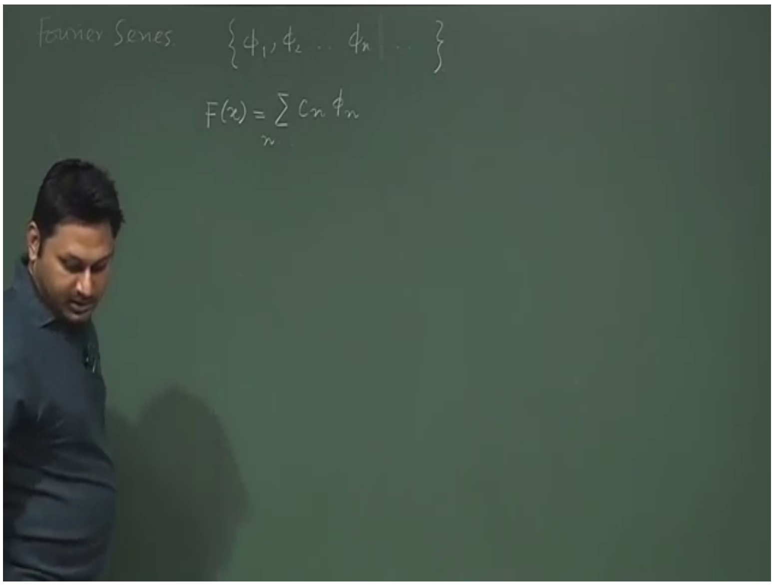
Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

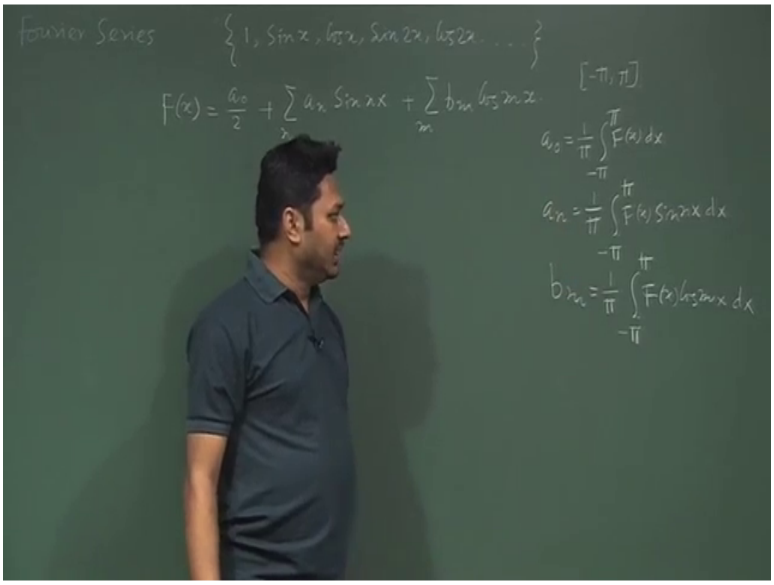
Lecture – 24
Fourier Series (Contd.)

(Refer Slide Time: 00:28)



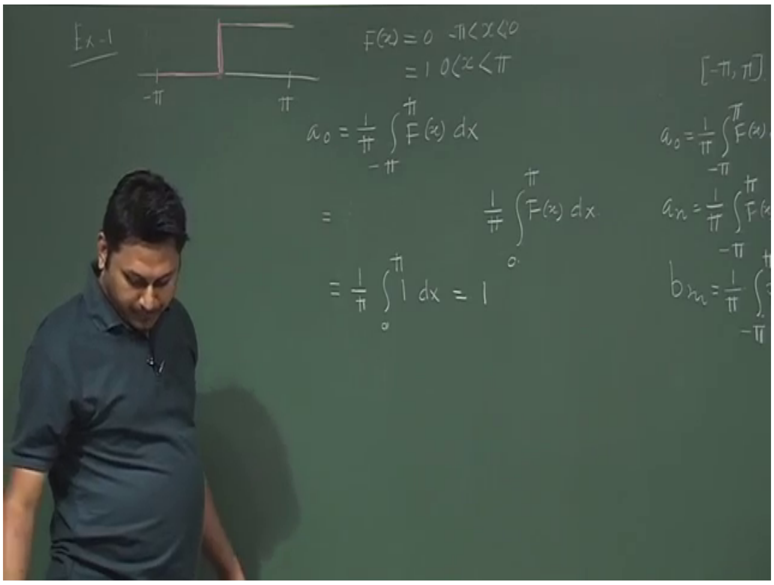
Hello student, welcome back to our class. So, in the last class, we started very important topic Fourier series. So, if you remember we started with a very important concept which is orthogonal functions. So, if a set of orthogonal functions are there, let me remind you once again, this is a set of orthogonal function in principle in function space, it can be extended up to infinity. Then any function F x can be represented in terms of this set of orthogonal function which behaves like a basis.

