Mathematical Methods in Physics-I

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Lecture – 22
Delta Functions, Completeness

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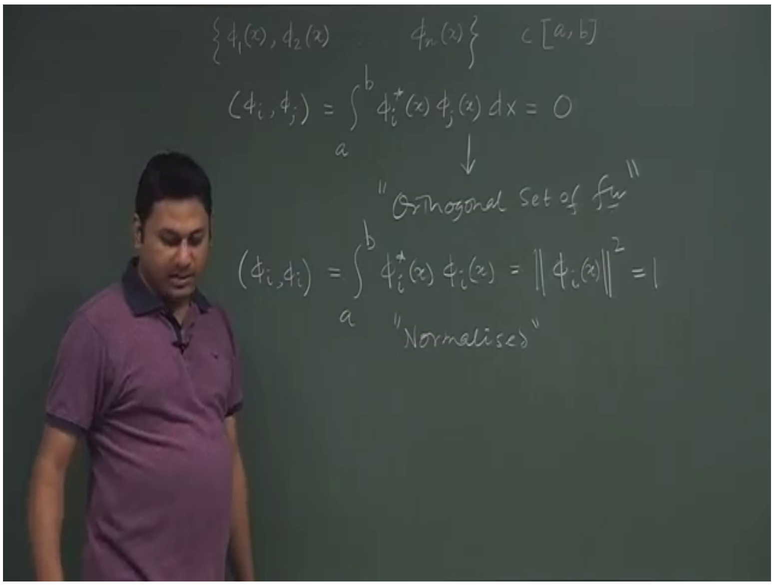

Welcome back students to our mathematical methods one course, where we are now learning about the function space and something called orthogonal functions. So, if you remember in our last class, we have suppose a set of function like this. Now, also we define something called inner product if they are continuous set of function in the interval a and b then phi i, phi j defined by this notation, which is a to b phi i star x phi j x d x is equal to 0. If this is 0, this is the inner product of this two functions i th and j th function if it is 0, then the set of functions are called this suggest me that it is orthogonal set of functions, they are orthogonal set of functions.

Now, apart from that if they have this property also, now I am taking the same function same two function within a product. And then in the right hand side according to the definition if this quantity normally we define this quantity as a norm of the function if this is 1, then we call these functions as normalized. Now, if both the condition are applied together, if they are orthogonal set of functions and also they are normalized then we can write in a single line.

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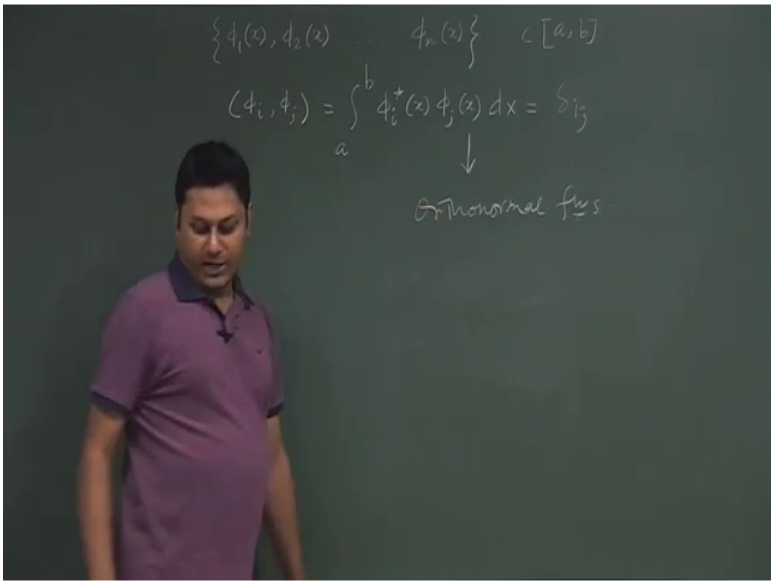

Instead of 0 by the way it is 0, when i is not equal to j this is 0 when i is not equal to j with this notation. Now, I slightly change this thing because they are now normalized, so I should write as delta i j if I write this is as delta i j that means, when i is equal to j then it is 1; and if it is not equal to i not equal to j then it is 0. So, there then we call this as orthonormal function fine.

