Mathematical Methods in Physics-I
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Lecture – 26
Parseval Theorem, Fourier Transform

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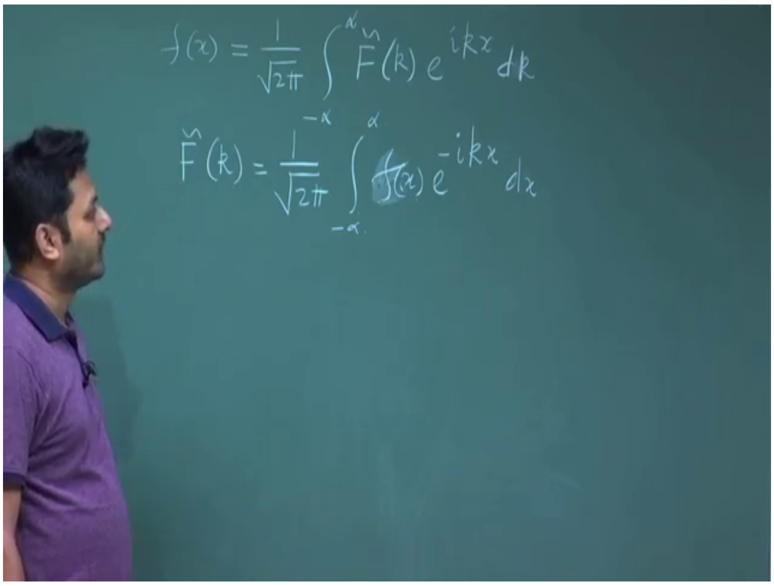

Students, so welcome back to our next class. In the previous class, if you remember we started a very important thing, which is Fourier transform. If f x is a function and related with this relationship and F k is related to the original function like this f x e to the power minus ikx dx then these two equations are related to each other with the Fourier transformation. And this is the corresponding Fourier function Fourier transform function if a function f x is given to you.

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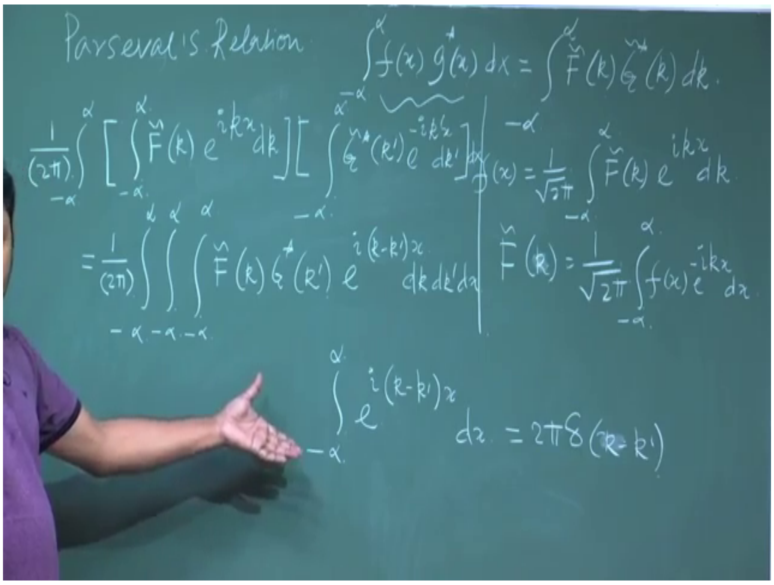

So, today we will extend something this is we know. Now, today we will like to transform like to know few identities which is called say Parseval’s relation. Parseval’s relation is directly coming from the concept of the Fourier transform where the relation suggests that integration of minus infinity to infinity f of x g star of x dx will be equal to minus infinity to infinity dk. Where this tilde stuff is the Fourier transform of this, I should put a tilde here also G tilde star is a Fourier transform of g function G.

