Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 02
Vector Space

So, welcome back to the next class. In the first class, we mainly discussed about set binary operation and then, we discussed something about a group which is a very important concept. Now, we extend this concept and let me define another thing which is ring.

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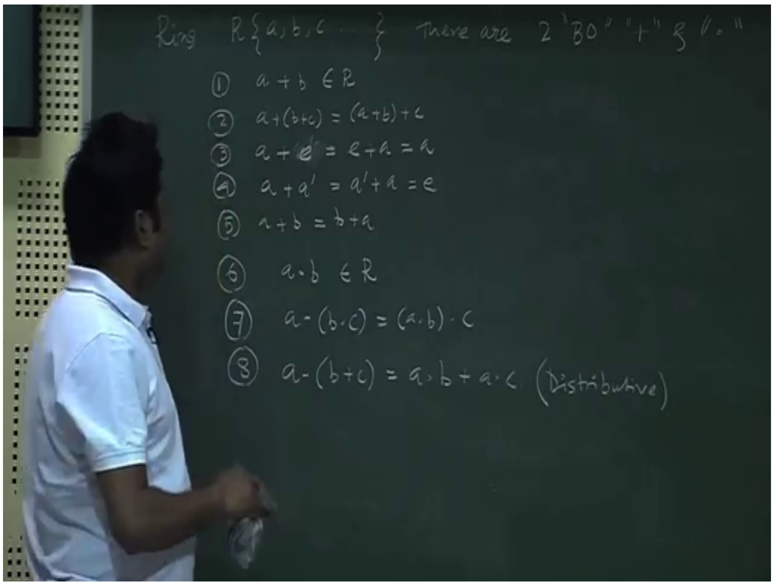

Now, ring is slightly different than group. So, let me first say R is a set of element a b c and all these things and here instead one binary operation, we should have two binary operation. There are two binary operation, I say binary operation BO because I need to write this binary operation word several time, so cut it short.

So, it is BO which is plus and dot. Now, if R is a set of all these elements with these two binary operation, then I should start writing the properties. So, what are the properties to form this set R a ring. So, the properties are a plus b belongs to R standard 2 a plus b plus c is equal to a plus b plus c associative. Third one is a plus a minus a say let us write it as a prime say e better to write it e, say e plus a is equal to a. So, this identity element is there I called it e 4 a plus a star is equal to a star plus a is equal to e 5 a plus b is equal to b plus a.

So, essentially what I am doing that I am writing all the properties that a group can form. So, R essentially forming a group with this plus operation, this plus binary operation, they are forming a group. What about the other thing because the dot is also there. So, 6a star b belongs to R quite trivial because a star b is another operation. Star is another operation. I am taking two elements. So, these two elements when they are integrated with this operator, this binary operation star I will get another quantity a star a dot b which also belongs to the set R a star b star c is a star b star c. In star, they are following the associative rules. So, first is a closer property that is followed by this and second thing is associative.

So, they are associative 8. This is very important. Why? This is important because the first time I am applying these two binary operation together up to 7, I operate only one operation here and from here, I will operate another operation star dot. Now, I am associating this plus and dot together and now if I do that, then I will say this is the property that it should hold. So, this property also has a name. It is called distributive property. So, they are distributive. So, now if a set R is following all these rules, then they are called ring. We need to know this concept of ring very clearly because ring is a very important concept. As I mentioned this is where I am associating two different operators. One is star and another is plus. After the concept of ring, I will again try to form something different which is called field.

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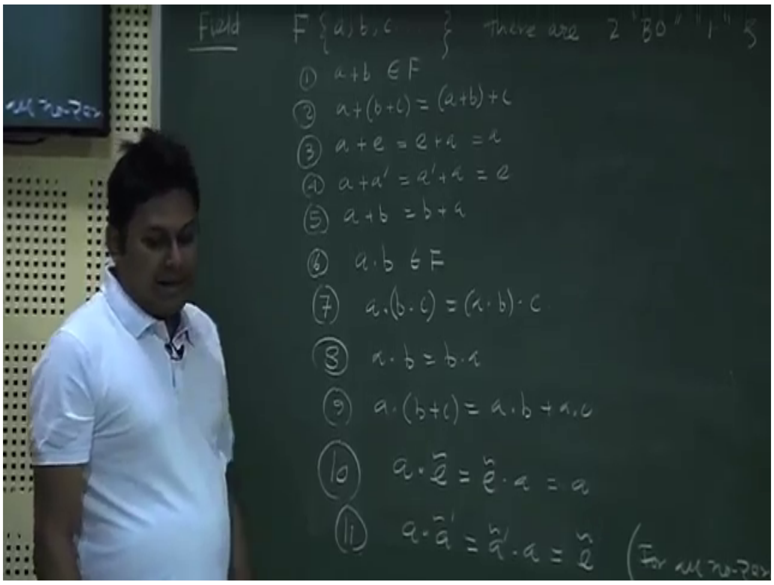

