Mathematical Methods in Physics-I
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Lecture – 23
Fourier Series

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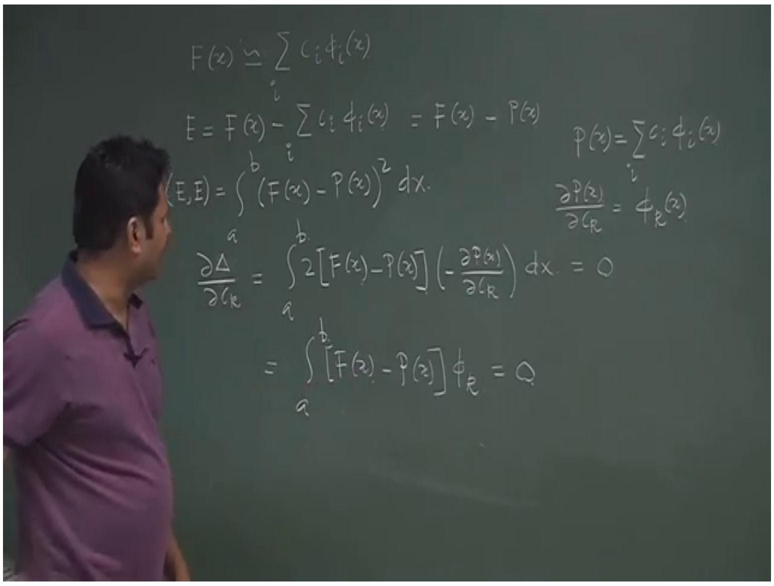

So, welcome back student. So, in our last class, we learn something about orthogonal basis and find something called Fourier coefficients. So, let me write once again these things. So, F x can be represented I do not write the equal sign right now here, so let us write the nearly equal to sign say C i phi i x and i. F x is a function I want to write this function I want to decompose this function in terms of some linearly independent set of function phi i which are orthogonal to each other. And if I do that then I want to find out what is my C i that calculation we have already done in the last class, but today what we are doing that we try to find out what is the error and how the minimization of the error can be calculated. So, minimization of the error will come with the C i. So, if I adjust my C i in such a way that this error can be minimized for a given set of function then readily I can understand what is the value of C i.

So, let me find let me say E is a error function which is F x minus this quantity, please note that since F x is not equal to this. So, error has some value so that means, error function is something here it is not equal to 0, since I am saying that this is nearly equal to zero or if it is of the order of these things, but not equal to F x completely so that means, there is some error associated with that. So, let me write this function F x and we have some function here also which is the linear combination of these things, say I write this function as my P x is a function, where P x is summation over C i phi i x i. Now, E function is given by this minus F x the given function minus P x another function which is a linear combination of the set of functions which is orthogonal to each other we called it basis.

Now if I want to find out what is the inner product of E by E that means, what is the norm of E what is the magnitude of E. So, I will have to do this operation, which is essentially a to b this quantity multiplied by this quantity. So, it is something like this. This quantity is this I try to find out this quantity which is which I called say delta which is a number this delta is inner product of the error function that means delta give me the value or the norm of the error function. If I do that then the right hand side, I will have this. Now, I want to minimize this quantity, how I minimize which is the variable here which quantity is variable here F x is a function that is given fine, and P x is another function which is given by this now this is the basis function it is already given. So, I will not going to change that.

So, what thing I want to target I want to target that how I fix my coefficient C i, so that this error become minimize, so that gives me readily what is the condition or what value of C I should be there to minimize the error. If I do that then I need to find out this quantity. So, in order to minimize the error with respect to C i need to find out this quantity the way we find a minimization the function is minimized. So, I will be need to make a derivative here C x is the changing thing changing parameters if I need to do that with C x. If I do then the right hand side has something like 2 F x minus P x derivative of these things over C x. So, two of these things multiplied by this and derivative of this quantity with a negative sign so minus.

