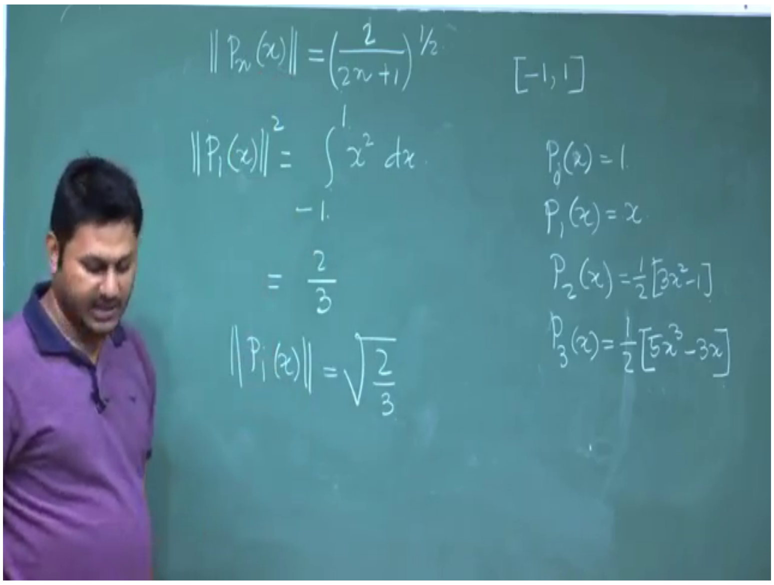

So, what is the norm square of P 1 x norm square? So, it will be minus 1 to 1 x square dx which is how much 2 over 3 if I integrate then it will be x cube divided by 3 if I put x cube minus 1 to 1 then it will be 2, so 2 by 3. So, my P 1 x seems to be root over of 2 by 3. Let us try to use this formula if I put n equal to 1 then it will be 2 divided by 2 plus 1 3. So, 2 divided by 3 whole to the power half 2 divided by 3 whole to the power half. So, so far the formula is matching.

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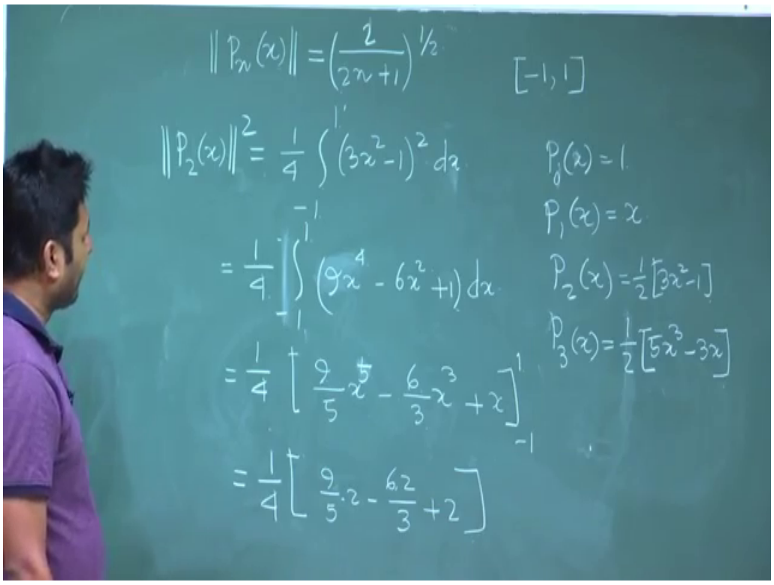

So, finally, we will check for P 2 and then. So, then we conclude. So, P 2 again I am doing the same thing, when I make a square of this quantity. So, 1 by 4 will be coming outside minus 1 to 1 it will be 3 x square minus 1 square dx. So, it will be 1 by 4. Now integration minus 1 to 1. So, it is 9 x square let me write it clearly 3 x square (Refer Time: 20:01) making square, 9 x to the power of 4 minus 6 x square plus 1 dx over this quantity. So, 1 by 4 minus 1 to 1 this quantity, let me direct integrate that why I am doing the integration sign or integrating. So, it is 9 by 5 x to the power x cube minus 6 by 3 I am making the integration sorry. So, it will be 5 it will be 3 and finally, I will have here x minus 1 to 1.