(Refer Slide Time: 01:28)



Now, if you remember for Fourier series we use a specific function here the set of orthogonal functions were very specific and they are 1, sin x, cos x, sin 2x, cos 2x and so on. In the previous class, I wrote 1, sin x, sin 2x and then cos x, cos 2x, but this is the same thing. Now, if I want to write the function in terms of Fourier form then this is my expression of the function that means, I expand this function in terms of this orthogonal set of functions, which behave like a basis. Here my function is in the limit pi to pi. After having that we also find what is the coefficient and we will find the coefficient of this was minus pi to pi F x dx a n minus pi and b n or m whatever it is just the damn index. So, we should not bother about this something like this. If the function is defined in minus pi to pi if it is defined minus l to l a similar kind of set was there we just replace something here, it will be replaced by n pi x divided by l; and this limit will be minus l to l, and it will be just 1 by l.

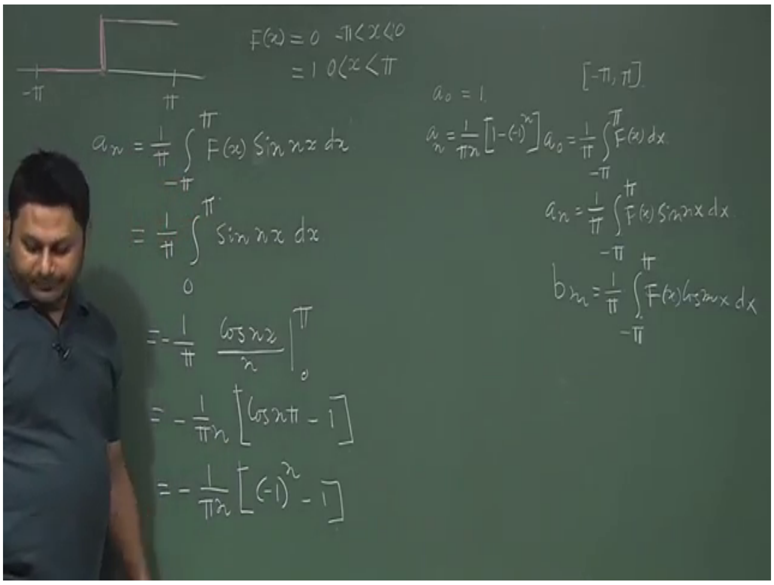
(Refer Slide Time: 03:47)



So, we know that. So, now, we will going to use this concept to define a function say example-1. So, example-1, let us take a function this is say minus pi, this is pi the function is defined like this. If I use another colour to define the function, it should be something like this; this red one is a function. So, mathematically if I want to write this function, it should be F x is equal to 0 in the limit this a n equal to 1 in the limit pi there are two limits. The function is and there is a discrete discontinuity here the function is suddenly have a jump I should not say this is a discontinuity, but there is a jump of this function. And because of that we will have some kind of average value that we discussed last day, but before that let us try to find out if I want to write this function in terms of this Fourier coefficient then how the function will look like.

So, first my F x the way it is defined is over the two different limits. So, let me first find out what is my a 0, a 0 is 1 by pi minus pi to pi the function dx by definition. So, now we will going to put the effects here to find out my a 0, 1 by pi minus pi to 0 F x dx; I just divide into two part and try to do that explicitly this. Now you can readily understand when I divide this to minus pi to 0 the value of the function here is zero, so we should not bother about this term at all. So, this term will readily cancelled out only remaining term is here where this function has some nonzero value. So, what will be the value very easy one by pi integration 0 to pi the value of the function if this limit is 1 dx is 1 and then dx. So, eventually we have something this is pi and 1 by pi is there. So, we should have something like 1. So, a 0 is 1 in my case.

