Mathematical Methods in Physics-I
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Lecture - 21
Orthogonal Functions

So, welcome back student. So, in the last class we stopped with a very important concept by saying that a two functions phi i x and phi x suppose these 2 function, if the inner product of these 2 functions is 0 with i not equal to j, then these 2 function we say that these 2 functions are orthogonal to each other.

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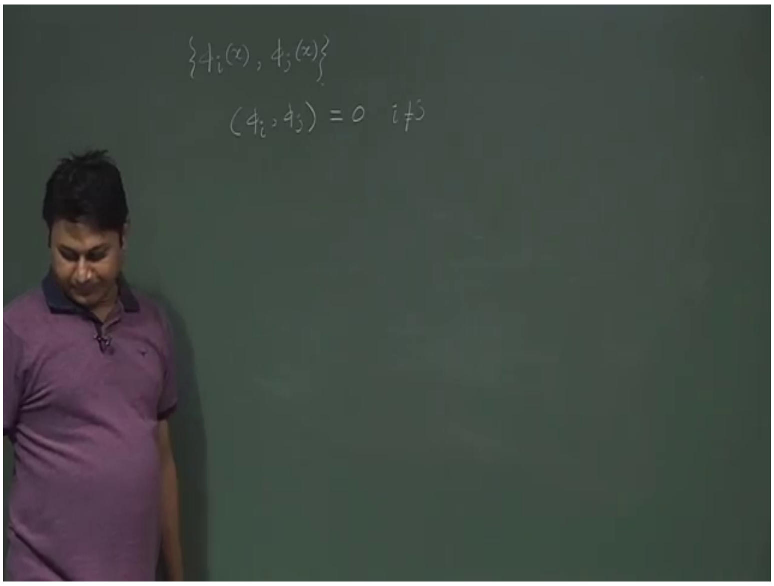

Before going to further this orthogonal thing in detail let me do one thing once again that which is related to linearly independent and dependent functions.

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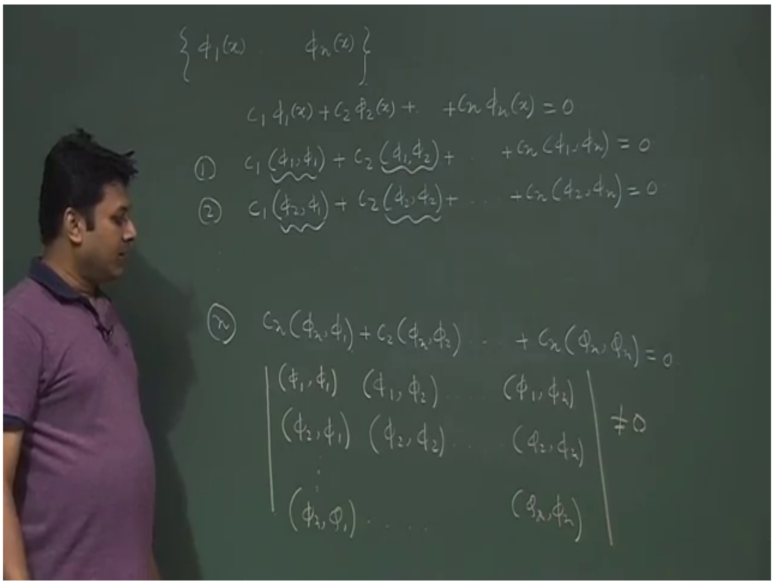

So, say this functions we have a set of functions now we know what is the meaning of inner product of a functions, so that why it is a correct place where I can put another thing to find out another recipe rather to find out whether the functions are linearly dependent or independent. Now these things normally people I mean there is a process we know, but it is a this process is little bit trickier than the previous process the Wronskian process, but still I believe it is important that you should know that this way also you can find out whether the functions are linearly dependent or independent. So, let me do that you will really understand what is the meaning of that.

So, I have c 1 phi 1 x plus c 2 phi 2 x plus c n phi n x is equal to 0 that is my equation. So, what I am doing here, I am again try to find out whether the set of this function is linearly dependent or independent, but not using the Wronskian method that we discuss in the previous class rather we are doing something different because now we know what is the meaning of inner product. So, now, what I will do that I will make another equation by taking the inner product of both the side with phi 1. So, if I do that then I will have something like this, then something like this I am making the inner product of both the side when I make inner product of both the side as if I am doing this operation both the side phi 1 I am operating over this.