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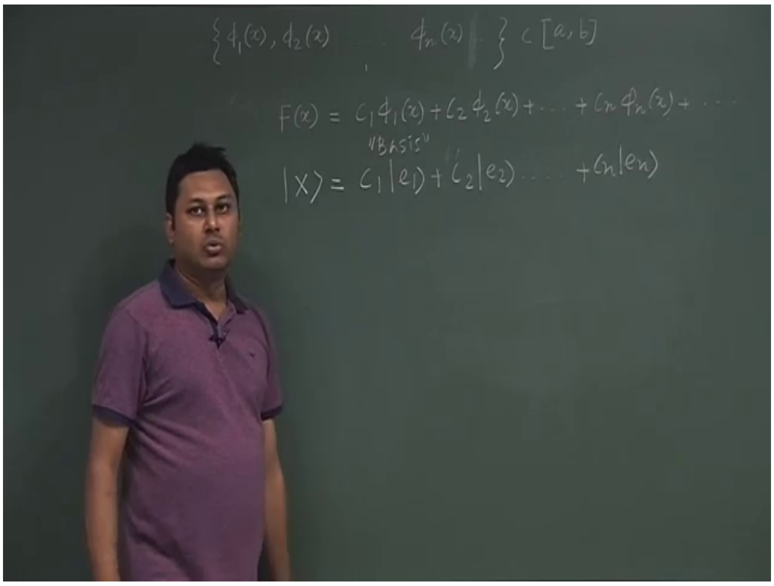

So, now we have a set of functions we considering that they are orthonormal. Then I can write any function f z if x in terms of this function like this, but you should remember here in the function space that the dimension is infinity, so I should not stop here it should go with infinity so that means, this set is in principle is infinity. So, now I can represent a particular function in terms of this phi 1 phi 2 and this things where the relationship between phi n the relationship between phi's are this if they are orthonormal then.

So, now, I can correlate these things with my vector notation any vector for example, x can be represented like this c 1 e 1 plus c 2 e 2 and c n e n, if it is a finite dimension then it is ok. But there is a possibility that vector can be infinite dimension then n tends to infinity then I can write this things and this things will go on I mean I can write more terms. But the thing is that now these and these are correlated to each other if you try to find out that a function can be represent a vector can be represented with the basis here a function can be represented with some functions other functions which can behave like a basis. So, here this can behave like a basis that is the first thing last we mentioned that. So, these are a special function which can work as a basis this one.

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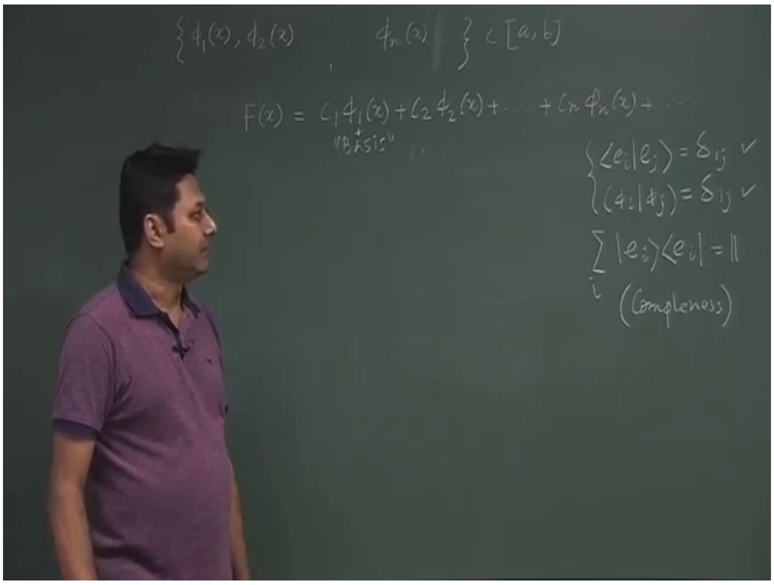

Now, if they are a basis then they are need to follow for example, in function in vector space, if I write somewhere here in the right hand side say e i e j inner product of this two is delta i j that is also true for here in this case. So, here we have the similar kind of thing phi i phi j is delta i j, but apart from that there are another important property in vector space which is this. If you remember we also derived these things and we called it as completeness or complete completeness. The basis should follow these things, so completely expand a function in terms of this basis if say I say e i e 2 e 3 are the basis. So, this is the completeness.