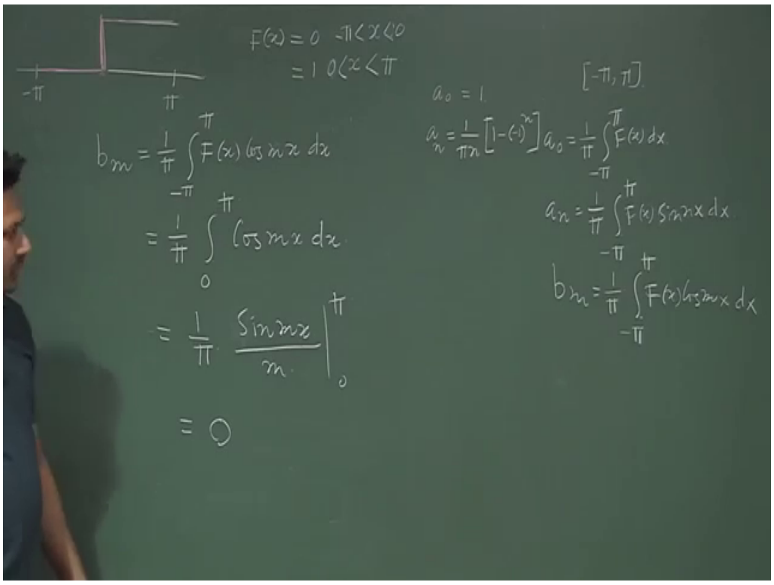
(Refer Slide Time: 07:13)



So, if I now calculate the other coefficient then other coefficient will be calculated in the same way. So, let me do that. So, let me write somewhere the value of a 0, a 0, I figure out it is 1. Then the next coefficient I would like to find is my a n which is by definition using the formula rather I need to multiply sin nx dx this is the total thing. Now, in the similar way I need to divide this integral into two part, and minus pi to zero part since it is 0, so I will not going to bother about that part at all. So, directly I write this limit because the function will be zero. So, where the function is nonzero I am writing this. So, it is sin nx dx, I need to evaluate this integral to find out what is my a n.

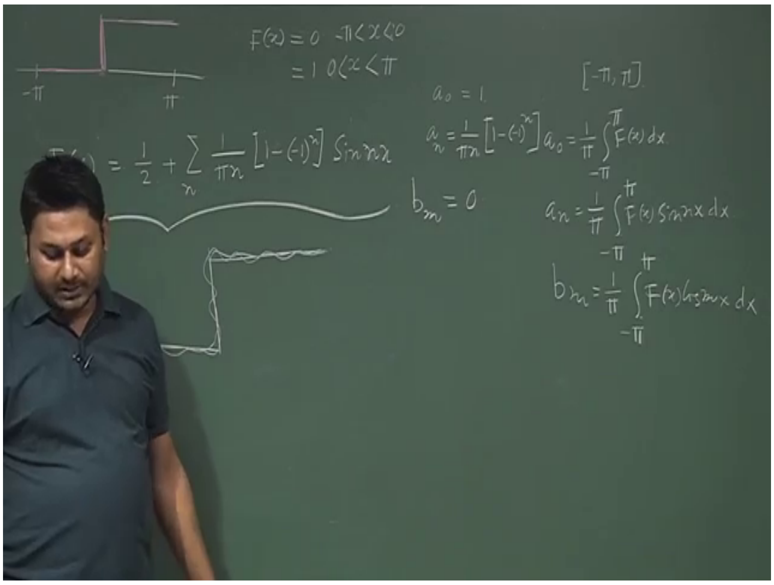
So, this integral if I do the integration of sin, I will have cos of nx divided by n with the limit 0 to pi. Since I am integrating sin, so there will be a sin here with a negative value, so minus sin will be sitting here. And then I will have minus 1 by pi n cos of n pi minus 1. This n, I will take outside then cos of n pi because I put the limit here and then 1. So, you should know that cos of n pi the value of cos of n n pi minus, so 1 by pi n; cos of n pi is nothing but minus of 1 whole to the power n minus 1. When the value of n is odd, we have minus 1; when it is even then we have plus 1. So, this is also oscillating with respect to n. So, I have some kind of oscillating value in terms of n here in this. This minus sign I also can absorb, so that I can write 1 minus of minus of 1 to the power n. So, here somewhere I should also write what is my value of a n and it comes out to be something like this.