So, when I am making c 1 and this quantity is nothing, but this inner product notation in functional space then finally, I have c n phi 1 phi n is equal to 0 right I have one equation this is my equation one. I also have equation 2 with c 1 now you can readily understand that in the first line if I make a inner product with phi 1 then in the next line I will do that for phi 2. So, it will be phi 2, phi 1 plus c 2 phi 2, phi 2 this and go on I will go on up to my n equation my n th equation will be c 1 phi n phi 1, plus c 2 phi n phi 2 plus c n phi n phi n. So, now, I have n equation in my hand it is the same way we use the thing we use in forming the Wronskian here.

We will had the same thing. So, these are nothing, but the numbers these are the number for non-trivial solution I will just have this matrix in my hand. So, I need to find out this quantity. So, this matrix whatever the matrix on I have, if I make the determinant not equal to 0 then I can readily say that there is a this function whatever the function I have is linearly dependent or not dependent it depends on whether it is 0 or not exactly the same way that we have in the previous case. But the determinant in the previous case was the function and their derivative, but here the inner product of the functions. So, in this case extra you need to do that you need to find out the inner product of the entire thing, but it is not that combustion in the know if you note that this quantity phi 1 and phi 2 and phi 2 and phi 1 is not different because if they are real quantity because this quantity are the same quantity.

And this particular determinant is normally defined by G this is called the Gramian like the Wronskian this is called the Gramian and this method is called the Gramian method.

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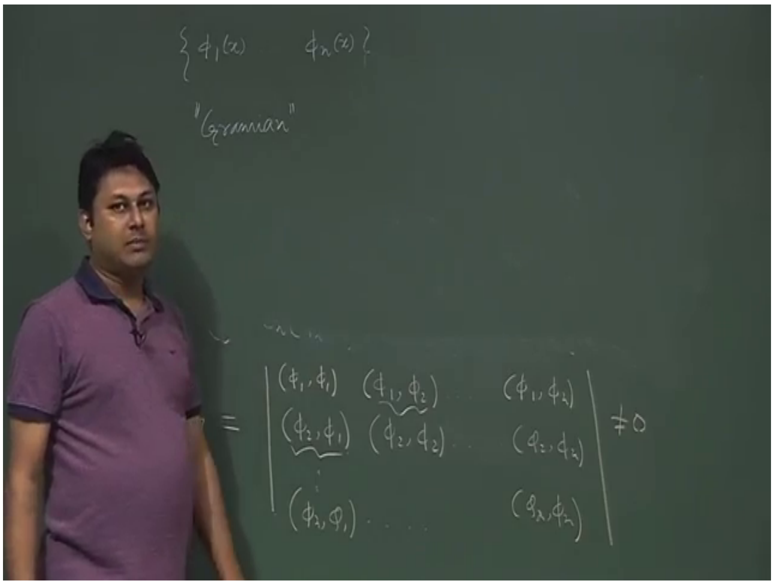

So, this is called the Gramian determinant this is nothing, but the Gramian determinant. So, I can have my function I can make a inner product series inner product of this things and then I find the derivative, then I find the determinant of these things when I find the determinant is equal to 0 or not equal to 0 readily I understand we have a non-trivial solution or the trivial solution and this things is called the Gramian method, you can use also the Gramian method to check whether I can have the linearly dependent or independent solution. If it is not equal to 0 obviously, it is linearly independent if it is zero then there is some kind of linear dependency over that, I wanted to say because in many places in many books it is not there.

But this is another way to find out this is Wronskian is not a only way to figure out whether a function is linearly dependent or not this is another way to find out whether the function is really linearly dependent or independent, but the only thing is that in Wronskian you do not need the concept of the inner product, but here you have the you need to have the concept of the inner product. So, that is why I show that after learning the inner product what is the inner product we have. So, that is the thing; now go back to our original problem which was the inner product problem.