Now, if I want to prove that then we need to use as I wrote earlier this relationship let me write it once again here in this part of this board. And F tilde k is we know that this transformation we know that. Now, I like to prove this thing I just by just putting the stuff here. So, minus infinity to infinity, if I write this quantity, this quantity if I write in terms of this then it should be 1 by 2 pi, because for f x I will have 1 by 2 pi when I put g star x I will have another term 1 by root pi. So, root pi root pi I write it here as 1 by 2 pi minus infinity to infinity.

Now I should write my f x with this which is minus infinity to infinity k e to the power of i kx dk this is one term. And another term is something which is g star, so g star means I should write here in this way minus infinity to infinity G tilde star I write say k prime because already 1 k is used. So, it is better to write because this is a dummy index it is better to write another indices e to power of since it is a star I should write minus k prime x dk prime. These two terms are there. This is the standard way to write these things. This part is a different part, so put a line here.

So, now if I go forward with this I will have this quantity with three integration one dx is already there. So, outside there should be some dx. So, there should be three integration, the first integration is already minus infinity to infinity dx; and in between there are two functions; for these two function, I should have another two integrations. So, now my total integration is three both are running from minus infinity to infinity this I write like this and then e to the power i a minus k prime x dk dk prime dx, this is my total integration.

Now, inside this integration you will have one quantity something like this. And if this quantity I can extract, this quantity is essentially this, this is identity, this is a delta identity. And if you want you can try to prove by yourself I consider this I mean I should consider this as a homework for you and you can please try it out and then you will find that this is coming readily. So, I will going to use this thing without proving that, but I will expect that you will going to prove by try to prove at least yourself that how it is coming.

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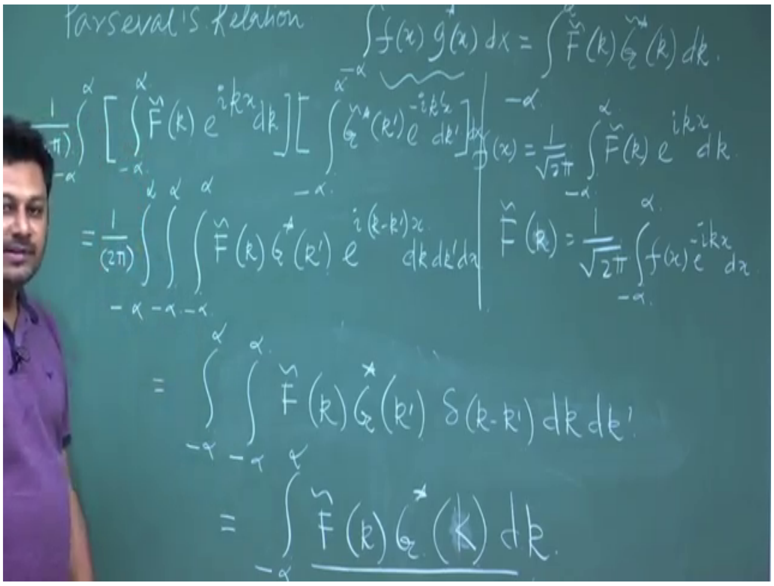

If I use this identity here then this 2 pi will cancel out and finally, we will have something minus infinity to infinity minus infinity to infinity. If I carry forward this integration over dx, then I will have something like this, this quantity is this. Finally, I end up with this equation. So, now if this is my equation, the delta function is sitting here, I just take a integration over that. So, this delta function will have the value one when k is equal to k prime; otherwise it will vanish. So, when I take the integration for k prime then this will be one when k prime is equal to k. So, then we finally having our result. So, I should write F k G star in the argument I have k prime, but I should write it as k because I am changing my integration when I done the integration for k prime when it is k then this is one. So, I should put k prime to k and then dk.

So, this is the thing I wanted to prove and these things this quantity if you look at this part of the board is same. So, I want to prove this equal to this. So, I already show very with using some simple step and using the identity of this delta function that this is possible we can show that, but interestingly something different. So, now, we know that this whatever I have written in the top of the board let me erase this. So, from that I can also conclude something very important.