Now, the question is what is the counterpart of this completeness in terms of this phi 1, phi 2, phi 3. So, I know that this is inner product and this inner product both are by definition they are different, but they are correlated to each other, but what should be the correlation of these things in function space that we try to figure out which is interesting and which is important.

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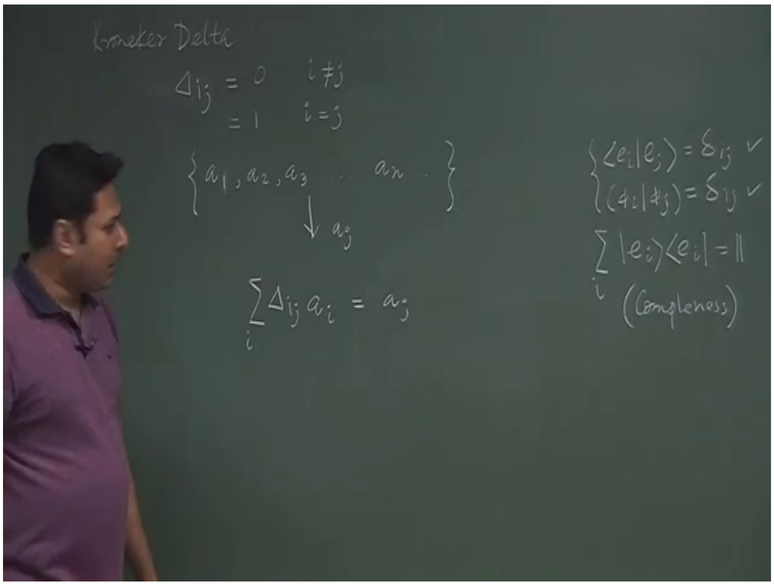

So, in order to do that we need to first we need to know what is delta function. So, let me first write something called Kronecker delta define as delta i j. So, this is the property of this delta i j is that this is equal to 0 when i is not equal to j; and equal to 1, when i is equal to j something like delta i j, but Kronecker. So, I mean what is the significance of these things suppose we have a sequence of quantities say a 1, a 2, a 3, a n and so on, a sequence of numbers we have. Now, if I want to choose a particular element say a j from this I can do one operation and this operation is I can make this quantity over i.

So, I can multiply this quantity with the varying, so here i is the time index which is running. So, taking different, different value I am getting in the right hand side I am getting different, different quantities. But please note that this delta function has a property that when i is equal to j only then this become 1; other case it will going to vanish. So, what happened that this will going to extract this a j when i equal to j then this quantity become 1, this i becomes j then eventually I am getting this quantity back. So, I can extract these through this by using Kronecker delta.

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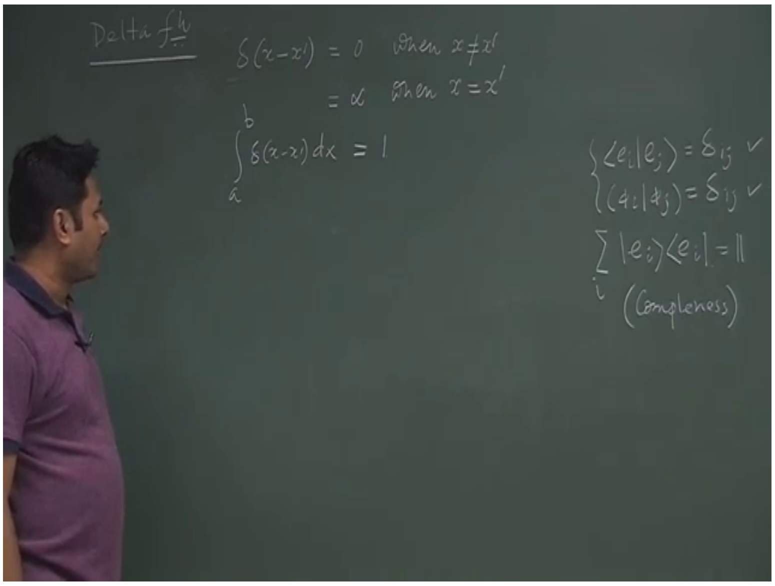