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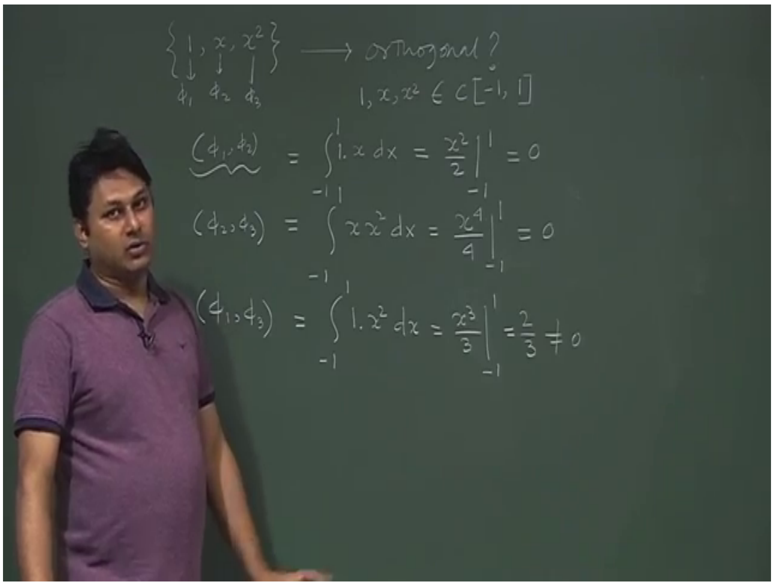

So, let me have some function one x, x square. Now try to find out whether these functions are orthogonal to each other or not these are linearly independent function that we proved in the last class. So, now, my goal is to find out whether they are orthogonal or not.

I should put a question marks here not that marks they are orthogonal the question is a set of function is given to me I want to find out whether they are orthogonal or not, but before that also I mention one important thing that they belong to the continuous function with some limits here I need to put the limit definite limit because I need to do the integration here. So, say minus 1 to 1. So, first I need to check these 2 functions say this is my phi 1, this is my phi 2 and this is my phi 3. So, phi 1 phi 2 I check whether it is 0 or not if it is 0 then these 2 are orthogonal to each other, if other things are 0 then all of them are orthogonal to each other individually we need to figure out whether they are orthogonal or not.

So, this quantity is nothing, but integration minus 1 to one phi 1 is a function here one multiplied by phi 2 which is x d x this quantity now this quantity is nothing, but x square by 2 minus 1 by 1, but before that you can already find that this is the odd function with minus 1 to 1 so; obviously, it will be 0 if you do will be having a result 0 so; that means, phi 1 and phi 2 here phi 1 is 1, and x they are orthogonal to each other with at least with this whatever the limit I have the definition of the function is within this closed limit closed interval what about phi 2 phi 3 minus 1 to 1 phi 2 is my x phi 3 is my x square d x if I do then what I am getting here again x x cube is here. So, it is odd function 2 to the power 4 by 4 minus 1 to 1 again I have 0. So, this is a odd function with limit minus 1 to 1. So, I should have a value 0 here what about phi 3 and phi 1? This things then I need to do the integration with one multiplied by x square d x.

Now, here you can see one important thing that this 2 quantity here if I multiply. So, we have a even function in our hand. So, it will be x cubed divided by 3 minus 1 to 1 which I suppose is 2 by 3, but not equal to 0; that means, even though this functions are orthogonal these and these are orthogonal these and these are orthogonal, but these and these are not orthogonal. So, according to the definition according to the definition, it is not that these and these are orthogonal and these and these are orthogonal then this has to be orthogonal to these 2 it is not necessarily in vector may be that is possible, but in function theory it is not the case. So, with this kind of treatment you can find out a set of given set of functions are orthogonal to each other or not let me find out another set of function.

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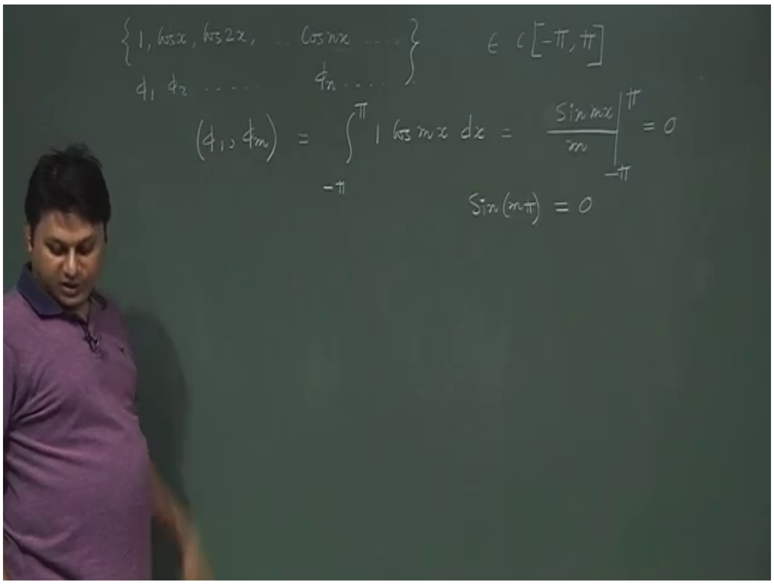