This quantity if I do from that notation whatever we have del P x del C k is what quantity it is a summation over that. So, one ck is here sitting and I make a partial derivative over C k, I will only have phi k in my hand here. So, this quantity again this quantity is 0, because I know that this is minimized. So, in order to minimize, I need to this is my condition that should be minimize. So, finally, I will have a to b these two will be cancelled out, I mean these two and this minus sign will be absorbed because left hand side right hand side is 0. So, I will have F x minus P x, this quantity I have already find out which is phi k is equal to 0, this equation I have.

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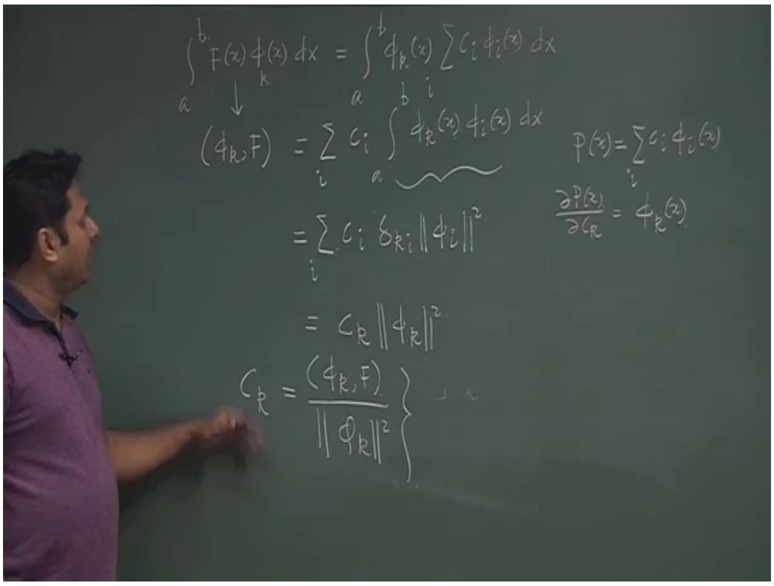

Now, if I put that one by one, then I will have something. So, then I will have a to b F x phi k x dx is equal to a to b phi k x multiplied by P x, P x I will put this so summation of i C i phi i x this. Now, left hand side is F x multiplied phi x dx a to b it is nothing but the inner product between the phi k and F, so I write it is phi k and F in inner product notation this I can write in this way no problem in this. Then what I will do I will take the treatment the logic of the treatment is exactly same one dx should be sitting here, so phi i x dx. I will take this summation sign outside C i is not there, so I can put this inside the integration inside, where this function is associated and I can write this.

So, now, this quantity you can readily understand what is the meaning of this quantity it is C i, this quantity is nothing, but delta k i mod of say i square, this. Now, if I run this summation over this quantity, then this quantity simply comes out to be C k mod of phi k square. And my C k comes out to be phi k, F. This quantity we have already figured out in our last class the same quantity, but here we show that this is the Fourier coefficient and if I want to find out the minimum error then for the minimum error thus coefficients should naturally come in this form. And if this form are there then we can say the error associated error when I expand the function in terms of the set of orthogonal function phi i, the error will be minimum if I choose my c in this way. So, this is the natural choice of C to minimize the error.

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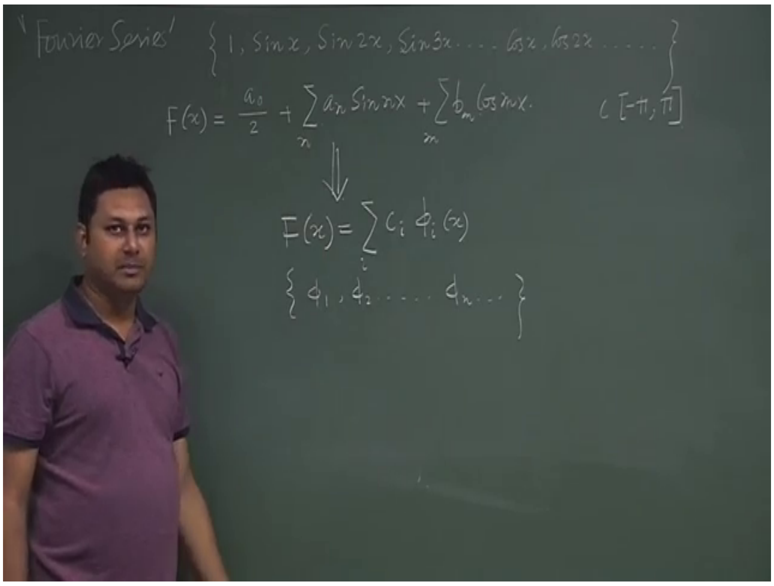