(Refer Slide Time: 10:09)



Next I need to write in the similar way I need to figure out what is my coefficient b m. And if I do it will be simply be something like minus pi to pi F x into cos of I am writing the same thing it is a good habit to write the same thing over or over again, so that you can familiar with this terms. I believe I do not need to explain why it is this, because it is a similar way if I put this limit minus pi to 0 it will be vanish and then only the contribution of the function is coming from this here one, so one multiplied by cos of mx. When I am integrating cos, it will be sin of m of x divided by m with the limit 0 to pi. Now, we know that sin m pi is 0. And if I put 0 here in the lower limit also it is 0. So, this contribution is essentially 0. So, I have a contribution which is essentially 0, so that means, I calculate all the coefficients Fourier coefficients or Fourier in order to expand the Fourier series.

(Refer Slide Time: 11:34)

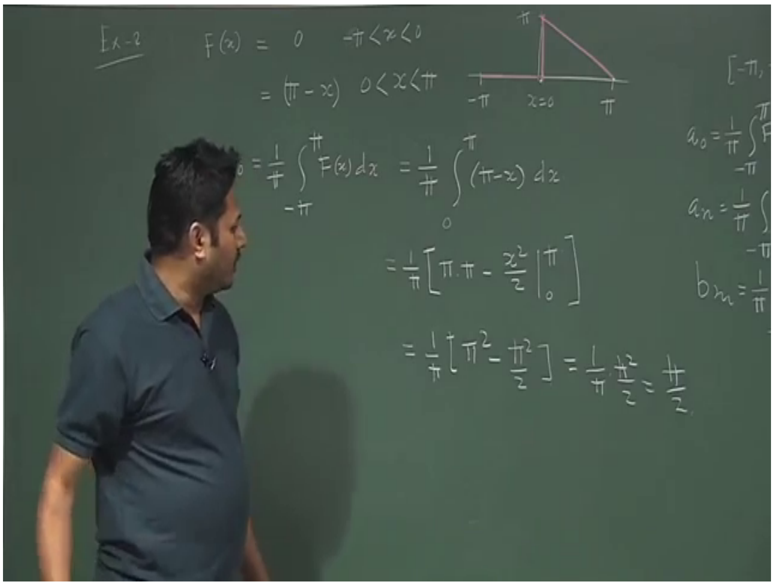


And now if I write the total form total function it by using that coefficients my function F x will be something a 0 by 2 that means, half plus summation of a n is not equal to 0, but b n is 0, so there will be no b n part here. So, n 1 by pi n multiplied by 1 minus minus 1 whole to the power n of sin nx, this is my functional form the way I write in terms of Fourier series the function it should be something like this. Now, you can see one thing that in order to match this function because this is the sin term last day we are discussing, this is the important you should note this fact that this function is a step like function. But still I can expand this function in terms of a function here sin x which is sinusoidal in nature. But in order to accurately represent this function, this limit go to infinity that mean means I need to put infinite number of terms here to match this function.

So, here you should remember that if you restrict your term to few things then say with adding ten term you will have something like this, this kind of function. If you increase the number of term then gradually you will find that you are really grating something very close to this, which is quite interesting and you should appreciate that. Even though you are expanding a function in terms of sin or cos, which is sinusoidal in nature, but if you add sufficient number of terms like this then you can come across to the function which can corresponds or which can be equal to the given function, which is like a step function, here is a step function. So, since it is a step function there is no sinusoidal variation, but still you can have that.