So, say 1 cos x, cos 2 x, cos n x and so on I can extend this function up to infinity, I can extend this set of function up to infinity I can anyway I can do up to three, but in general you can extend this functions.

So, now I need to find out whether. So, this is my phi 1, this is my phi 2 if I note in terms of phi this is my phi n and so on. So, first I need to find out whether any general component if I take say phi 1 and general component phi n this quantity by the way this functions belong to the continuous functions with minus pi to pi limit. So, now, I have this. So, again I need to go in I need to check individually whether they are orthogonal to each other or not, but here what I am doing since this function is cos I take the general function and then check for any value of m or n this functions are orthogonal or not.

So, first my definition minus pi to pi and then phi 1 is 1, I have cos m x d x this is the function I have what is the value of this cos m x the integration is sin m x divided by m with minus pi to pi. Now we know that sin of m pi is always 0 where m is an integer here 1 2 3 4 5 6 all these things are integer. So, if I put this here. So, if I put m as integer then this quantity is nothing, but 0; that means, phi 1 and phi m are orthogonal to each other whatever the m you take it will always remain orthogonal. So, I will invariably I can check take any functions that will be orthogonal to each other now the other things because this is the one so that is why I need to do in this way.

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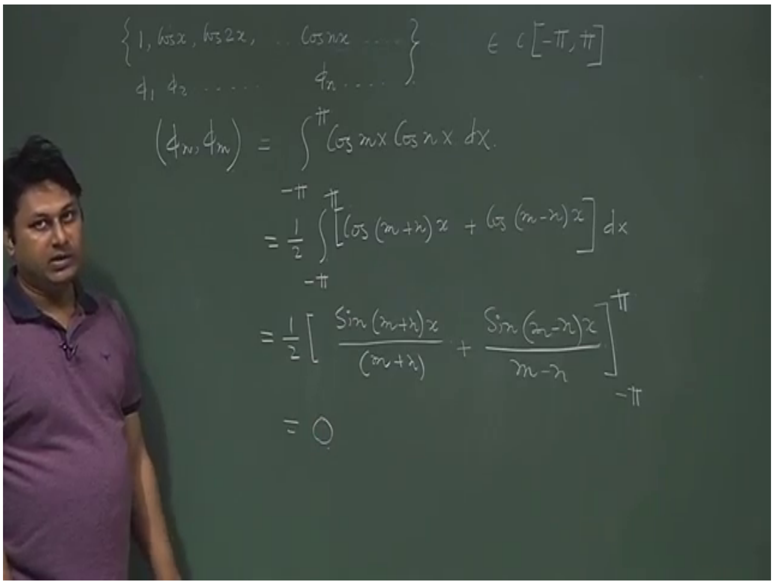

Now, I need to do a general thing phi m phi n phi m is where m and n not not equal to one this is something defined value. So, my integration will be something like that cos of m x multiplied by cos of n x and these things. Now we know how to do this integration I if I take half here minus pi to pi here, then 2 cos this things can be split it 2 cos m plus n x plus cos m minus n x like this. Now, it is easier for us to integrate this quantity if I integrate will be sin of m plus n over m plus n and that quantity will be again sin of m minus n with minus pi to pi with the limit this. When you make the limit here like this again the same thing the sin of m plus n which is again an integer multiplied by pi. So, this quantity will go in to vanish and this quantity will go in to vanish. So, overall I will have a 0.

So; that means, I have a set of functions cos x cos 2 x and all this things when I make the inner product of the entire functions, I find that any m or m value they are 0 they are always 0 so; that means, this is really a set of orthogonal functions.