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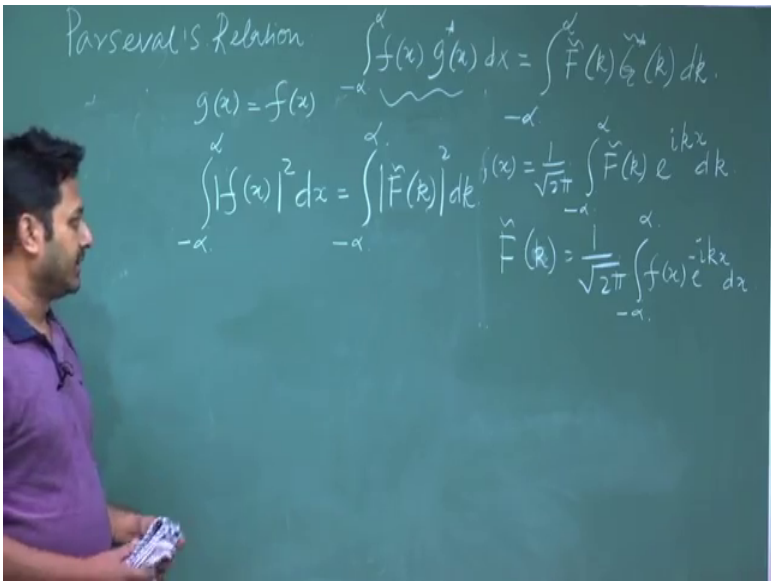

So, this and this is same which I called the Parseval’s relation, but apart from that I can, so if g x is equal to f x, g x is any arbitrary function, now I want to take g x is equal to f x same function then this identity is coming like this. That means, if I take a function and if I take a Fourier transform of that function f x is my function and f prime F tilde k is the corresponding Fourier function. If I take a mod over that mod square over that some sort of some sort of function which is giving some something like norm, but here it is not like that, but anyway, if I take this function then these things and these things are identical. If I take the integration minus infinity to infinity of this things dx, it will be the same value of whatever the Fourier function I have when I integrate this in Fourier domain with the dk. This is basically the Parseval's relation that I wanted to show with these things.

So, after that we will go to our next thing which is important also very important which is called convolution theorem. Next is our very important theorem, which is the convolution theorem. We suggests that if two functional related with some kind of response function there are the concept of response function you will know in future I guess, but if it is related some part of response function, then you can correlate this function in Fourier domain very easily.

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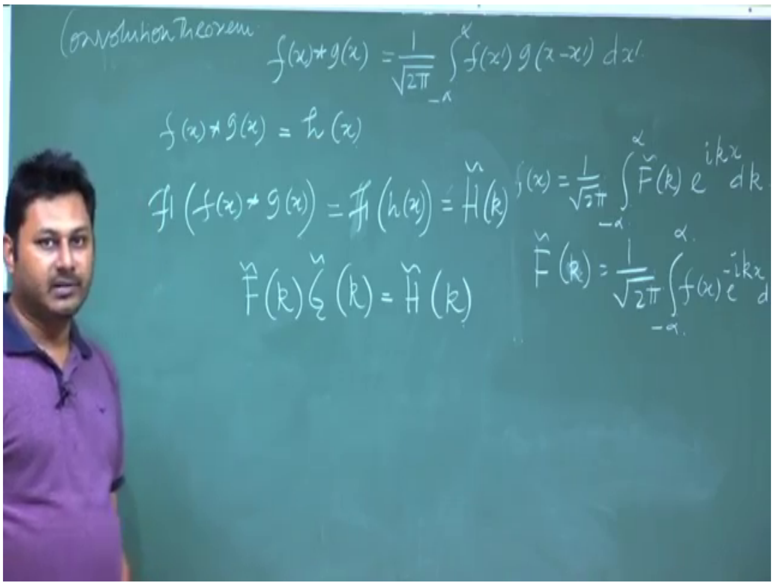

So, let me first define what is convolution theorem. The theorem suggest that what is the meaning of convolution. So, f x star g x we normally write the star as a convolution of two function f x and g x. By definition it is something like 1 over root 2 pi integration minus infinity to infinity f of x prime g of x minus x prime g of x prime, by definition this is my relation my convolution relation by definition. That when I make a star here then that essentially means right hand side I have this integrated form. So, this is thing.