So, 9 x cube x to the power 5 divided by 5 x cube divided by 3 1 in x 1 by 4 now if I put the limit. So, this is odd this is the odd this, this is odd minus 1 to 1 so; that means, this will give 2 this will give 2 this will give 2 because, 9 by 5 into 2 minus 6 into 2 by 3 plus 2 ok.

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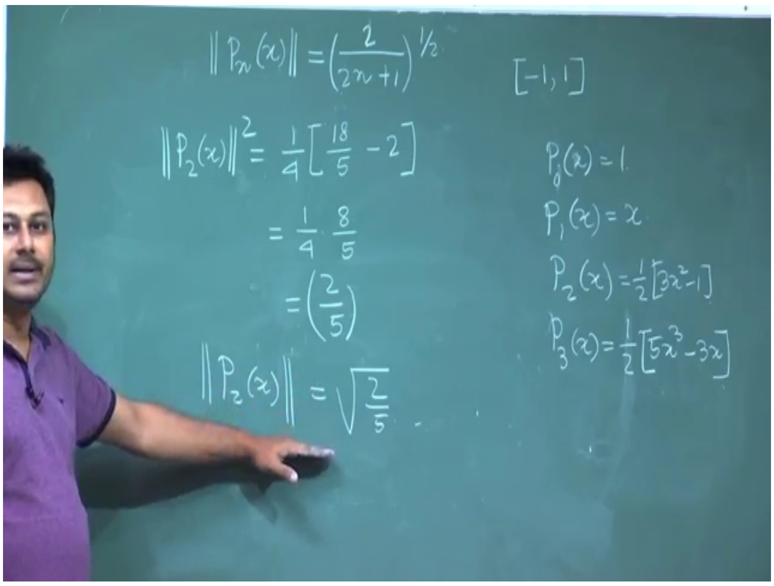

So, finally, this value comes out to be 1 by 4, 18 by 5 minus this 3 will cut to this 6 with 2. So, this is 4. So, this is minus 2 this is 4 and this is plus 2. So, minus 4 plus 2 I will have minus 2.

So, now I will have 1 by 4 into 5 into 2, 10. So, I will have here h divided by 5 and it will be nothing, but 2 divided by 5 this value finally, I am getting. So, this 2 divided by 5 is the value of P 2 square P 2. So, norm of P 2 x norm of P 2 x is nothing, but root over of this quantity and this formula that I have wrote here 2 divided by 2 n plus 1. So, if I put n equal to 2 here then it will be 2 into 2 4 plus 1 5, 2 by 5 root over that. So, I will have 2 by 5 root over of that by doing all the extensive calculation by calculating the inner product of that.

So, we will roughly check that there really this formula is really working good and also we find the normalized value of this quantity. Now if I divide the every function with this quantity then this function potentially be a orthonormal basis there are actually another way to find out whether they are forming orthonormal basis or not and this form I have already described which is called Gram-Schmidt orthonormalization, Gram-Schmidt orthonormalization.

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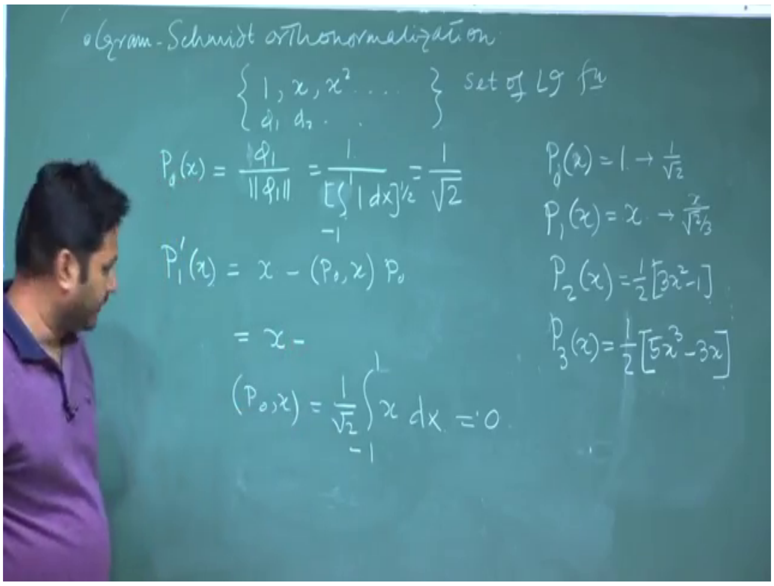

So, Gram-Schmidt orthonormalization suggests that if you are given a set of linearly independent vector if you remember this is a set of linearly independent in gram schmidh say there it is giving like a vector, but here this, this is a linearly independent set of functions.