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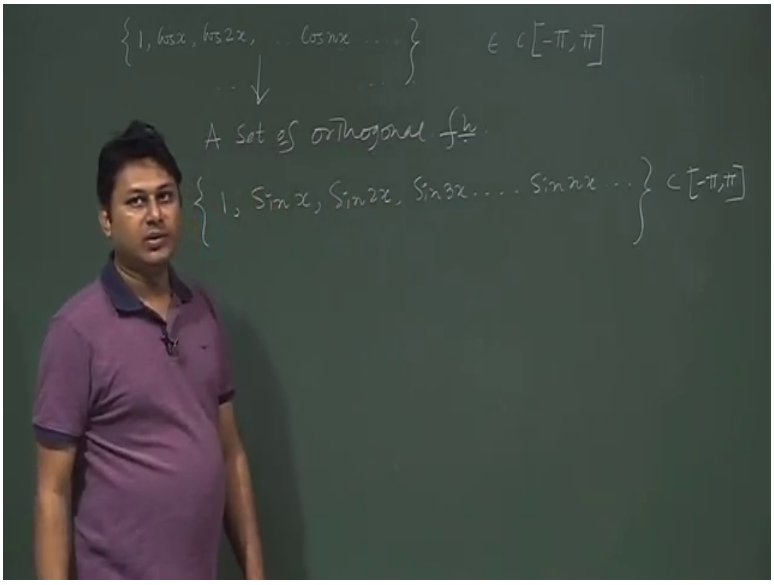

So, this is indeed a set of orthogonal functions this is a set of orthogonal functions, with this note I can give you another homework since this is done. So, please check I am giving one another set of functions like this, please check whether this is a set of orthogonal functions or not I check that this is a set of orthogonal function, but this is a homework for you people who are taking this courses please sit down and do the same calculation and find out whether they are orthogonal set of function or not. Again they are in same close interval they are continuous in same closed interval.

So, that is valid here also well now I know the state of functions are orthogonal to each other, but one thing is important that whether they are normalized or not that is important whether they are normalized or not that means.

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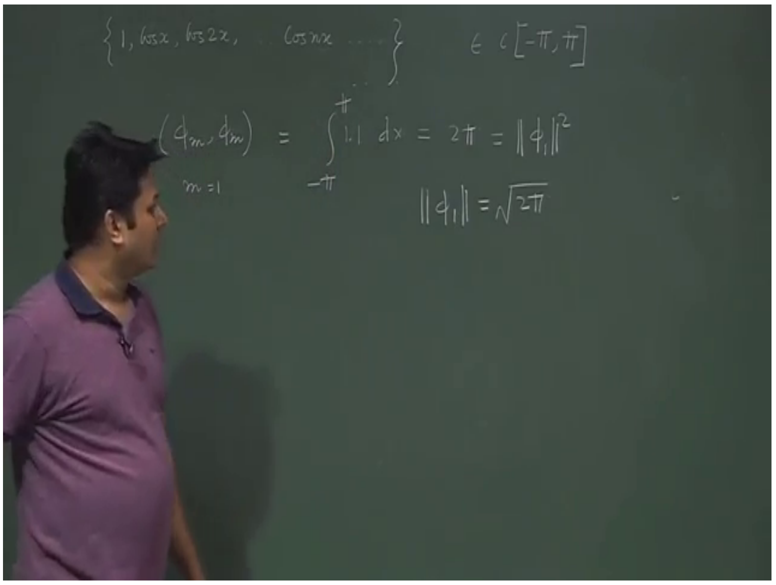

If I take this quantity whether it is one or not, so in order to check that again I need to do the same thing, but now I will take any phi m to phi m. So, when m is 1 that means, I will do the integration this is for m is 1, that is 1 in to 1 minus pi to pi d x I am making the inner product for the same function so that I can find out the norm of this one. So, if I do then I will find it is 2 pi. This is x. x is pi 2 minus pi then it will be 2 pi this 2 pi is basically the value of phi 1 square. So, what is the norm of the first function is root over of 2 pi root over of 2 pi is the norm of this function well I should note it here somewhere.