So, next is something very important after having this knowledge and all these things. Now, we are in a position to write something, which is the Fourier series. So, in Fourier series, for example, I have a set of function which is something like this and so on. This is a set of function, infinite set of function. And in some previous class, we already show that these functions are linearly independent to each other that is point one and also they are orthogonal to each other. By the way, this function is belongs to this set continuous set of function with the limit minus pi to pi. So, for this limit whatever the function is given is linearly independent to each other that is point one. Second thing is that they are also orthogonal these functions are orthogonal to each other.

Now I know a set of functions, which are orthogonal to each other orthogonal functions. So, any functions can be represented in terms of this set of functions according to whatever we have learned so far this knowledge. So, I write this in this way, the same thing. This is nothing but if you consider F x this is the same thing by writing this quantity I am writing the same thing but by splitting three part, one is associated first part is associated for this one, the function which is one. In this case, what happened my function was given as phi 1, phi 2, phi n and so on. So, in general I write in this way this is a summation form I write. This form and this forms are eventually same only thing is that i divide this sin function with another one summation and cos function with another summation with the some coefficient associated with the sin a and some coefficient associated with the cosine we call it b with some indices. Exactly like I just change the indices the same thing.

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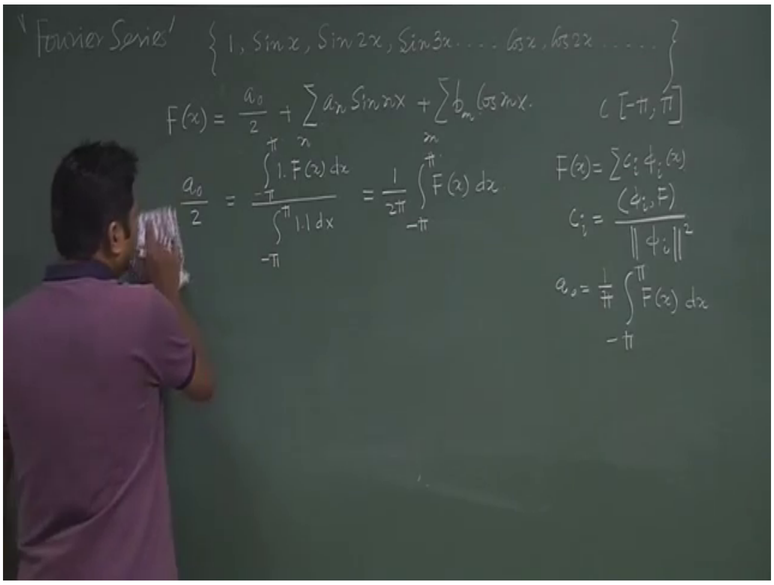

Now, if I have this now readily I know what is my coefficient this. So, what will be my there is a reason why I put two, you will check that. What will be my a 2 by 2? If you remember let me write it here once again what we find just few minutes ago that if the function is represented like this, my C i will be the inner product of this quantity divided by the norm of this, this quantity that I figure out, this is the Fourier coefficient. So, I am trying to find out this coefficient which are the Fourier coefficient. So, a 0 by 2 will be represented by this divided by this. So, this means what is the function associated with this coefficient 1 multiplied by whatever the function is given to me. When I am taking the inner product it is minus pi to pi is the definition of this divided by norm of this; that means, minus pi to pi 1 into 1 dx this. So, my a 0 will come out to be simply this quantity is how much 1 dx integration of minus pi to pi. So, it is 2 pi. So, 1 by 2 pi integration minus pi to pi this quantity here I should write a dx F of x dx fine.

