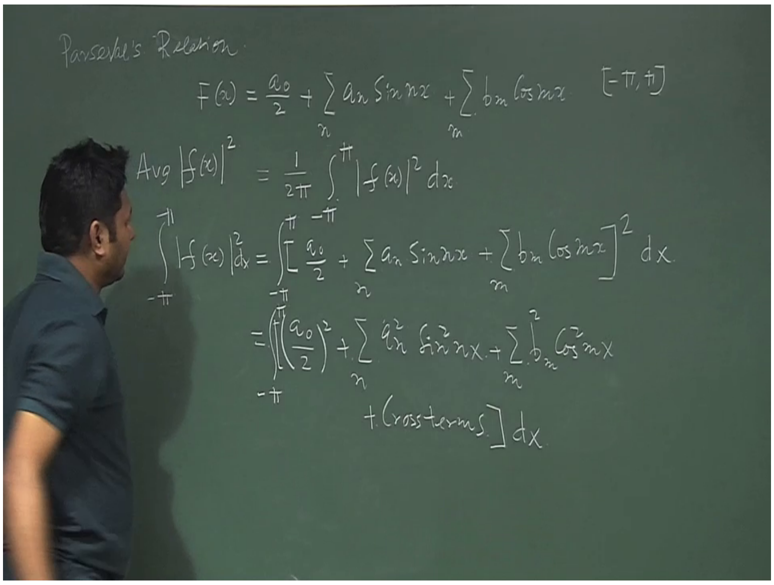
Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture – 25
Parseval Theorem, Fourier Transform

Welcome back student. In the last class, we will learn how to write a function in terms of Fourier series. And today as I mentioned in my last class that we will learn few identities few theorems which is important and you should know.

(Refer Slide Time: 00:39)



So, let us start with something called say Parseval's theorem; no it is first Parseval's relation. First let us write first Parseval's relation. So, it is a straight forward relation if a function let me write a function in terms of Fourier series, fine.

So, I write a function in terms of Fourier series where you should always remember that these coefficients are very important we are more enthusiastic should be more enthusiastic to find out what is the coefficient here the Parseval's relations suggest that if I try to find out the average of this function say this function is in the limit minus pi to pi as usual the Parseval relation suggests that if I want to find out the average of this function there should be some relationship between average of this function and this coefficient that we need to find out.

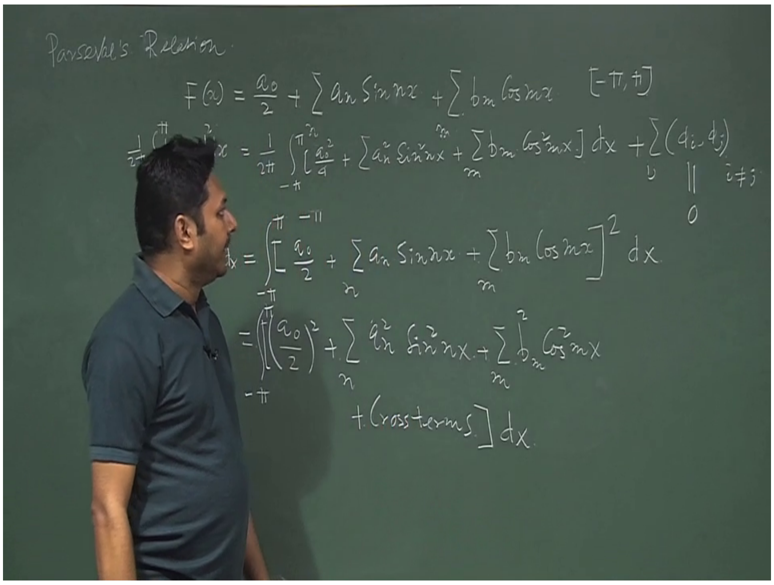
So, what is average of basically, I try to find out average of this quantity because this function may be real may be complex. So, average of this quantity which is nothing, but the norm of this function norm square rather, then it should be written in this form this is the way we write average of a function this function because in this case we have taken the norm square of this function with a limit minus pi to pi and this is the square of these things. So, this is ordinal function multiplied by dx.

So, now if I want to find out what is the value of this with this limit then readily I can understand what should be the value of this average. So, the question is what is this? This things is nothing, but a 0 by 2 plus an plus square of that straightforward square because I am con for the timing; I am considering this is the this coefficients are real to make the life simple, but even if it is not real if it is complex, if you take the mod the procedure will remain same the way we will do is same.