I write F. So, here let us consider F is a set of element a b c and so on which is a different set. Also, two binary operations are associated with this set like the ring. Now, if they follow this rule, I should write the same thing. Once again let me first write and then, I am going to explain what exactly is going on here up to this. This is exactly like ring. So, all the properties that ring follow, the field is also following the same property when I am applying the plus or addition operator over the elements, then the rest of that closer property, associative property.

These two properties also hold, they are commutative also and also, I should say they are distributive. So, all the properties that a ring followed here up to here at least they are following the same things. So, nothing new; only thing is that in order to form a field, I have this thing. Also, I have in which star operation I have an element say e here. If I write e, should not write it here e. So, it should be different say e tilde e tilde i. Just make sure that whatever the identity element I have in plus operation; it is not necessarily the same thing, in same element for this dot operation.

So, in order to distinguish this and this, I put it tilde over that, but this is a different element. You should know that you should recognize that this is a different element. It may not be the same. So, these things gives me, these things return back and finally, I will have something which is again a prime tilde a prime tilde because a prime I have already used here as an inverse element. So, inverse element here is for dot operation.

You should know that e tilde where these elements have to be. So, for we need to make a comment on that because this is a very important property that even under multiplication, I can have inverse. This is only possible for all non zero elements in the field if there is some zero elements.

So, normally our normal perception is that whenever I will try to multiply some number with the inverse of that number, so that I can write back 1 here. Normally 1 is a multiplicative identity if I multiply a with 1, then I will write back a. So, if I want to write back 1 here, I need to multiply something which is 1 by a. So, a tilde prime is nothing, but 1 prime 1 by a. In normal sense if that is the case, then if a is 0, then never have this inverse. So, that is why specially I need to put a condition that for all non zero elements, this is true. For all zero element, this is true. So, if all these properties hold, then the elements are forming fields. Now, let us go back to the example it is better. So, all these theoretical things is but unless we put some kind of example, then it is difficult. However, the examples are quite trivial and quite simple.

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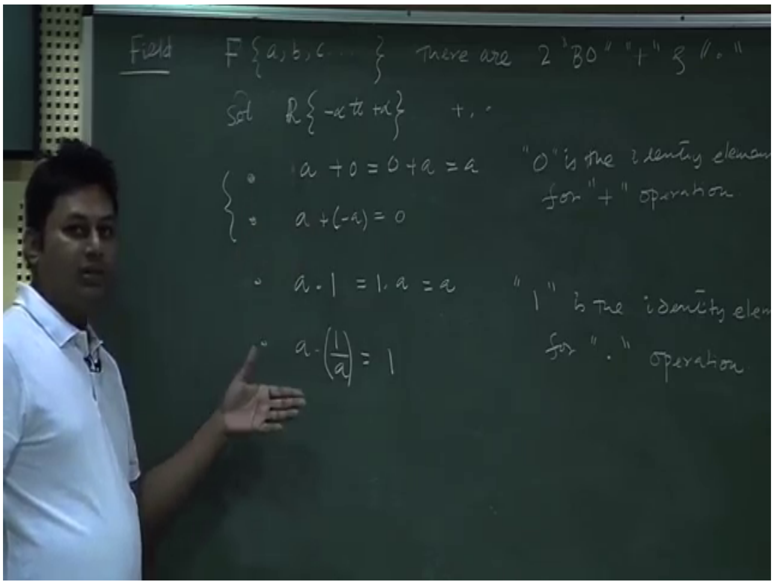

So, let us say my set is again the real number. All the real number minus infinity to plus infinity this. So, now if this is my set and my operation is plus and dot, where plus is my additive addition, standard addition and dot is multiplication, then I can check that all these properties are following. So, let me start with this existence of identity and existence of this inverse which are the critical property that a set has to follow to be a field because these are things, it is difficult to find other things comes naturally. So, let us concentrate on that.

So, it is quite trivial for R when I try to find out the identity element. It is nothing, but 0. So, whatever the quantity say a plus 0 is equal to 0 plus a is a. So, here 0 is the identity element for plus operation, fine. Now, what about the inverse? It is quite trivial that if I take a and if I take minus a which is again a quantity real number, whatever a you think 1 2 3 4 5 6 according to your choice, then I will have this. So, this quantity is minus a is a inverse of that particular element. So, these two follows what about the multiplicative identity and multiplicative inverse. So, as I mentioned a star 1 is equal to 1 star a is equal to a.