After this example, we will go to another example and this is the straightforward way to calculate the Fourier series. In the exam or in your exercise maybe you are asked to calculate the Fourier coefficient or you are basically asked to expand a given function in Fourier series then this is the process you should follow. That you should you write the function, you calculate the coefficient of a and b and a 0, you have find that because the function is given to you. And once you find that you just write the entire term hat is all.

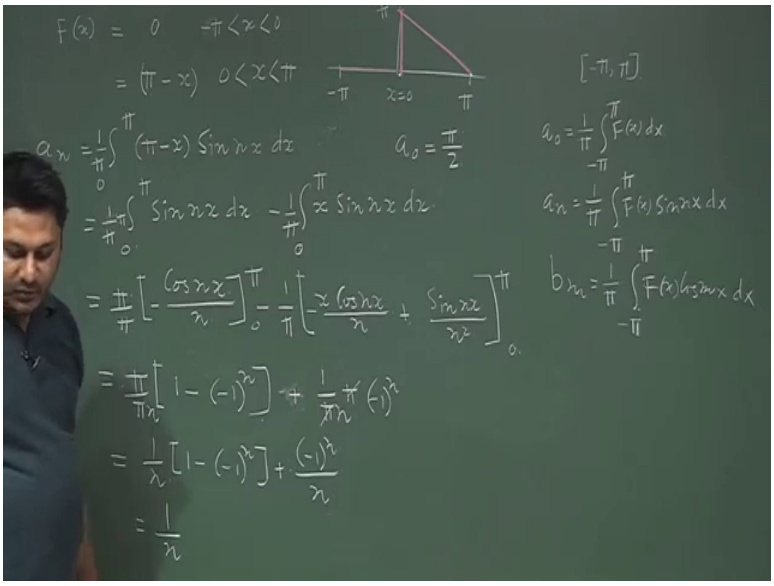
(Refer Slide Time: 14:51)



So, let us go back to the next example. Here the function is defined something like the function is 0, when the limit of x is a minus pi to 0 and pi minus x the function will vary like this, when x is 0 and pi. So, before drawing that before doing anything we should need to find out how the function will look like. So, let us do that here. So, this is my minus pi, this is pi, the function is 0 from this minus pi to 0. So, it is something like this. And then what happened then it is pi minus x in this limit. So, it should be somewhere here and this. So, the function will look like this. So, this is x equal to 0; at x equal to 0, the function is pi. So, this value is pi and then it is gradually comes to the value minus pi the value pi, because when it is a straight-line kind of function, when x is equal to pi then the function is view and gradually this is the y equal to m x plus c kind of stuff. So, it will come like this. So, the given function is something like this.

The next thing is the same need to find out for this function what should be the value of a 0, a n, and b m, so that I can expand this function in Fourier series. So, a 0 if I calculate the thing I have already reading there, but still I am writing because you may not see that part the board. So, it is simply 0 to pi because minus pi to 0, it is 0 exactly like the previous problem this I need to evaluate this integration as simple as that. So, 1 by pi pi into this will be pi minus x square by 2 0 to pi. This integration will be x square by 2 and my name it will be 0 to pi, so it will be x square by 2 0 to pi 1 by pi it will be simply pi square minus y square by 2 or 1 by pi pi square by 2 or pi by 2 straightforward is pi by 2.

(Refer Slide Time: 18:21)



So, let me write somewhere what is the value of a 0, in this case a 0 is pi by 2. Now, I need to write the value of a n, a n is straight when I write 0 to pi because there is no need to write everyday minus pi to pi, because the function is 0 here. So, this multiplied by the function multiplied by the sin function sin of nx dx. There will be two function 0 to pi pi sin of nx dx, this is one function. And the second function is zero to pi x sin of nx dx step by step we will do. These things will be simply pi integration of sin which is cos of nx divided by n with a negative sin zero to pi this and the right hand side since this is a multiplication of two functions. So, first function into integration of the second function. So, it will be cos nx divided by n minus differentiation of the first function and then integration of differentiation of this function, then this multiplied then integration of this function once again. So, eventually you have two integration of this. So, eventually one for one integration you have cos and then other again another integration you will have sin. So, eventually you have sin nx.