So, now I will know what is my a 0 coefficient. If my function is given then I will just integrate and then if I calculate this I will find what is my a 0 by 2. So, my a 0 is I should write it here a 0 is 1 by pi integration minus pi to pi, whatever the function is given to me with dx first coefficient I calculate and which is something like this.

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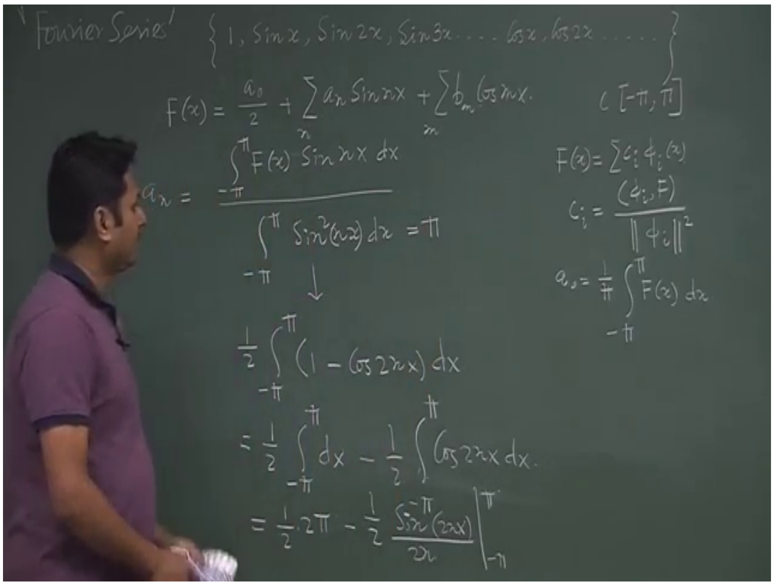

Now, what about the other coefficients again I want to find out a n coefficient. So, a n applying the same thing; that means, the function associated with A n will be linearly will be the inner product of this quantity. So, minus pi to pi whatever the function I have, I need to multiply it with sin nx dx. This is nothing but the inner product of this function with sin x divided by integration of minus pi to pi sin square x nx dx this quantity. If the function is given, I can calculate this, but this I can readily calculate. How to calculate that? So, this part only this part. So, it is if I take half here minus pi to pi then it will be two sin square of these things; that means, I can write it as 1 minus of cos 2 nx dx. 1 is sin square theta plus cos square theta minus cos square theta plus sin square theta it will be 2 sin squared theta this quantity, because there is a half sin sitting here.

So, now if I do the integration it will be half this integration is something let me do that meticulously minus half integration of minus pi to pi cos two nx dx, this I know what is the value. This is half into 2 pi. What about this quantity, this quantity is minus of half cos will be when I integrate this cos then what happened it will be sin of 2 nx divided by 2 n with minus pi to pi. Now, we know that whatever the value here we put because this is a sin, so if I put pi here in the place of x, so this will going to vanish. So, eventually what I will have that this quantity is nothing but pi contribution of the first term.

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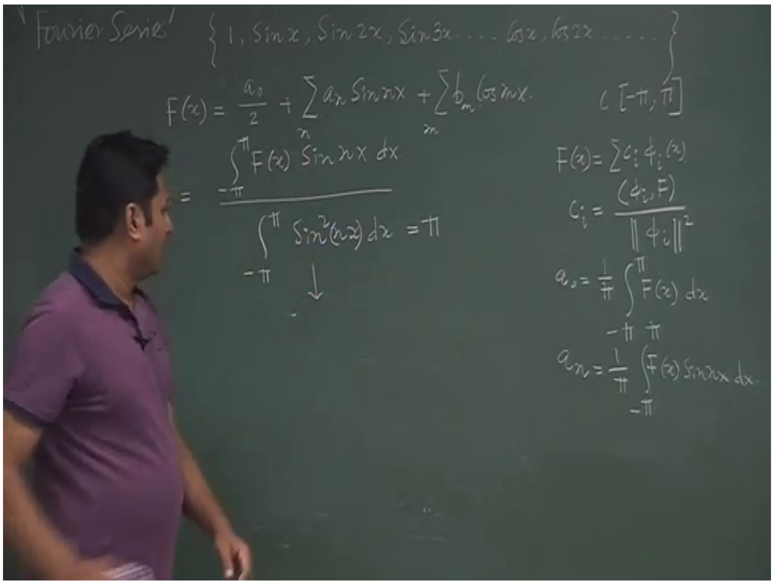