Now, for the function which is the very brief idea of these things, I can do the same thing for function which is called the delta function. So, suppose this is my delta function you know about these things. The property is this quantity is 0, when x is not equal to x prime x is varying parameter, it is sliding. And when this value is equal to x prime then this value is not equal to x prime all the value of x which is not equal to x prime is 0. And equal to infinity or tends to infinity, when x is equal to x prime. Not only that if I take integration of this function, so I will have say some interval a to b where this function is defined I should have the value one here. So, this is the property of a delta function. Now, with this knowledge of delta function, I believe all of you have some idea about the delta function. So, I am not going to be very detailed about this delta function and all these things rather I try to use this delta function to find out how this completeness is related to that quantity.

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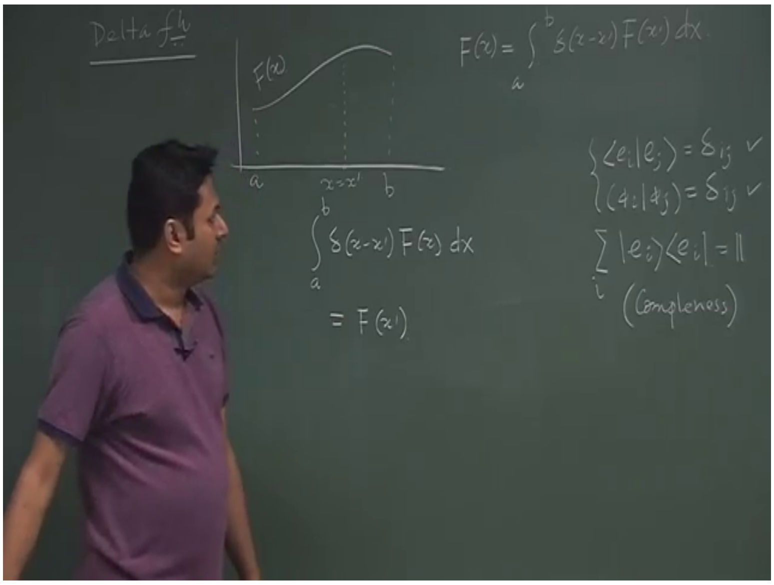

Suppose we have a function in the interval scale a to b this is my function of x, which is distributed over that; and a particular point is x equal to x prime, this is prime. So, exactly like the way Kronecker delta is used, I can use delta function here to find out the value of the function as at x prime with the same logic. So, what I will do that I will multiply this quantity over x and integrate, because this is a continuous function there is no discrete quantity there we need to make a summation over that. So, here in place of summation, I need to put the integration because this are continuous things a to b. And when I am doing that, then what I am doing that I am sliding this delta function over that x is changing, this is also changing, because this is a behave like a demy indices if you correlate that to the previous notation which is given in terms of summation sign.

Here what happened that, it will slide. When it will slide, what happened it will take only one value when x is not equal to x is equal to x prime in that case I will have this quantity in my hand. So, this is the beautiful property of the delta function which is correlated with the function F x with this I will get this. So, I can write it in a different way I can write my F x can be represented something like this. Up to this is fine. So, I know how to extract a particular value of F x at x prime with the use of delta function with the property of the delta function.