Now, if I use the Gram-Schmidt orthonormalization procedure I will return back these functions. So, let me do that quickly. So, what is the Gram-Schmidt orthonormalization I am using the Gram-Schmidt orthonormalization process for a given set of function this is a set of function which is 1 x x square and in the previous class we show that these functions are forming in linear independent because if you do the wrong scan of that it will not equal to 0. So, if I start with that. So, what should be my P 1 P 0 x first function I know that this first function if I write it as phi 1 phi 2 and all these things then if you remember it is phi 1 divided by mod of phi 1 this quantity. So, here phi 1 is 1 and mod of phi 1 is mod of phi 1 is minus 1 to 1 one dx inner product of these things in minus 1 to 1 limit. So, it will be 1 square is 1, so this.

So, first term is 1 divided by and square of that half of that because I am having the norm of these things. So, if I do that then I will have it is 1 by root 2. So, my first function is 1 by root 2. So, these are the polynomials I wrote without the normalization factor if I write this in the normalization factor it should be 1 over root over of 2 it should be x divided by root over of 2 third and so on and this value I have calculated just before. So, here I will have my first value when I make a normalization I will have the first value which is the first Legendre polynomial.

What about the second thing? So, I have the first thing to second thing how do I calculate. So, P 1 prime x if you remember I put the prime because I still now I did not normalize that it is the given function second one minus of the inner product of the function I will have. So, P 0 x multiplied by P 0 this was the technique.

So, first function is first function what is the inner product of P 0 and x it is minus 1 to 1 P 0 I have already calculate it is 1 by root 2 if I write it here it will be 1 by root 2 x dx this is the inner product this inner product x is how much integration minus 1 to 1 x is nothing, but 0 because is the odd function. So, inner product is 0 here. So, I will eventually have my function second function just only x, so now I can normalize.

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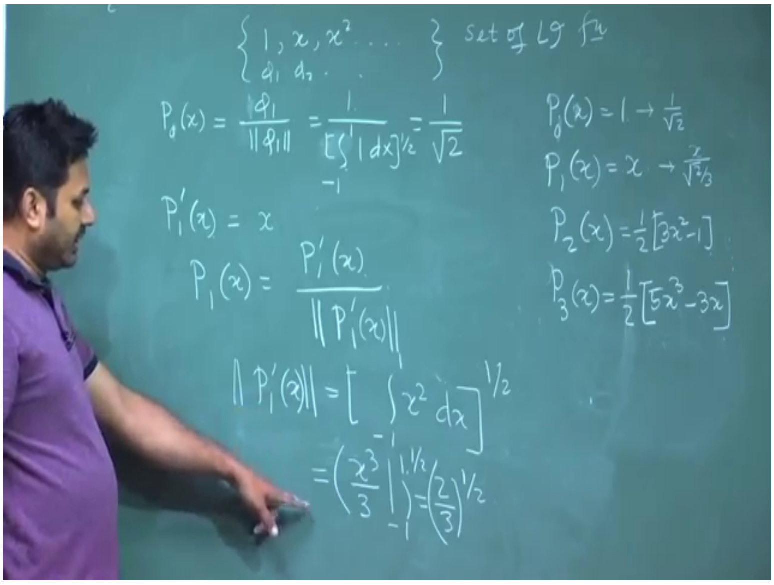

So, my original P 1 x will be my P 1 prime x divided by the norm of P 1 x and what is my P norm of P 1 P 1 prime norm of P 1 prime. So, norm of P 1 prime x is integration minus 1 to 1 it is x. So, I will have x square dx whole to the power half of that when I am making the norm.

So, now if I do that it will be x cube by 3 with minus 1 to 1. So, it is nothing, but 2 over 3 with whole to the power half definitely. So, it will be whole to the power of half. So, my second function is normalized by root over of 2 by 3. So, finally, I will have. So, let me write it here in the right side. So, P 0 I am having P 0 x I am having 1 term root over of 2, P 1 x I am having x divided by root over of 2 by 3 this is my second term I am getting.