Every time when you integrate, you have n square term down here. When you integrate these things then you will have one negative sign associated with this, so this sign so seems to be plus with the limit zero to pi. Now the way that this function is here you can understand that since sin will become cos and then cos will become sin whatever the multiplicative term. When I put the limit zero to pi this term will go into vanish. So, I should not bother about this term because this is coming like a sin, so the limit is zero, but this term will not.

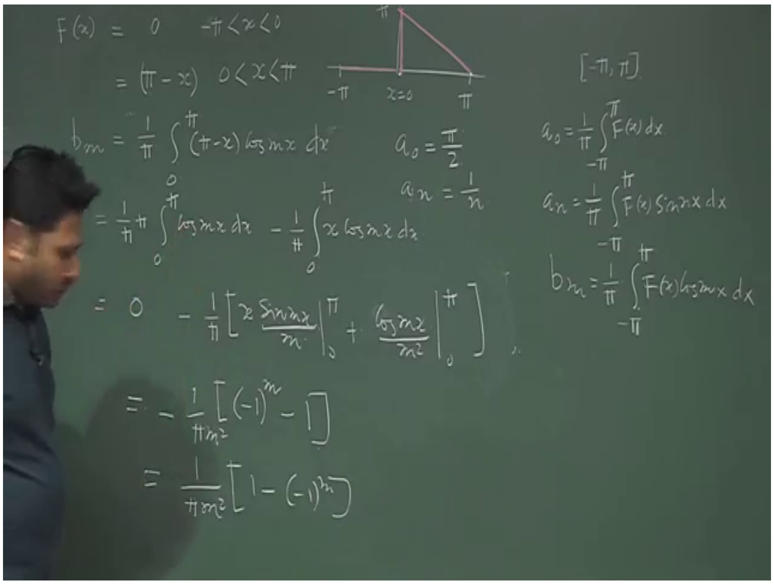
So, let me do that step by step. What about the first term the first is pi divided by a n. So, one by pi should be there. So, when I do the integration this one by pi, 1 by pi should be there. So, I should miss that by pi by pi. And also here 1 by pi should be there, 1 by pi should be there, you should be careful about this because when I find out a n, this a n value is associated with this formula, so 1 by pi was already there. But when I was doing that I just miss that, but right now I find this one by pi is still there. So, let me check everything once again if it is ok or not then n will be outside. And if I change the limit 0 to pi then this minus sign will going to absorb then we have 1 minus minus 1 whole to the power n like the previous way it is something like this.

What about the next term it is minus 1 by pi this is x multiplied by this. So, n will be there sitting, it will be x multiplied by this. So, x the first limit will be there pi, and then minus 1 whole to the power n, but there will be no second limit, because there will be no second limit because this x is sitting here. So, when you put the zero this will vanish and this term is already vanish, it seems to be something like this. So, let me check once again everything is fine or not, they are seems to be one when I change that, so it will be minus one. Let me check once again everything is correct or not in order or not then I will go forward. So, it seems to be one by n pi pi will cancelled out, here also this pi pi will cancelled out and we have one term here one minus minus of 1 whole to the power n minus minus 1 whole to the power n divided by n something like that.

So, let me check once again if everything is correct or not it seems that this sign not be minus, but this lets check everything is correct or not pi minus x sin x, so zero to pi i integrate. So, when I integrate zero to pi, so 1 by pi is will be there. The first term is pi sin, so pi sin, sin when I make a integration of that it will be cos, it will be cos with a negative sign. Then 0 to pi, when I make a 0 to pi then this term will be minus of this it will be this minus sign will be still there, because minus sign I put pi. So, it will be this minus sign.

So, this minus sign will be sitting here and if I put it will be minus of these things by no which is. So, then what happened here the next term this term seems to be fine for me. And then the next term is 1 by pi x sin. So, if I expand this, it will be two term. So, first is this multiplied by this. So, cos here when I make a derivation of so one negative sign should be here that I was missing. So, now, I will put here a plus sign here I am making one mistake. So, this is plus. So, if this is plus, then this term and this term will going to cancelled out, and eventually I will have 1 by n. So, I suggest all the students to do all these calculations very carefully, because it looks quite straightforward calculation, but this plus minus sign always create some kind of problem and then you can get different results. So, here I know that this should be 1 by n that is why I can find out that where I am making mistake, but you should be very careful about doing when you are doing this problem. So, my a n comes out to be 1 by n fine.