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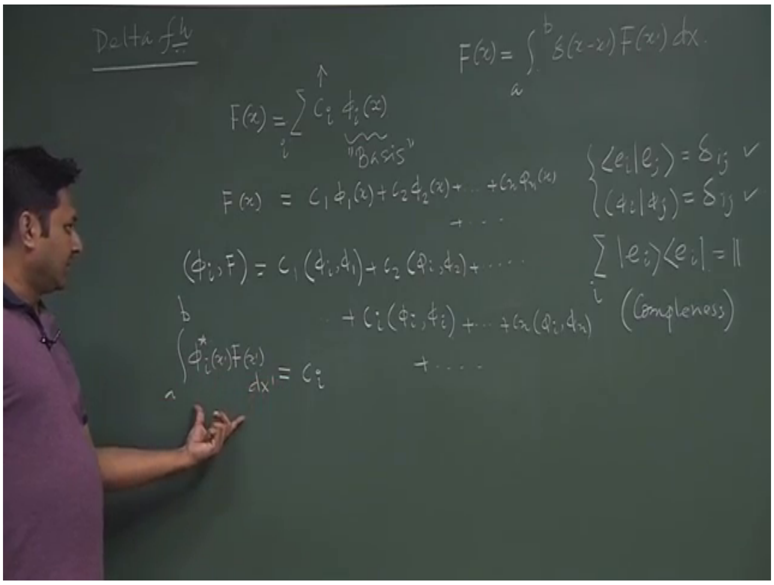

Now, I know F x can be represented completely if I able to represent this completely then I can write it as C i phi i x sum over i. So, I can write the function in terms of basis function. So, now this is the basis function and this quantity is the corresponding coefficient exactly like a write a vector, the way I write a vector here I am writing the same thing that I described. So, this is the basis function say basis and I can write it in this way. If I write in this way, then I can calculate c i by using this. How, so I can say I can write this as this way say C 1 phi 1 x plus C 2 phi 2 x, phi n x and so on. In the left hand side, I have F x in my hand. So, what I will do, I will make a inner product both the side with respect to say phi i. So, if I do that then if I make inner product with phi i, so left hand side will remain this and then the right hand side i should write something like this phi i phi 1 plus C 2 phi i phi 2 C i phi i phi i and C n phi i phi n and so on.

But the interesting thing is this quantity this quantity this quantity whatever the quantity we have with i and this things, so it eventually gives one thing that since the functions are orthogonal to each other this gives you 0. So, I know this is true here, so that means in the right hand side I will only have C i like this C i. So, if I expand this thing. So, what is the definition of the inner product? So, this side I can write at a to b phi i x prime, I need to put a star here if this function is in general a complex function then and then F x prime d x prime this is in the left hand side which is equal to c i. Because left hand side is inner product and I know what is the definition of the inner product, I just use the definition of the inner product for the function in the limit a to b for which the function is defined and I can extract what is my C i here with this. So, now I know what is my C i, so I will replace this mind it when I do the when I calculate the c i, I integrate over this thing with prime because this is a damn index. So, whatever I like, I can do. So, there is no problem by taking this as a prime because this is nothing but a number.

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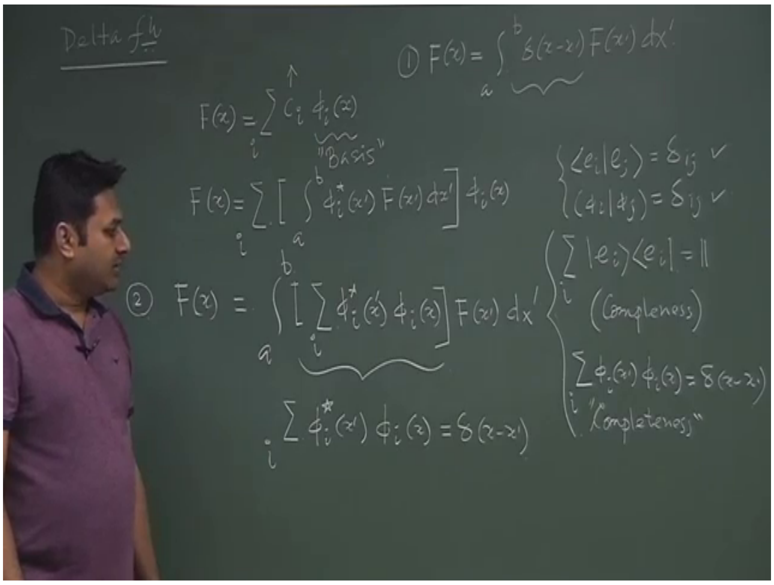