So, now I want to find out, so for example, f x star g x I can write a function say this. If I make a Fourier transform of the entire thing want to make a Fourier transform of this quantity this curved f suggest that I am making a Fourier transform of entire stuff is equal to the Fourier transform of the function h x, which I write as H tilde k. Now, the convolution shows that if I make a Fourier transform of this quantity, which is H tilde k is nothing but multiplication of these two functions in Fourier domain this. Where this is a Fourier transform of this function, this is a Fourier transform of this function, so that means, whatever the function is giving to me, I make a Fourier transform of that. When I make a Fourier transformer, I have some function this function is nothing but the linear the simple multiplication of the individual Fourier transform of this function. So, this is basically the statement of the convolution theorem.

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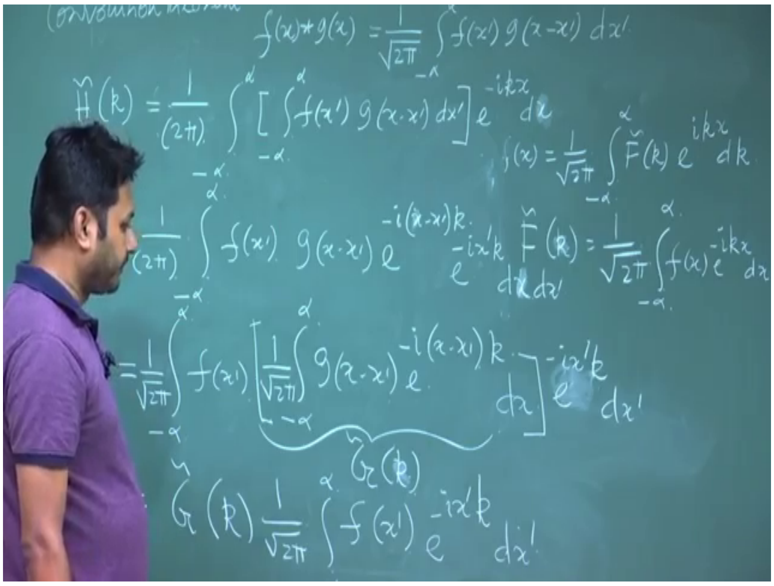

Now, after having the statement of the convolution theorem, so let us try to apply our knowledge to find out to figure out whether really it can be done or not so that means, first I need to make a Fourier transform of that. So, if I do then according to my notation it should be H k which is the Fourier transform of this quantity; by definition it is 1 over root pi integration of minus infinity to infinity this function f x this function I should write. So, I should write 1 by root over 2 pi once again. Let me because my space is limited. So, 1 over root pi is coming for this function and 1 over root pi is coming for the fact that I am making a Fourier transform, so that is why let me write it as 1 by 2 pi straight away I believe you can readily understand why it is 2 pi. 1 by 2 pi is originally in the function and 1 by root over 2 pi is coming because I am making the Fourier transform of that function that is why it is 1 by 2 pi.

So, now this function is something by definition it is this dx prime this function ended here and because I am making a Fourier transform, I should write here e to the power of minus i k x dk right I can do that. So, now, I will write this as this way f of x prime I take in this way g x minus x prime e to the power of minus of i, I write it as x minus x prime k right. So, I take minus i k x here an additional minus ik prime here. So, I need to return back this extra term which is ix prime k integration dk dx prime fine. So, once I have that then this integration is not this will be x, I am making one mistake because this is the Fourier transform. When I am making the Fourier transform this my integration will be dk here, we have a dx.