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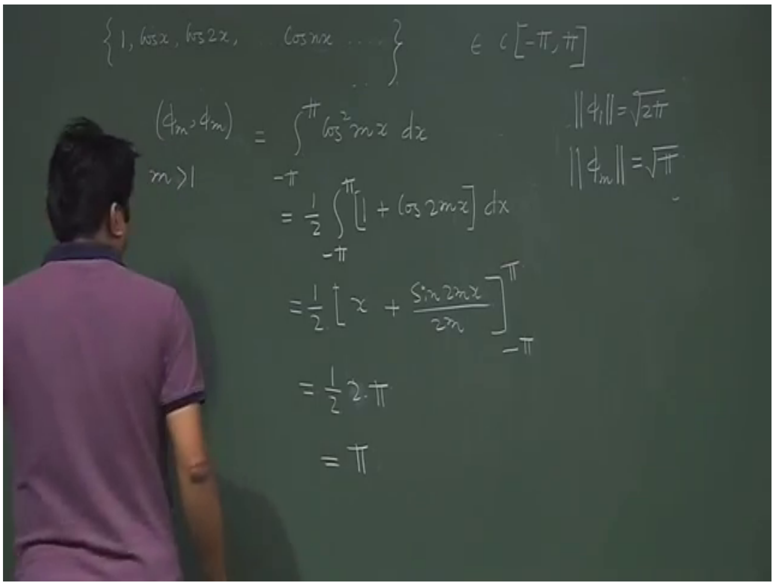

That my phi 1 was root over of 2 pi next I will do for other cases. So, I will do the same quantity I will do the same calculations, but for here m is not equal to 1 greater than 1 so; that means, when I greater than one means I will take down the cos and all these things so; that means, it is minus pi to pi cos square m x d x any m whatever the m you can consider this is the case. Now I can do in this way I can take half and make it 2 of cos square m x and 2 of cos square m x, I can write it as 1 minus 1 plus rather cos of 2 m x d x. The first case you will have half the integration will be simply x in the second case the integration will be sin of 2 m x divided by 2 m with the limit upper limit and lower limit minus pi to pi, if I do that then I will have half x will be simply 2 of pi, but this quantity when I put the limit will going to vanish.

So, finally, I will have something equal to pi this quantity, invariably for all the values of m so; that means, the norm of other quantity is root over of pi. Once we have the norm then I can readily say that my normalization factor is in my hand. So, what I will do that I will just change this function.

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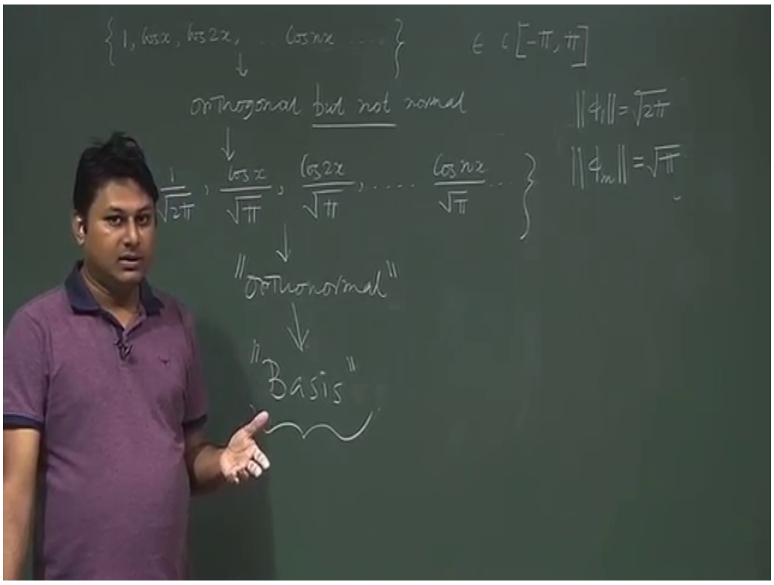

So, initially if you remember this function was orthogonal, but not normal because if I take the norm of these things I will find this value is not equal to 1, now what I will do that because the norm now I find I can change the set of function like this.

My first function in state of 1, I put one by root over of 2 pi in state of cos I put cos of root over of pi and then cos 2 pi and so on finally, we have cos n x root over of pi and so on. So, I modified this function by putting some constant term and when I put this constant term, now this set of function become ortho normal; that means, it is orthogonal to each other that is 0.1, second point is it is also normalized so; that means, this function can behave like a basis which is very very important. So, now, they are potential basis they are potential basis.