So, once I have this, so I can readily find what is my a n. So, my a n is 1 divided by pi integration minus pi to pi F x here, but here I should have F x into sin x, so F x into sin nx dx. So, also I will find what is my coefficient Fourier coefficient a n. In the similar way exactly the similar way, I can also figure out what is my b m.

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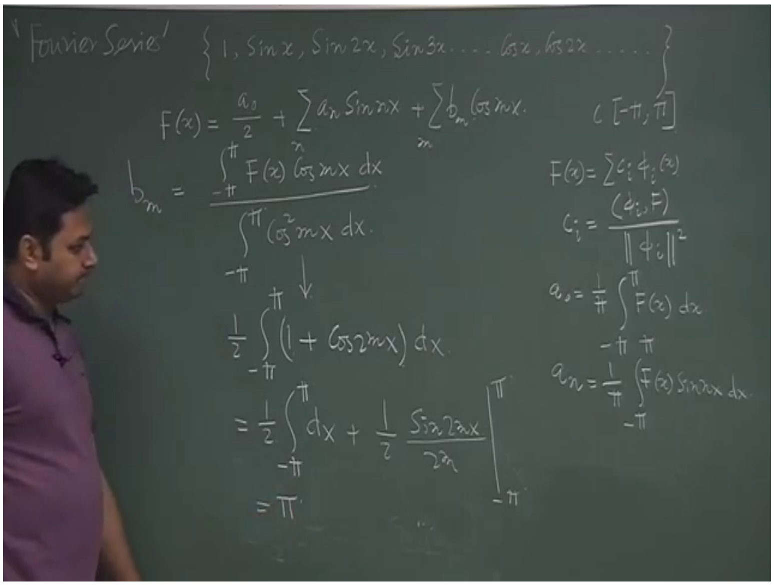

Let me do that meticulously, so that b m should be integration of minus pi to pi inner product of F x into cos a mx dx divided by norm of this quantity or norm square so that means, integration minus pi to pi cos square mx dx. Again this quantity I need to find out once I have the value of F x. So, one the value of F x is given then only I can calculate that, but here this quantity I can find out readily. This quantity exactly in the similar way if I make a half minus pi to pi then this half gives me 2 of cos square this quantity, which is 1 plus cos 2 of mx dx, this things.

Again the same procedure half integration minus pi to pi dx plus half of let me do the integration here cause integration is sin of 2 mx divided by 2 m with the limit minus pi to pi. If I do this limit, if I put this limit again in the way I find in the previous case that it is 0, here also it will be 0 because sin of pi and sin of minus pi both are zero sin of n pi whatever the coefficient here we have we should not bother. So, here we should again have something pi. So, I can write clearly let me write it here. If I can expand a function in term terms of sin and cosine series which is a orthogonal function in the range minus pi to pi it 1. If I write this, this is an expansion of a function then I can find the Fourier coefficient here.