So, 1 is here identity element for dot or multiplication operation dot operation. Again I can have this quantity which return me back one. So, I can have one element 1 by a. If I multiply these two a, then I return back my identity element. So, this is the inverse. So, 1 by a will be inverse this. Again note that if a is 0, then this is not going to happen, but when I define field, I say for all non zero element this is true. So, since it is true for all non zero element, I simply say that this a is non zero. Since a is non zero, this is true. If it is zero, then it is difficult, but my definition also exclude this point zero. No problem with that.

So, we find that real numbers set of real number can form a field. This is very important concept.

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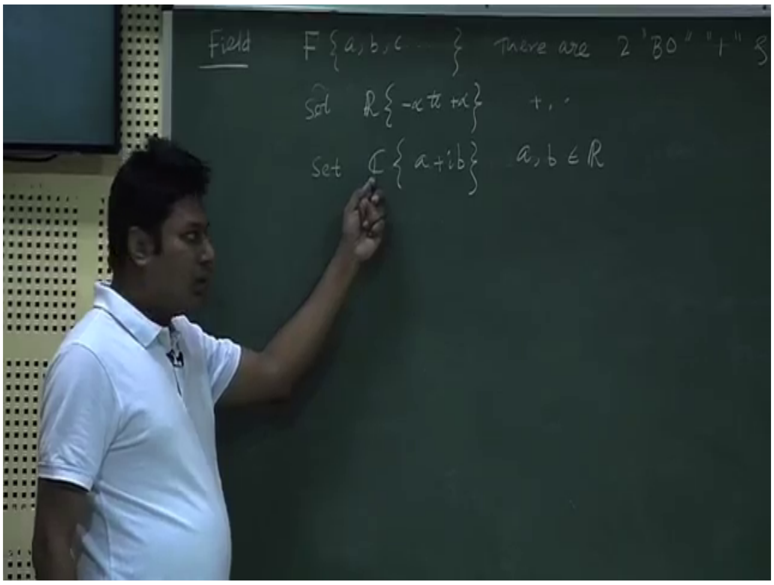

Also, if I do then we will find that set of all complex number, where a b belongs to real numbers set of all complex number, this is a set of all complex numbers, they can also form a field. So, real number can form a field and the same way I need to check all I mean all the properties, so that they can form a field, all this stain and element properties and if you do that, you will find that c which is the set of complex number can also form a field. This is also we need to know that the real number can form a field as well as the complex number can also form a field.

Now, after having the knowledge of this field, we are now very close to something very important which is a vector space.

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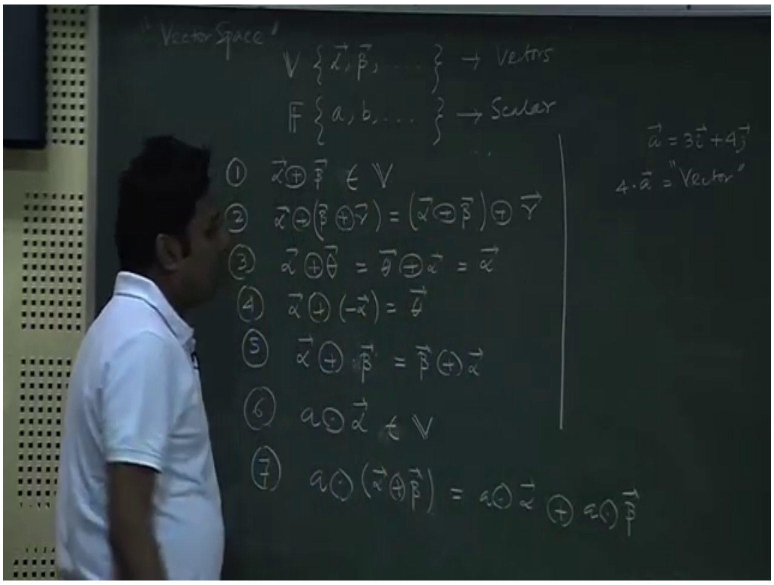

So, now how the vector space will form? So, let us check. So, now vector space so far whatever we have learnt is relative to set, then binary operation, then ring, then field. All of these things has some properties. Now, I am saying that after having the knowledge of all these things, now I am going to construct something called vector space and let us try to find out how to define a vector space using the concept so far we have learnt. So, say v is a set of element alpha beta. I call this element as a vector element. That is why I put a vector sign. This please do not confuse with a normal vector. This can be anything.