So, once I have this then I can rearrange this term a little bit. And I can write it as minus infinity to infinity 1 by root over of 2 pi f x prime as usual and then in this bracket I write minus infinity to infinity 1 by root over of 2 pi and write g of x minus x prime e to the power of minus i x minus x prime k dx whole out dx prime. So, inside this bracket whatever I have written is nothing but G of k, because this is the Fourier transform, when you shift it when x is shifted to x prime, I can put another coefficient another variable here say y; if I put another variable y it is y and the integration over dx, so it will be dy. So, it is a shifted thing, but I can still have my Fourier coefficient or Fourier value here.

So, once I have that then it should be something like G k 1 by root over of 2 pi integration minus infinity to infinity this quantity minus infinity to infinity and the rest part. Rest part is one e to the power stuff is still here which because I write these things and one e to the power i x prime k is still sitting here. So, I need to write it here. Once I do that it will be G tilde of k the thing already we have then f x prime. And this part I already taken out because this is now is function of k only which should not be inside the integration inside the integration we will have something which is f this e to the power of here let me check once again it will be minus this thing. So, in order to compensate that, I should put a negative sign here because this is minus ikx and plus ix prime k. So, I need to put a minus sign to compensate this x prime term. So, it should be minus please careful about these things, it will come automatically. The beauty of this kind of calculation is that if you do some kind of mistake by doing this in hurry then it will the problem suggest you that you are doing some mistake because it should come automatically. So, now it is x prime k dx prime. So, now this entire stuff 1 over 2 pi this and this is nothing but the Fourier transform of F tilde k, F tilde k which is the Fourier transform of this.

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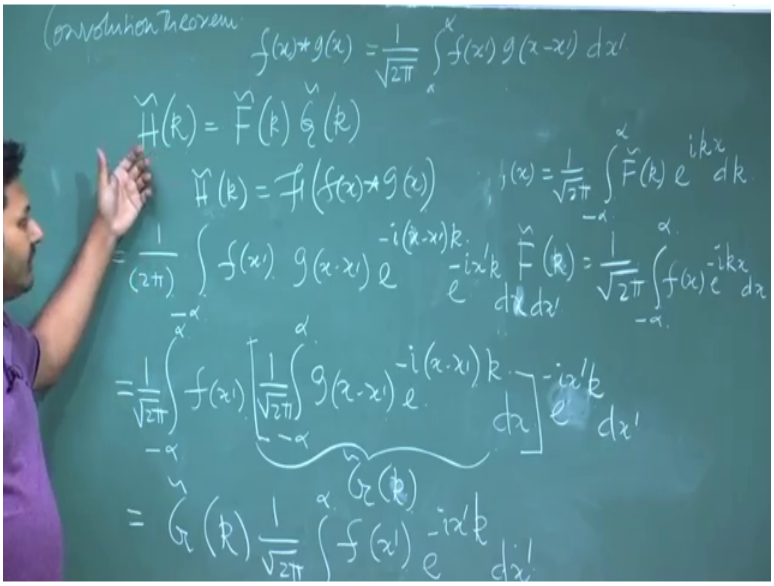

Now, if I write it here then my H tilde k is something which is F tilde k multiply of G tilde k. F tilde k and G tilde k is the Fourier transform of x and g. And H is the Fourier transform of H tilde k is nothing but the Fourier transform of this convolution f x star g x. So, with that we prove a very we show a very important theorem which is the convolution we called it convolution theorem. And from this theorem, you can readily find out if some function is given some odd looking function is given like that as I make an example in physics, you will have this kind of equation in many places. For examples for response function if something is responsive then one response function will be there and this response function which will have some kind of delay. So, this delay is associated with this x minus x prime term.