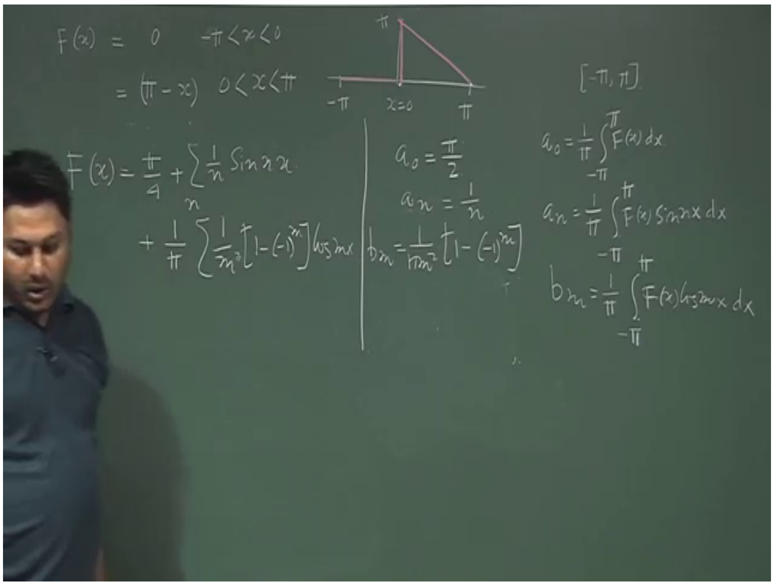
(Refer Slide Time: 26:20)



So, next I will do the same for bm. So, b m is 1 by pi 0 to pi the function is pi minus x then I will have cos of nx dx. So, 1 by pi then pi integration of 0 to pi cos of mx dx minus integration of 1 by pi 1 by pi integration of 0 to pi x cos of mx dx. So, this pi pi will cancelled out cos will be when the cos term will be there, it will be converted to sin, and when you put the limit this term is also going to vanish. Here if I do then 1 by pi first term is this and then there will be a. So, first term integration of these things. So, it integration of these things, I am making things quite first because I believe you understand these things. So, when you make the integration of these things x first function into integration of these things it will be sin of m x divided by m with the limit 0 to pi. So, this term will cancelled out.

And then the next term minus then there will be a plus sign and then you will have a derivative integration of this term twice, because this is the first function you make a derivative and then integration of these things twice. When you make a integration of these things twice, it will come to be cos of m x by m m square cos term will be sin then sin will be cos one negative sign will come, this negative sign will be absorbed here. So, it will be 0 to pi. So, finally, when you put this value, this will go into vanish, but here this term will not go into vanish. Here we will have something seems to be minus of 1 by pi m square and then minus of 1 whole to the power m minus 1. If I rearrange it will be 1 by pi m square 1 minus minus of 1 whole to the power n, it should be something like this. So, if I put it here my b m sounds to be something like this.

(Refer Slide Time: 29:22)



Now, I have all my a 0, a n and b m. Now, my total calculation is complete. So, now, I am in a position to write it is a Fourier component with a Fourier series. So, if F x will be a 0 by 2 that means, pi by 4 plus summation of 1 by n sin of n x over integration n plus 1 by pi it should be come outside, 1 by m square with a oscillating term that appears in the previous problem also and cos of mx. So, this is the entire Fourier series I have for the given function.

Well, so I will like to stop here students. So, in today's class we did something which is important because this kind of problem you will always face because of now we know what is the how the Fourier series is formulated and how it is working. Now, you can understand if a function is given to you, you can always write this function in terms of Fourier series once you know what is the coefficient of a a 0, a n, b m. the way you find the coefficients already giving here this is the recipe and we figure it out. So, whatever the problem whatever the function is given to you whether it is a discrete or finite having finite discontinuity, but still you can able to evaluate this function in terms of this coefficient. And this is essentially the Fourier series of a given function. With this note let us conclude here. In the next class, so we will start there is few important we will learn few important identities, few important theorems and with this note let us conclude here.

So, see you in the next class. Thanks.