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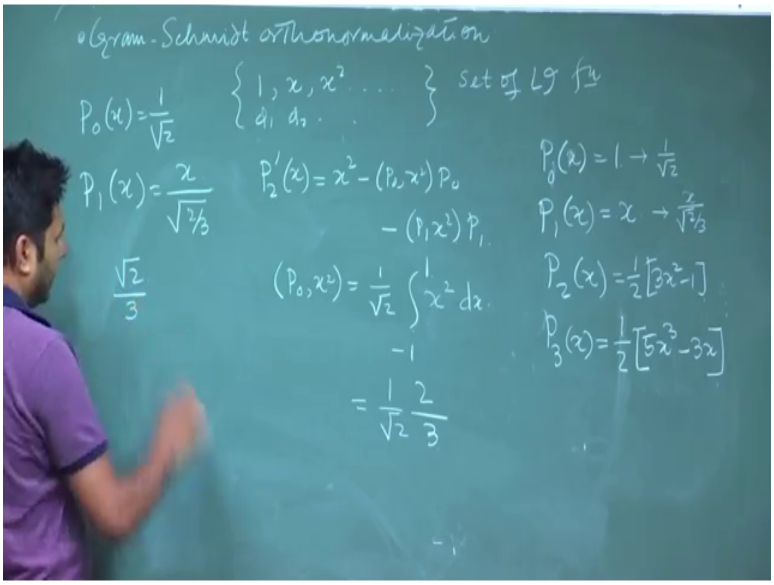

In the similar way I can do the find the third term if I do let me do that quickly. So, my P 2 prime x is how much? x square minus of P 1 P 0 with x square inner product P 0 minus P 1 of x square inner product P 1. I suggest you to please go back to the Lacian of where the Gram-Schmidt orthonormalization process were described and this is the procedure we are following exactly the same procedure that is used in calculating in the vector space when the sum linearly independent set of vector is given to you are calculating the similar way how to find a normalized orthonormal vector out of that. So, I am doing the same thing.

So, let me find out what is my P 0 inner product x square my P 0 I have already calculated it is 1 over root over of 2 minus 1 to 1 and it is both the and it is x square. So, x square dx it will be 1 root over of 2 x square means it will be 2 by 3 x square will be x cube divided by 3 when you put minus 1 to 1 it will be 2 by 3.

So, I will have the first value of my inner product which is let me write somewhere so that I should not forget. So, root over of 2 by 3 is the value of this quantity.

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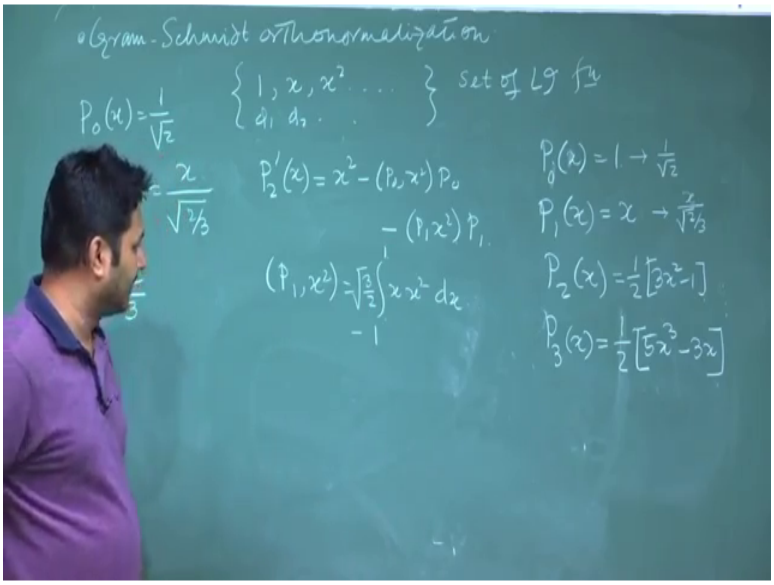

What about the next quantity? The next quantity is P 1 x square inner product which is minus 1 to 1 what was my P 1? P 1 was my this value. So, let me write root over of 3 by 2 outside then x multiplied by x square dx x x square is x cube which is odd function. So, it will go into vanish. So, I will I should not have the contribution of this.