If we have this kind of delay thing with the impulse then when I make try to make some kind of calculation try to find out what is the value of this quantity. Then it is easier to go to the Fourier domain and then calculate what is the value of the thing in the Fourier domain and return it back. So, in physics we really use this kind of calculation according to our use that is why it is important.

And since we are dealing with Fourier series Fourier transform though the Fourier transform is not entirely in our syllabus this syllabus because this is entirely a different subject and you need to put more emphasis on that. But this is not a subject of linear vector space, but since I am going through these Fourier series extensively, then I thought that some identities or some kind of concept of Fourier transform I should provide you, so that you should aware of this kind of things. And when you use that you should remember in the linear vector concept that linear vector space, and this concept is the orthogonal function basis and all these things. You should correlate these two things together; when you correlate these two things then things will be easier for you to understand.

So, after having the convolution theorem, we should go to now something different something. Let me start these things and not sure today I will able to finish that or not in this class, but let me start.

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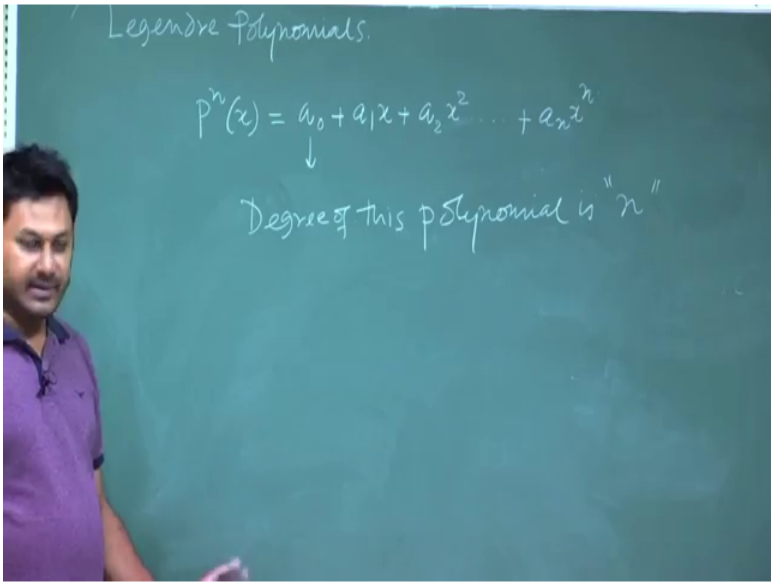

So, the next thing is something called Legendre polynomial. So, in the last class I mentioned something about polynomials and I said that polynomials also form some kind of space which is called the vector space, which is the some sort of vector space or rather they form some kind of space called polynomial space because they are formed by polynomials. Before that let us try to find out what is the meaning of polynomial then we will go to Legendre polynomials. So, normally the polynomials are represented in this way P n x, it is nth degree of polynomial. And I can write this polynomial in this way this is a more standard form to write a polynomial. And these are the coefficients and x axis for excess this is and degree of this polynomial is n this is any degree of polynomial. So, now after having any degree of polynomial I can do few quick things, which is important, before starting the Legendre polynomial. Let me do that.

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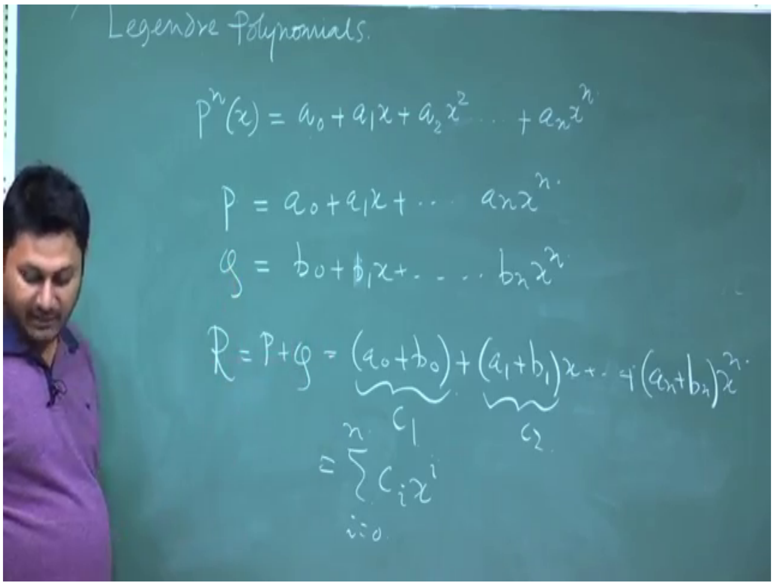