Now, if I make the square of this quantity it should be something like a plus b plus c, it is if you consider like this, then you will have something some term square is the first part the square term because if you start multiplying you will getting this square term this into this; this into this; this; there are an infinite number term you should mind, but apart from that there are other terms which are cross terms I should write it cross why I write it cross terms because all the terms that will be multiplied by this, this or this, this or this this cell multiplication term will be this and cross term will be there.

So, why is this cross term, I write this cross term here because this cross term the contribution of this cross term eventually going to be 0 and that is the most important part in this particular formalism why it is 0 because I am taking the limit minus pi to pi which is nothing, but the which is nothing, but the inner product of whatever the basis function I am using. So, if I write that; then if I put this here and if you just try to appreciate this case let me write it let me write it from here. So, now, I want to integrate these minus pi to pi and dx in the right hand side also I need to do that thing minus pi to pi dx because I am writing this part only 1 by 2 pi I am not considering for the time being I am doing these things and I will integrate this minus pi to pi whole this term with a third bracket dx. Now I can separate out this contribution and the rest of the contribution which I write cross. So, let me write this term minus pi to pi all this cross term as something phi 1 this term as something by definition; let me do it here.

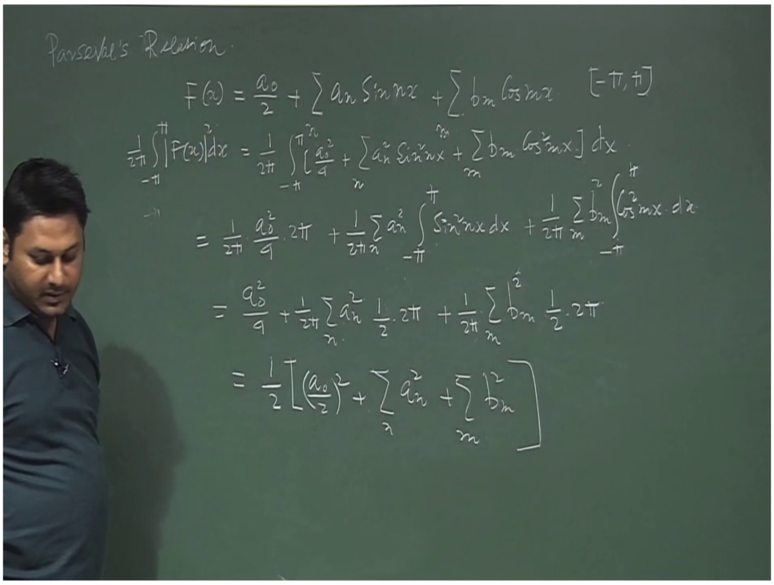
(Refer Slide Time: 06:57)



So, I will finally, have. So, this quantity 1 by 2 pi minus pi to pi fx dx mod squared dx rather; it is something like 1 by 2 pi this quantity. So, minus pi to pi a 0 square by 4 plus summation of an square sin square nx plus summation of bm m cos square mx is the first part with dx plus the cross term with the integration this. So, I should write this as the cross term phi i phi j with integration say ij. So, this phi i phi j is nothing, but all the cross terms of the set of these things; it will be sin sine x multiplied by cos x; it will be 1 multiplied by cos x; it will be cos x sin x multiplied by 1 all lot thing.

So, this thing is nothing, but with i not equal to; since it is across term it is i not equal to 0. So, this term will be 0 all the cross term why because they are orthogonal to each other and we know that when the functions are orthogonal to each other the inner product will going to vanish and this is the summation of the inner product of the cross term so that thing that means this term will going to vanish. So, only left over term we have is these square things and now we need to evaluate this term this thing term by term.

(Refer Slide Time: 08:49)



So; that means, what is my first term; 1 by 2 pi will be there a square by 4 multiplied by 2 pi this is my first term.