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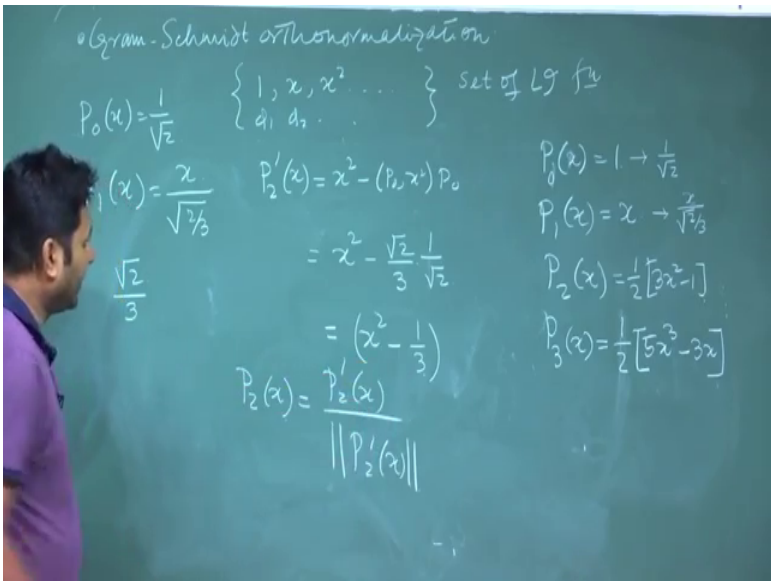

So, finally, what value I am getting for P 2 I am getting x square minus this quantity the inner product is root over of 2 by 3 and P 0 P 0 is my 1 by root over 1 by root over of 2. So, I will have something x square minus 1 by 3 I have something like this. So, now, I need to normalize this quantity if I do then my P 2 x will be simply P 2 prime x divided by the norm of P 2 prime x.

So, norm of P 2 prime x P 2 is given P 2 P 2 prime is given. So, I can calculate quite easily. So, let me do that quickly extensively I am showing how to calculate these things that believe you can do that.

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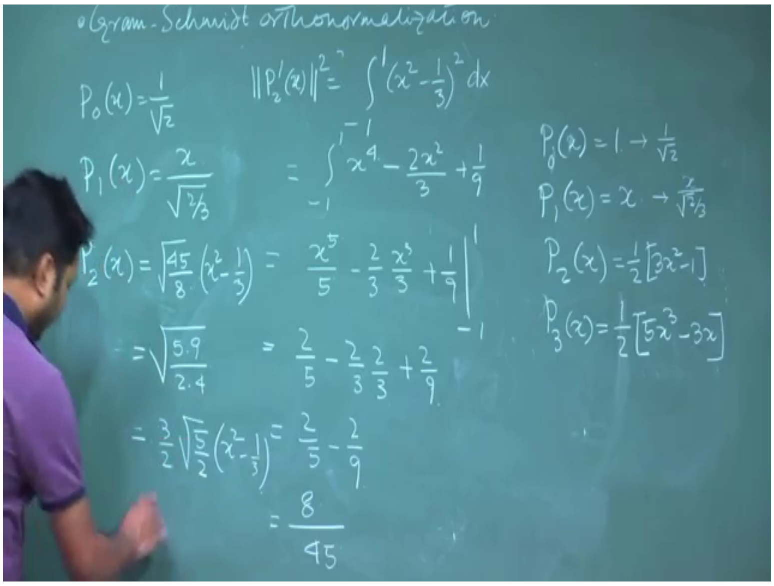

So, I try to calculate first P 2 prime x square it is nothing, but minus 1 to 1 this quantity x square minus 1 by 3 whole square dx. So, this is minus 1 to 1 x to the power 4 minus 2 of x square divided by 3 plus 1 by 9, so my original function. So, if I do that then it will be x to the power 5 by 5 minus 2 third x cube by 3 plus 1 by 9 with the limit minus 1 to 1.

If I put that it will be 2 by 5 minus 2 by 3 into 2 by 3 and plus 2 by 9. So, it will be 2 by 5 minus this is 4 by 9 this is 2 by 9. So, I will have 2 by 9. So, it will be 45 and 8, 8 by 45 this quantity. So, this quantity squares if I make a root over of that. So, if I write it here my P 3 P 2 then P 1 P 2 is this quantity root over.