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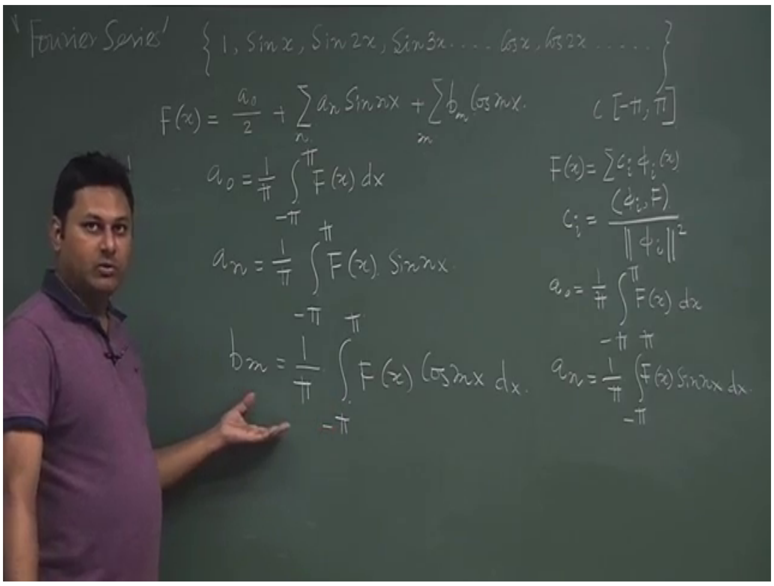

And the Fourier coefficient for this is a 0 should be 1 by pi integration minus pi to pi F of x dx; a n is 1 by pi minus pi to pi whatever the function I have multiplied by sin nx and b of m is 1 by pi minus pi to pi F x cos of mx dx. So, all the coefficient I know please note that there is I mean I put it here n, I put it here m. So, this n and m has the same I mean this they just suggest that they are the coefficient of some order. So, this is 1, 2, 3, 4, 5, 6; here also 1, 2, 3, 4, 5, 6. In some books both the same both the things are same. So, no problem with that we can also write it as b n is equal to cos of m. So, it is not going to change anything that does not matter whether you are putting, but just to make the difference that this is a series of sin series this is a series of cos series I just put this integer different in name here. But you can also put n here because I have already put b, so it is a different from a, so you can also put and do the same thing nothing will going to change . So, now, I know what is the Fourier series and how the different coefficients are related to each other, I mean these coefficients I can figure out one the function is given.

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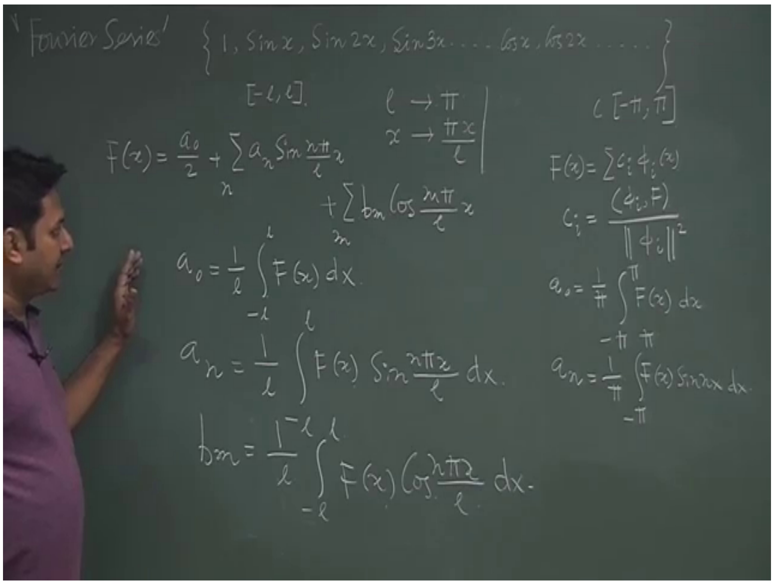

Sometimes what happened that the function is given in the range for example minus l to l instead of minus pi to pi then the Fourier series something will be same. So, the function of F x will be represented mind it here now the function is changed to the limit of the function is change to minus l to l not the pi so that means, l is tends to pi, x will be change as this quantity. So, when we put the value x here, when the limit is minus pi to pi; when it is l i need to put in terms of in place of x i need to put these things. So, F x let me write it F x is a 0 by 2 an sin n pi by l x I cannot write sin nx because my limit is minus l to l; please note that when l is x is l, then it is come to n pi. So, I just rescaling this quantity plus m b m cos m pi l multiplied by x, sometimes the limit is given in this way in some problem we will try to target few problem where this kind of limits are given.