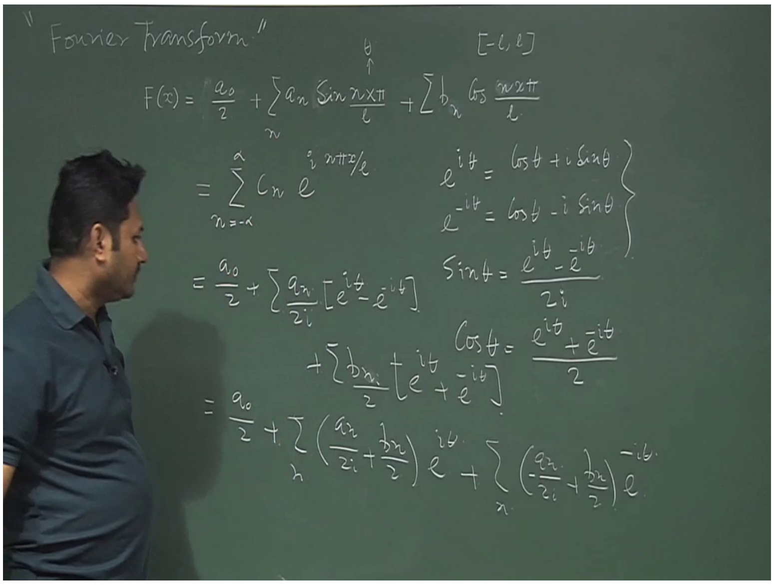
What is my second term second term is 1 by 2 pi summation over n a n and I put this integration minus pi to pi sin square nx; here dx separate it out; the integral sin is over a n I take it outside and put this integration inside. So, that I can evaluate this quantity plus 1 by 2 pi into summation of m bm cos square mx dx with the integration minus pi to pi the first term I have already evaluated. So, it will be simply a 0 square by 4 second term is 1 by 2 pi summation over n an and I need to evaluate this term and I believe you have already an idea; how do you evaluate this term this is a simple thing and we have already done in the previous class take half of that then 2 of sin square; I can divide 2 of sin square as 1 minus cos 2 nx with a half; I replace this function here and when you integrate this function this cos will remain sin; when this is sin 2 pi minus pi 2 pi limit, this term will going to vanish and only you have this thing and half. So, you will finally, the integration of these things will be simply half of 2 pi.

What about the last term exactly the same thing exactly the same thing integration of this quantity I replace this as 1 plus cos of 2 mx here maybe I write 1 minus cos of nx, but it will be cos of 2 nx it a half term this term is also represented by this half, but this minus sign will be plus because this is a cos square the straight forward the thing and when you integrate this cos will remain sin when it will be sin, then if you put the limit, it will going to vanish only this contribution will be there and as a result we will have the same thing because this integration I have done earlier. So, I am not meticulously doing all the integration it will take much time and I believe it is not a very hard thing to understand. So, it is something like this square is missing here.

So, this is the final average value; I am looking for this is the average value I am getting. So, what value I am getting average is half of a 0 by 2 square plus summation of an square plus bm square. So, average value is related to the coefficient and if the average value is related to the coefficient; it will be something like this. So, if I have the value of coefficient if i square and add and make divided by 2 the value I am getting is basically the average of this function. So, this relation is called the Parseval relation.

Now, we need to go to another important thing before that we need to find out what is the meaning of this Fourier transform.

(Refer Slide Time: 13:19)



So, let us one important thing Fourier transform. So, far we are dealing with Fourier series now we are saying something Fourier transform this word is not a very new thing new word to you, but we like to in this class, I like to show you how from the Fourier series you can ended up with the Fourier transform and what is the condition for that.

So, let us write in this way. So, fx is something a 0 by 2 plus summation of an sin of nx pi by l; I write it because of some because my function here is defined minus L to L I am writing this notation because this notation we have already; I have already introduced it is the same thing, but when the limit is not n pi, but L, then I need to rescale this thing. So, and I rescale that that is the only difference between the previous thing and this things.