So, in the vector space if you remember there the concept of the basis is coming in exactly in the same way and what is the meaning of the basis the basis of meaning of the basis in the vector space was if I have a set of function in which I call basis, which can expand or through which I can write any vector then this function whatever the functions we whatever the vectors we have called the basis so; that means, any vector can be represented in terms of the basis. In the similar way in the function space I can potentially write any function in terms of the basis functions. So, let me generalize this before concluding this class.

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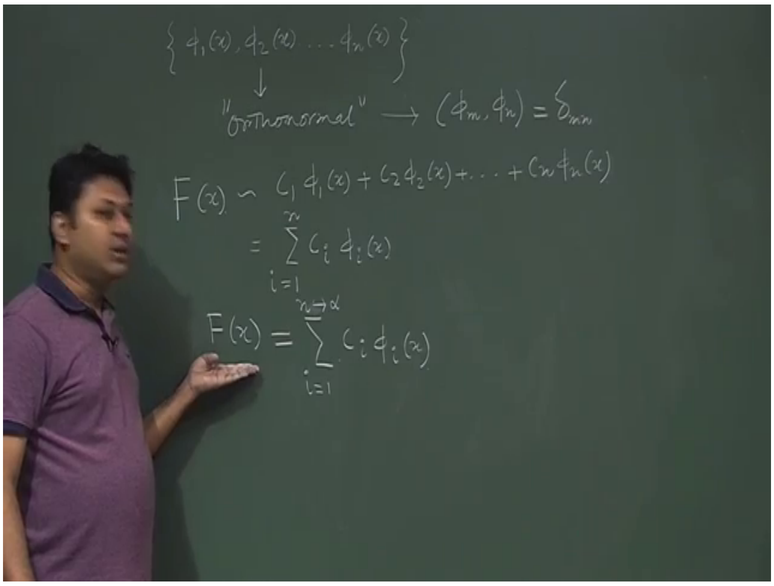

So; that means, I have a set of functions phi i x, phi 2 x, phi n x like this say finite set of function for the time being this is an in very important thing because this is this should not be finite this should be infinite, but let us consider I am taking a finite set.

This functions are orthonormal when I am saying this functions are orthonormal that essentially means phi m phi n is equal to delta m n these things is the inner product notation. So, inner product of any function with m and n is delta m n, next I am saying any function f x potentially represent it as c 1 phi 1 x plus c 2 phi 2 x plus c n phi n x this form or in more compact way c i phi i x, i running from 1 to n please remember that this equal sign I should not put here like this way because I am taking only the finite set of functions, here the dimension of this function is dimension of the function space is infinity. If the dimension of function is infinity potentially I should have infinite number of basis functions.

So, this function if I able to write here. So, basis function also go to infinity if I say I will like to write this function up to n number of functions basis functions then this f x will never be equal to the right hand side whatever. So, this is a very important thing you should remember in principle if b tends to equal to will be equal to this when n tends to this n tends to infinity; that means, I will try to find out what is my f x by taking 3 or 4 terms; obviously, the right hand side will never be represent the original function.

Now, I gradually increasing my number of functions or number of basis, if I do that then I will gradually go to the original function whatever the function I have. So, that is the gradual process if I go to more and taking more and more and more and more function, I will go more and more and more and more towards the original function that is F x. So, there is a error; obviously, there is error associated with that. So, this is a very interesting problem and it is related to the Fourier series. So, in the next class gradually we start from here that how a function can be represented in terms of the basis function and the number of basis if I take the limited number of basis the functions, which is represented by the basis functions should not be same, but if I take that to up infinity then we will find that this these things gradually give you the original functions.

So, that is nothing, but the Fourier series the well-known Fourier series that you know that any function can be represented in terms of the basis function, here the basis function whatever the basis function we will take is nothing, but the sin and cos, but you can take the different kind of basis function also and in that case you will get the different polynomials. Some polynomials are there which can also be a potential basis function we will also going to learn that.

So, with this note let us conclude that in the next class. So, today’s class very important thing we find that what is the basis of the function what is the meaning of basis of a function and next thing is that how if function can be expanded in terms of basis and this is basically the building block of the Fourier series that we will cover in the next class. So, with that, see you in the next class.

Thank you.