So, eventually I have F x is equal to sum over i C i I should write the complete form a to b phi i star x prime then F of x prime then d x prime bracket closed, C i I represented in this form and then multiplied by phi i x which was already there. Now, in the next step I just change this integral sign take it outside say a to b, I write it in this way a to b in the bracket I put the summation sign inside which is over i. I write two quantity phi i star x then phi i x because this is over i. So, I take all the item together and I will get these things bracket close then F of x prime d x prime. Here I should write it prime here, that time I mistake. This is my outcome.

So, now if I compare say this is my equation 2 and this is my equation 1 that I have already figured out. If I compare this and this two, then I have a relationship between delta function and my basis function the rest of the things are same. So, this quantity is equal to this quantity. If I write these things here, so basis functions if they are complete now I put a prime here I miss that. This quantity is equal to delta of x minus x prime. So, this is the counterpart whatever the thing we have in vector space by using the vector this completeness I can get the similar kind of thing here for the completeness and this is the counterpart. So, I should write that like here I have the inner product thing, I should also have completeness here. And this case the complete in the case of function the completeness should be represented by this phi i x star phi i x should be equal to delta x minus x prime this is the same thing as this we called it as completeness.

That means if I want to find out the basis functions, which is in general infinity then the thing is that this basis function should follow this orthonormal property that is for sure; and also for completeness if I want to describe a function or write a function completely then this completeness is also should be the basis function should also follow this completeness nature or this completeness property. So, with that we have an idea about the basis functions and this basis function should follow this completeness things fine.

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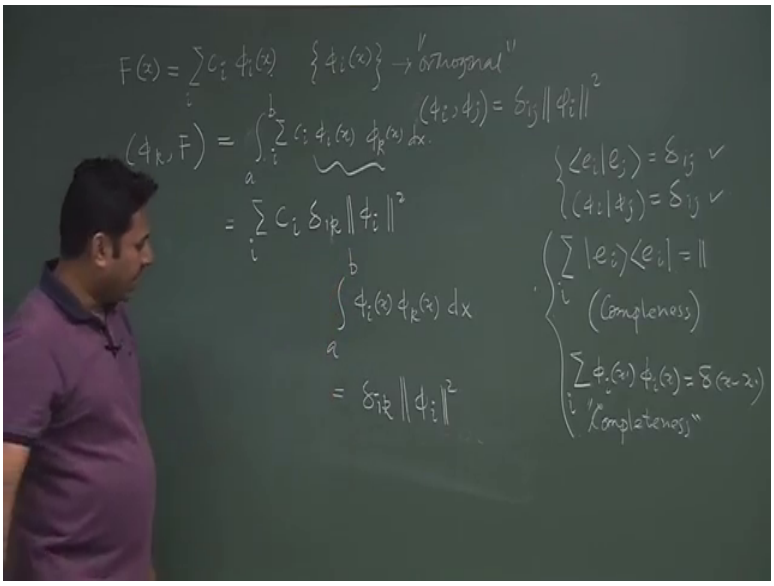

Now, we define a function in terms of this basis function. So, now the same thing I can write this as C i this, a function can be represented completely in terms of this. Now, how can I find C i, I have already shown. So, if I do the same thing for example, here phi i the functions set of functions is orthogonal. Just orthogonal I am not saying that this is normalized this is orthogonal that means they follow this rule. I can write this function in this way also when this is one then it is delta i j if this things are not normalized then when i equal to j, I will have mod of this things i or j whatever. So, this is the mod of this function. So, I write it in this way.