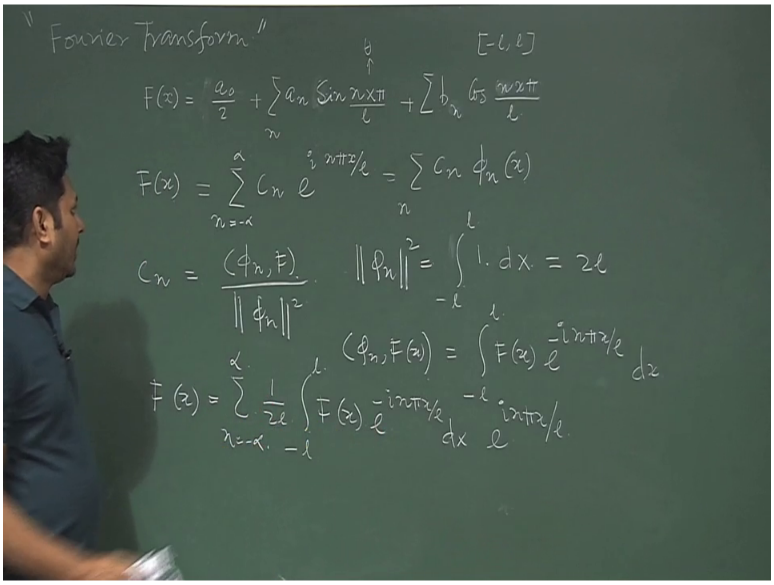
So, now, I like to write these things in a different way and you can readily understand; what is the thing. So, I write this as a whole say cn e the power of i n pi x by L; I write this thing in this way with n is equal to minus infinity to infinity. So, you can understand quite easily why these things and these things are same because I just replace this sin and cosine you know e to the power i theta is equal to cos theta plus i sin theta and e to the power of minus i theta is cos theta minus i sin theta from t these 2 expression I can always write e to the power i theta into the power minus i theta in terms of sin and cos. So, sin theta is e to the power i theta minus e to the power of minus i theta divided by 2 i in a similar way cos theta is e to the power i theta plus e to the power of minus i theta divided by 2; what I do is just replace this sin and cosine in terms of e to the power i theta and it is the power minus i theta.

After doing that I can take if I do it; it will look like something like this sin theta I will replace just 1; let us put it; let us call it as the theta. So, it is theta minus i theta 1 series and another series is bn or bm by 2 e to the power i theta plus e to the power minus i theta. Now what we will do that please check one thing that this n and m is I just distinguish, but I can write the entire thing in terms of single variable. So, if I replace this m into n there is no problem at all there is no problem at all, then I can always write this theta and this also theta and this is this is bm.

So, when I write this thing entire thing, then what I can do? I can take this e to the power theta 1 side say something like this an by 2 i plus bn by 2 multiplied by e to the power of i theta; one term another term is an minus of an by 2 i plus bn of 2 e to the power of minus i theta right with nn. So, I can write this e to the power i theta e to the power minus i theta a 0 by 2 in this term; whatever the expression is giving to me which is straightforward, then I can accumulate everything in a single summation and the single summation is this by putting n tends to minus infinity why because the positive integer is there also this is minus sign. So, negative integer I can introduce and this limit will going to change. So, essentially I can write these things in terms of this also.

I will show explicitly how you can do that. So, I believe you will understand that it can be quite doable.

(Refer Slide Time: 19:11)



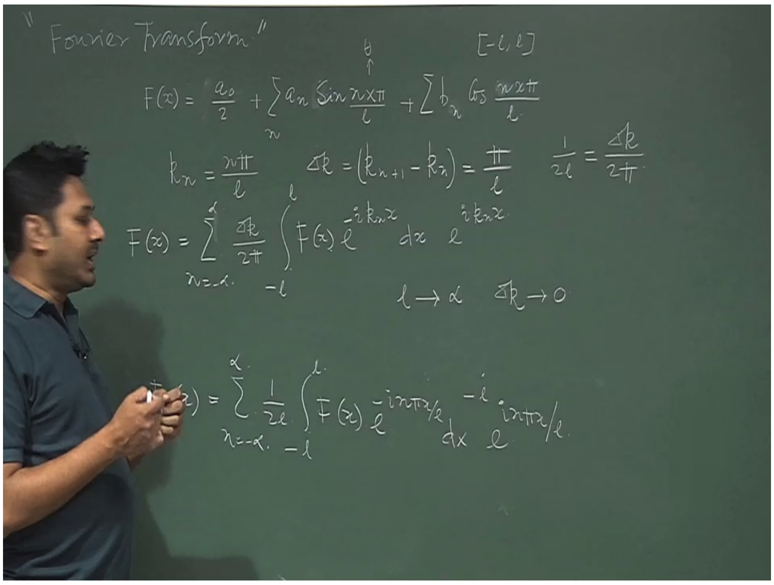
So, next thing is fx is this quantity. Now I can correlate these things to my original expression of orthogonal functions or bases. So, these functions is written in this way I can correlate these things to this so; that means, this is nothing, but this set of functions .