We will show an example that this vector space not only for the standard vector we know which has a directional magnitude, but the matrix 2 by 2 matrix or n by n matrix also form a vector space which are following this definition can be a vector space. This is not necessarily that all the vectors we know is forming the vector space.

Obviously, this is the element that will form a vector space and we will deal with that, but you should note that it is not always true that only these elements form the vector space. So, these are the elements we call as vectors and I also have a field with the element a b c and all these things to distinguish two elements. One is alpha, beta and another is a b c. So, just to distinguish that these two are different set, normally it is called the scalar. After defining these two, I need to define two operations which is very important. One is plus distinct which is an operation and another is dot distinct. I put a sign over that plus and then, a circle over that and dot circle over that to distinguish the fact that this is not a normal plus operation or a normal dot operation because in earlier classes, we show that vector addition is also defined by plus, but this vector addition plus sign whatever we are using as an addition is not critically the normal addition. We know it is not 2 plus 5 is equal to 7. It is something different. Two vectors when are added, they have a rule. So, this in order to distinguish these two things, I need to put a different sign instead of using plus. So, these rules suggest that if I take a element from vector and if I operate these things to another vector, it will give me a vector. This is the way to write the rule.

What is the rule of this plus? If I take a plus, plus rule is applied on this vector set of vectors. So, I apply this plus over these elements and as a result whatever the element I will get that should be vector itself. In the similar way, dot is something that I will take one element F and then, I will take another element from this vector. One element from field, one element from this vector and then, whatever the result I will get is a vector. This is a rule. So, that essentially means if I operate this star with this and this as a result what I am getting is a vector. This is the rule. You should remember.

Now, if V and F are following these rules with this operation plus and dot, then I should say that they also follow the others. We need to check what is going on for other rules. So, first rule is say alpha plus beta belongs to V which is trivial that alpha is a vector element, beta is a vector element. I am associating these two vector elements with my plus operation. According to the rule that if these two elements are from vector, whatever the result I will get should be a vector 2 alpha plus beta plus gamma is equal to alpha plus beta, the associative rule that we already defined.

So, 3 is alpha star theta. Theta is something, some element here. If I add these two things, I will have theta. I will return back alpha 4, I will return back theta. So, all that thing it is not a very new thing. Already we have a prior idea about what is going on here. So, I am just writing the same thing once again, but you should remember that this plus sign is now changed, one thing and second thing, one dot operation is also there which I need to associate it with this and this. That I will do within a minute and finally, all these five rules are followed. So, now what about the star thing? So, now I will associate this thing which is important. Now, I am associating one element from field and another element from vector.

So, now this operation is important because I am associating one element from field which I called as scalar with a vector and this is the rule that when I operate these two things, the result will be something which belongs to vector. So, this quantity is eventually in vector. So, a star alpha give me something gamma. So, gamma is an element of vector 7. Also, it is something if I associate a with some of these two things, so distributive things will also follow. So, I should write this plus this. So, what is the meaning of that? That two elements are there in the vector space a in this set of vectors, I should not call it vector space right now.

Now, if I add these two things, some element will be there which is also according to this rule, the element of vector now I associate with that vector with this sign and when I do that, this comes to be distributive. So, I will have individually one element and another element. Both these elements will be vector element and since both these elements will be vector element, they should be associated with this plus sign with a circle. That means, this is the vector quantity and according to our rule, those two vector quantity when they are associated, they are associated with this plus. So, that means this distributive things are there. So, now if these few features are followed, then we can say safely that is a very important statement I am going to make.

So, you can check that this plus under plus all these things are making and forming a group. So, vector itself is forming a group with this plus operation and also, some distributive law are associated with this field which we called the scalar. So, some distributive law is associated not only that when I associate one quantity from field to another quantity vector, the result thing become vector which seems to be quite trivial because for example a vector, normal vector say let me give a very simple example say 3 i plus 4 j. This is the vector. Normally we use in physics. This kind of notation are always used we will come to this point later, but let us consider this is a vector. Now, if I multiply any quantity, scalar quantity real number say 4 multiplied by a, this is quite I mean simple.

You can say that this is quite trivial that they will multiply and I will have another quantity which is vector. So, this quantity is nothing, but a vector with some values. We can calculate the values also, but this as I mentioned it is may not true for all the vector elements. It is this vector is not a normal vector. This is a standard vector which direction and amplitude it may not be true for all the cases. So, in that case this property is important. So, let me give you some example, but before that I need to make a very important comment.