So, this polynomial is forming as I mentioned this forming space which is called polynomial space. And this polynomial space has some kind of property say P is one polynomial and Q is another polynomial both are restricted to order n for example. Now, if I add R is P plus Q then this add additive thing will have another polynomial of the same form now this is also a polynomial of same degree if I write C 1, if I write C 2. So, R is nothing but summation of C i x i i tends to 0 to n, that means when I make addition of these two polynomial as a result I am returning back a polynomial. That means, with the addition they are they are forming the same in the same space they are forming another thing it is some sort of like vector addition. So, this is one thing.

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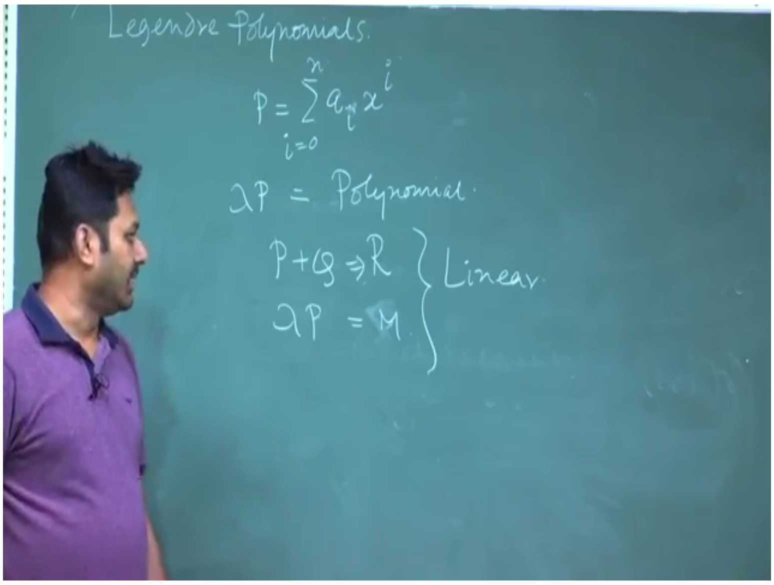

Second thing very quickly these polynomials is I say P is a polynomial, now I write a summation sign i i 0 to n this is a nth degree of polynomial. for example, if I multiply lambda P, it should also form a polynomial. So, this is also polynomial. So, two thing important thing we know that if I add two polynomial P plus Q, I will have another polynomial. If I multiply with these things, I will again have another polynomial some polynomials M, this structure is called the linear structure linear. So, this operation is linear and they follow the linear operation, so that is why this polynomial space is a linear space. So, it is linear in the nature because if I add two polynomial, we will have another polynomial; if I multiply then I will have still have another polynomial and this structure is associated.

For example in the linear operators, you operate over individually if you operate over the thing and you will add and then operate over that thing it will get the same thing. The similar if I multiply a scalar multiplier then operate over thing, and operate over these things and multiply it by the scalar it is same. So, linear operator we have already described earlier. So, here it is a very basic thing, but which it is important to know that their form is this linear structure.

So, today I will stop here. In the next class even though I am writing here Legendre polynomial I did not mention anything about the Legendre polynomial, but in the next class we will start with the Legendre polynomial and show that how they are forming a linear polynomial space. With this let me stop here. So, see you in the next class with this Legendre polynomial stuff.

Thank you.