Next what we will do next, I will try to find out what is my cn; cn is how much I know how to find out cn it is phi n f divided by mod of phi n square; this is the Fourier coefficient; we have write, we have wrote these things several we wrote this thing several times. So, we have shown this several time. So, what is the phi n mod square here it is one it is integration of minus L to L integration minus L to L the function is this multiplied by this? So, complex conjugate of this; so, it will be just one right it will be just one because I am multiplying the same function with the same function with the complex conjugate.

So, this complex conjugate will be multi this function will be multiplied by the complex conjugate. So, this plus sign will be minus sign and when I multiply you will get 1. So, you will have 2 L for that. So, this quantity is 2 L what about the other thing phi n f of x by definition, it should be minus L to L f of x multiplied by phi n. So, it will be e to the power of minus i n pi x by L dx this up to this is fine.

So, now I should write the entire thing here. So, fx. So, let me write it here. So, fx is summation of n tends to minus infinity to infinity cn, I figure out cn is 1 by 2 L integration minus L to L fx e to the power of minus i n pi x by L and then dx and of over that I will have 1 e to the power i term e to the power i n pi x divided by L. So, this is my entire term right. So, now, I will put something interesting here and may not be required that.

(Refer Slide Time: 22:24)

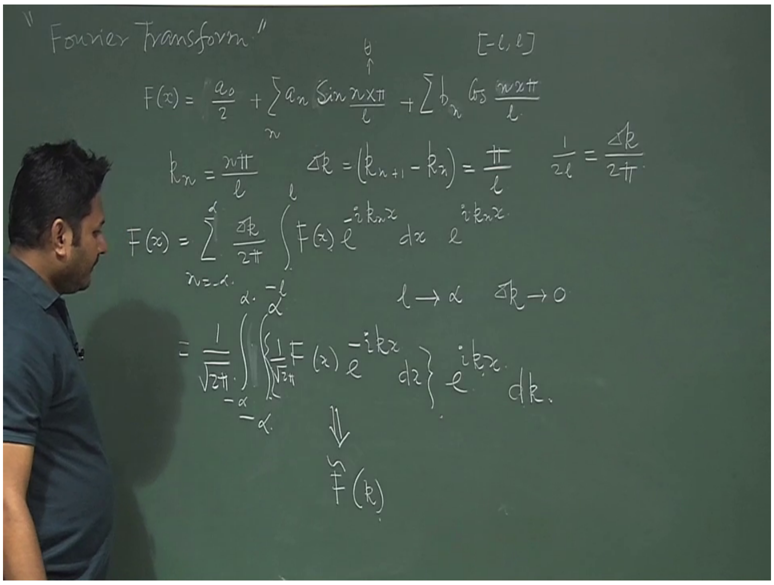


So, let us put k with a subscript n is equal to n pi by L. So, this quantity n pi by L can be represented by kn this. So, what should be my delta k delta k? So, n is the k is the discrete quantity because n is integer 1, 2, 3, 4 minus 1, minus 2, minus 3, minus 4 and so on. So, if I want to find out; what is my delta k which should be k n plus 1 minus kn this quantity; it is nothing, but pi by L pi by L. So, k is replaced by pi by L. So, I can replace 1 by 2 L from this expression; I can replace 1 by 2 L as delta k by 2 pi.

Now, I rewrite this expression once again f of x is equal to 1 summation of n which is minus infinity to infinity 1 by 2 L, I will replace this as delta k by 2 pi, then I integrate minus L to L f of x e to the power i; this quantity I replace kn x with a negative sign and rest of the term with dx and rest of the term is e to the power of i knx; this quantity well up to this; I will have something which is discrete because my summation is also from minus this to this. So, this summation is discrete summation.

But now one important thing; I will do that is I will change these things. So, what I will do that I will go L tends to infinity. So, I will extend my limit to infinity when I extend L tends to infinity what happened that my delta k will be tends to 0; that means, when I because from this relation L; L tends to infinity delta k is tends to 0 and as a result this discrete n will be gradually become a continuous limit because I am taking with the condition that my small delta k which is the difference between these 2 is pi by L and now I extend my L tends to infinity; that means, huge distance. So, that all the small segment of k delta k which is depend on n is become 0. So, this discrete n this discrete n tends to 0 so; that means, I will now have a relationship in a modified form and this modified form is something like.