And then the structure will be this and then a 0 will be something like this; a n will be represented by this, and b m will be represented like this. It is the same thing the previous case whatever the thing we have done, here we have doing the same thing just by changing the limit only this the argument will going to change according to the this limit.

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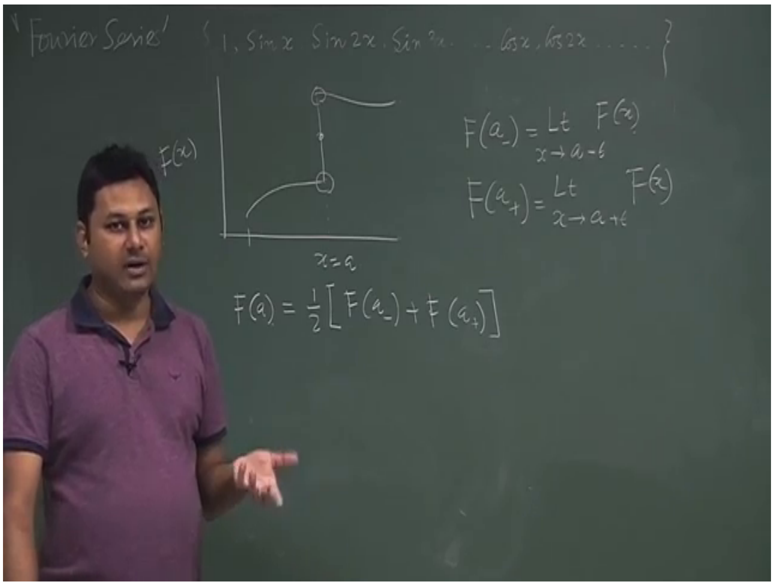

Now, after having this the knowledge of the Fourier thing today maybe I do not have that much of time to cover the problem in maybe in the next class, I will going to cover the problem. But before the conclusion of this class, I will also like to mention something for example, if there is some finite discontinuity. Say this is my F x, so I have some finite discontinuity like this. This is not infinite discontinuity some finite discontinuity some point x equal to a. So, the value of this discontinuous point will be the average of the value this.

So, this value is nothing but the left hand limit. So, this value what is the meaning of F a minus, this is limit x tends to a minus epsilon F x, this; and F a plus is equal to limit x tends to whatever the value I have. So, I will have the value here, I will have the value here and average is somewhere here. So, the value of F a will be something if there is of some finite discontinuity should come out in this form in Fourier series. So, this these things are we will going to apply or we will show naturally how these things are coming when we deal with some problem some definite problem.

So, today I like to stop here, because if I start the problem not going to finish that within this that within the time. So, next class, we will do some kind of example, we will show how by using this Fourier notation or Fourier ah coefficients I can expand any given function. So, now, one thing you should appreciate about this Fourier transformation that any function can be represented in terms of the combination of sin and cosine that is interesting because sin and cosine not a linear kind of function not a polynomial kind, this is a sinusoidal function, this is a periodically varying function all of you know about that.

So, now, how a function if it is not a sinusoid ally varying function can be represented by a sin series or cosine series is interesting. But our theorem or our theory suggests that it is really possible. And if I take sufficiently large number of terms which are feature associated with sin and cause even though they are sinusoidal by itself they can form as a combination as a linear combination, they can form a function which is not sinusoidal at all which is a straight kind of function.

So, in the next class I will show some example where you can find this kind of feature where some kind of straight line kind of function even that can be represented by the linear combination of these things. Only thing we need to find out the suitable coefficient that is a, a 0, then a n and b m this kind of coefficients will be associated with the Fourier series we need to find out how these coefficients are there and how this can be figure out with these things. So, with that let us conclude this class today. So, in the next class, we will start from here by this Fourier thing, and we will solve few problems, we will show that how this kind of problem will be solved if a function is given to you, how you make a Fourier series of this function using the knowledge that is shown in today's class.

With that, thank you and see you in the next class.