So, root over of 45 by 8 then multiply it by that quantity x square minus 1 by 3 this polynomial. So, now, if I slightly modify these things root over of 45 I can write 4 into 9 5 into 9 and then it will be root over of 3 if I write it say it will it will be something like 5 into 9 divided by say 2 into 4. So, it is this quantity I can take this 3 and 4 outside. So, it will be simply 3 by 2 root over of 5 by 9 5 by 2 multiplied by x square minus 1 by 3.

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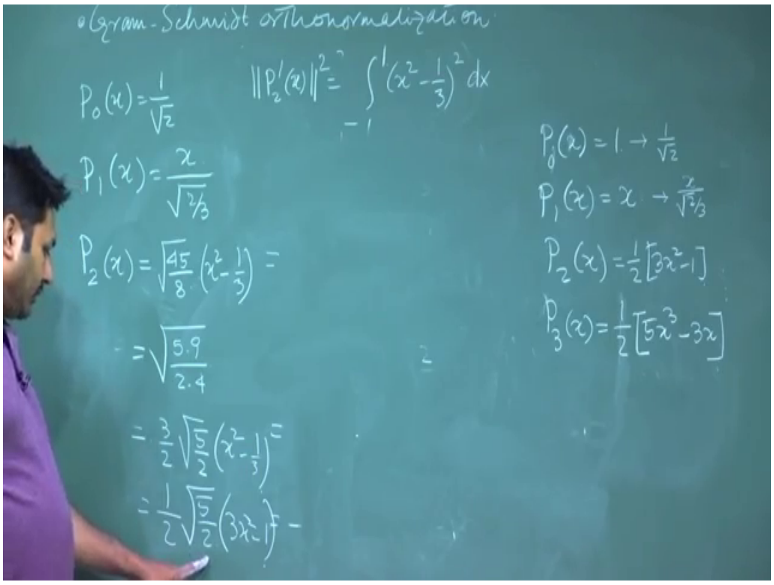

Now, these 3 if I am multiply it will come to the power original value that we start with 3 x square minus 1. So, this is lengthy, but straight forward calculation and I have the normalization factor here root over of 5 by 2 half these things are already there you can see it from here and I just multiply the normalized factor that is coming up the point is that if a set of vector is given to you which is set of function which has which has really independent to each other from that by using the Gram-Schmidt orthonormalization process Gram-Schmidt on process. I can have a set of functions P 0 P 1 P 2 which are if I do that which are linearly which are orthogonal to each other.

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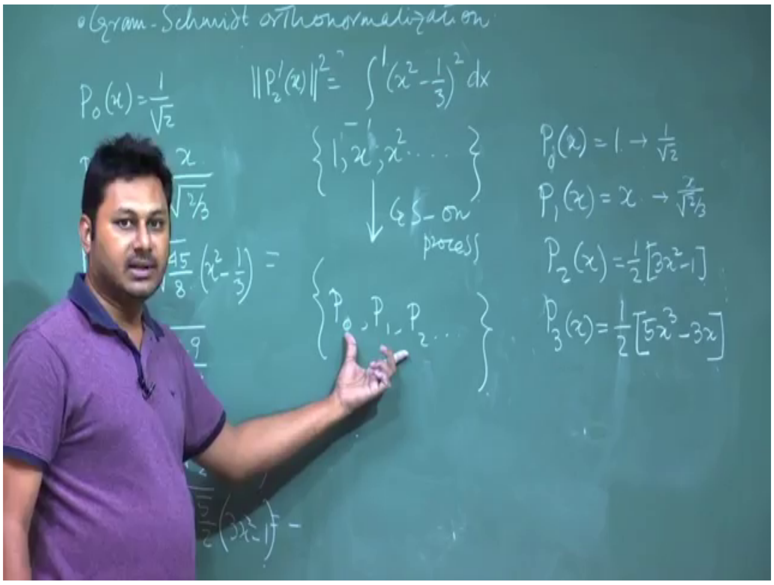

If they are orthogonal to each other and normalize to each other then we can say that these things can be used as a potential basis and this is the value we have calculated.

So, with this note today we like to conclude and in the next class we will try to do something regarding the polynomial space that we have started in the previous class how this polynomial space are there and all these things. And then we will conclude the first part of our course which is linear vector space, with that thank you very much for your attention, see you in the next class.