(Refer Slide Time: 25:59)

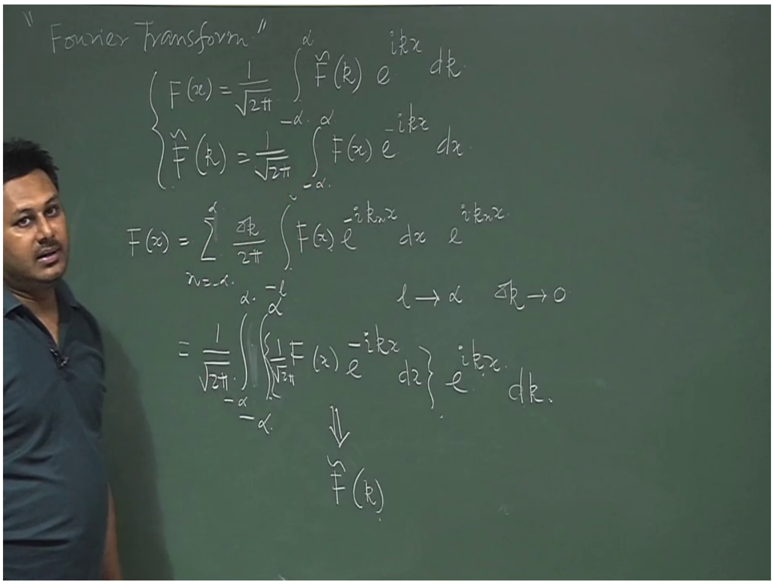


I can write it as this 2 pi; I just write this write this 2 pi into 2 different 2 pi, then minus infinity to infinity because already the limit is goes to infinity only things that n is now not become a discrete function it is something f of x 1 by root over of 2 pi L limit is also minus infinity to infinity. So, I will put this integration also and I put push this integration by putting L tends to infinity and minus infinity; so, here these things e to the power of minus I; since the discretization of n goes away.

So, I can now write k as my continuous k in dx over that I will have e to the power of i kx and dk. So, please note that this one by 2 pi i divided 1 over root pi and 1 over root pi 1 over root pi; I will take outside another; I will take inside fx all already there e to the power i, this is placed by e to the power ikx because the discretization of k is now removed. So, this summation sign which is over discreet thing is now, we will be replaced by integration. So, I change to add the integration sign. So, that is why this dk is now not a discrete dk delta k rather it is a dk I put it here dk. So, that this integration will go minus infinity to infinity and I will have these 2 terms here. So, I will put them here.

So, now I should write this term; say I write this term f of tilde of k if I write this readily you can have the form of Fourier transformation.

(Refer Slide Time: 28:11)



Let me write finally, what I am getting? I am getting f of x is equal to 1 by root over of 2 pi integration minus infinity to infinity f tilde k e to the power of i kx dk where f tilde of k is 1 by root over of 2 pi minus infinity to infinity f of x by definition e to the power of minus i kx dx.

So, with this, we landed up to a relationship between fx and f tilde k which are Fourier transform conjugate to each other. So, fx I can write in this way the Fourier conjugate f tilde k; I can write in also this way. So, now, this treatment from this treatment you can understand that the Fourier transform concept can be achieved by your known concept which is the Fourier series and when you discretize; when you extend the limit up to infinity then this descripting will be go away and you will have a continuous variation and under this continuous variation you can have a relationship between a function and its Fourier components which is nothing, but the coefficient of when we are doing these things over or the normal summation.

So, we if you know what is the; my f fx you can readily know what is what will be my f; f tilde k; if you know; what is my f tilde k; you can also find out what is your functional form fx. So, with that note I would like to conclude here. So, today we have learned a very important thing the relationship between the average function which we called the Parseval's relation and also we learn how to make a Fourier transform; from Fourier series. So, Fourier series and Fourier transforms are not a very apart thing, but they are somehow related when the continuous limit. So, under continuous limit you know how to do that.

So, in the next class, I will also like to show few important theorems like Parseval's theorem; Parseval's relations, persevals theorems. Today, I already show an another important thing say autocorrelation what is the meaning of autocorrelation regarding this Fourier this is the small-small things, but I believe you should know at least what is the meaning of that and how it is related to this treatment. And then we will go to another topic which is called the polynomial space or we will deal with something polynomial. After that we will finish our linear vector space course and then we will go to the next course which is complex analysis, but we will take another couple of class to finish them.

With this note let us conclude here. So, see you in the next class.