So, from here I can find out the C i exactly the same way just a few minutes ago I did. So, I can readily figure that my C i should be if I make a ok, let me do it in this way. I use phi k another k function, I impose this phi k over f to make a inner product and in the right hand side what you will expect it will you will expect that I will basically make the inner product a to b summation of i C i phi i x multiplied by phi k x d x. So, now, this quantity, if I do the integration I know that this is this quantity. So, readily I can understand that it will be just i C i delta i k mod of phi i square.

Because if I take the summation sign outside and integrate it over this, this quantity become this. By the way I am not taking the star here because assuming this is real quantity, but even if it is not a real quantity if it is complex then there is the procedure is exactly same, no problem with that even if I take star or non-star, you will not going to change anything these things. So, now, if I do that here then what I will get is basically delta i k mod of phi i square that already I wrote here. So, I just put this here. And now if I take the summation and this over i then when this i is equal to k, then I will have something C k pi k star.

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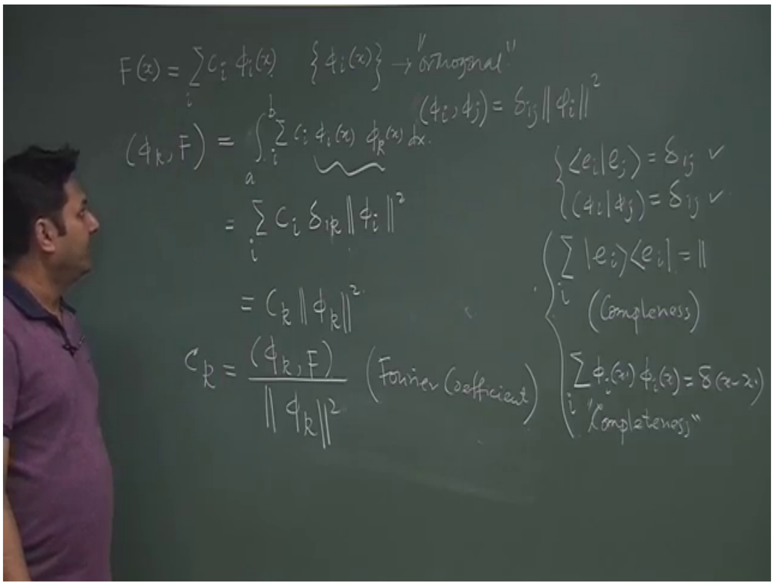

So, from here I can find this coefficient C k. So, C k is phi k F divided by phi k star; not a very new result I must say, but important thing is that you should you can write this entire function in this way when if this function of the property of orthogonality, then you can find out what is the coefficient associated with that. Exactly the same way you can find the coefficient in the vector space you can also find the coefficient here and this coefficient is normally called the Fourier coefficient.

Now, why this coefficient is look like this or we can do may be in the next class. That if I take this function is equal to this function if I say this now I know that I need to define this functions I need to decompose this function in terms of basis function. And basis function is of infinite dimension; that means, I need to go the infinite distance or infinite number of points, I need to take. So, if I take the infinite number then this value will be gradually equal to whatever the function is given to you, but if I do not take the infinite number of basic functions, then what happen right hand side will be nearly equal to this or of the order of this, but not equal to the same quantity.

So, now in order to calculate the coefficient I can calculate the coefficient in this form this particular form if they are orthogonal, if this basis are orthogonal. Like a vector space this not necessarily that the function can be represented in terms of only the orthogonal basis, any linearly independent set of basis can be used any linearly independent set of functions can be used as a basis like in the vector space. But I have some additional advantage over that, this additional advantage is that if I write the function in terms of orthogonal basis, I have the minimum error.

So, with that point I like to stop; in the next class, I will start from here and show that how the minimum because I want to have a minimum error. And if you have a minimum error then you will automatically find naturally you find that your c k will be this quantity. So, with that note let us stop the class here. So, in the next class, we start with this Fourier coefficient stuff and try to find out what is the condition to have the minimum error. When you find the minimum error then naturally it will come that my coefficient should be like this only. So, with this note, let us conclude here, see you in the next class.

Thank you.