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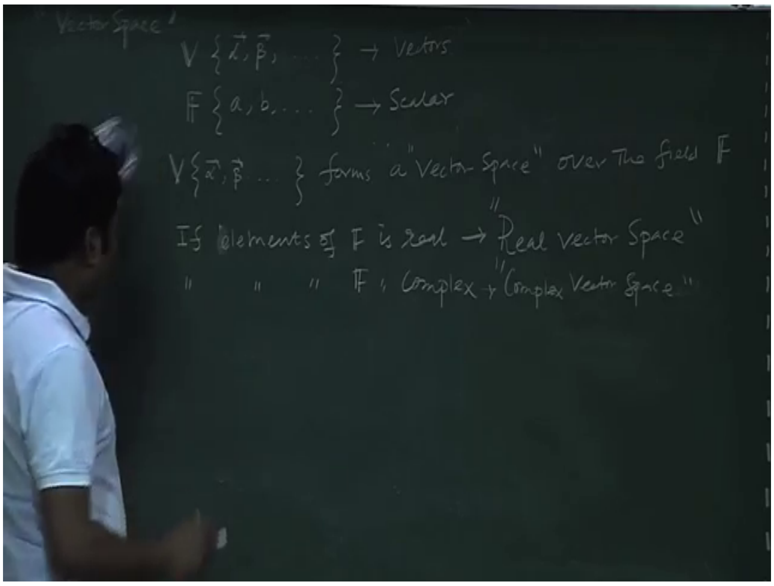

If these properties are followed, then we can say that vector, the set V forms a vector space. This is very important term vector space over the field F. Whenever I am saying that V is forming a vector space, I should mention over which it is forming, it is forming a field F.

Now, you remember that F can be two types; real quantity and also the complex quantity. So, this is important which is a set of few elements which are real. If rather elements of F is real, then the field over which it is generating the vector space is generating over a field F. Now, I am saying that my F is real, then this vector space is called real vector space. Now, if element of F is complex, we have already seen that F field can be formed by real quantity also and the complex quantity also. If the components of this field is complex, then we can say the vector space that is formed over the field F is complex vector space which is very important in quantum mechanics. In quantum mechanics, you should know that at this point I should mention that for physics, this complex vector spaces are very important because in quantum mechanics, the state function psi is considered to be a vector. Obviously, it is not a normal vector you think of. It is a state vector and this state vector are forming a vector space over complex field. So, that is why it is under complex vector space. So, this term is important in quantum mechanics, however in normal cases we always deal with this real vector space. So, before going to finish this class, I will give you two important examples of this vector space.

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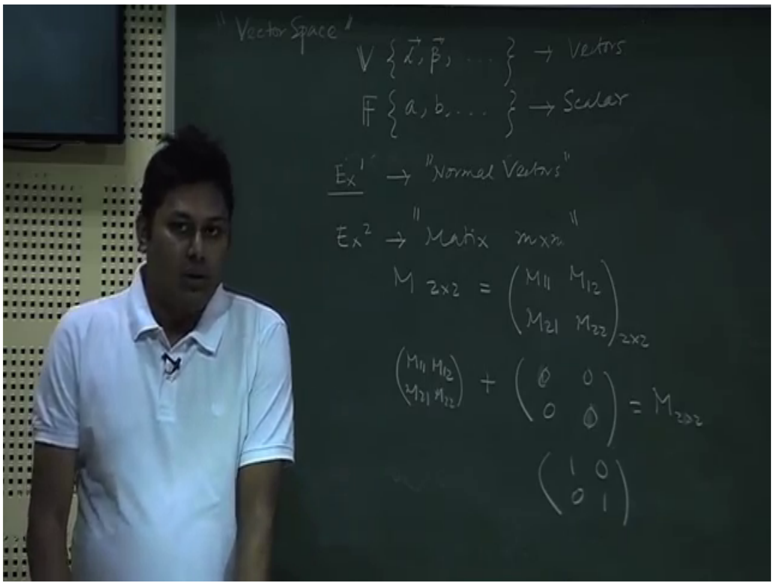

One example is as I already deal with these things, the normal vectors. Normal vectors means which has direction and amplitude. So, they are forming the vector space, they are forming the real vector space or complex vector depends on what the element of this field is. If there element of the field is real, they are forming the real vector space. If they are complex, they are forming a complex vector space. Apart from that I can put another example. This is example 1 and this is example 2. It is the matrix. Any n by m matrix can form a vector space. So, here I should put the emphasis that it is not necessarily that whenever you learned a word vector that essentially means the vector which has a direction and amplitude rather there are many other elements which can form a vector space.

For example, matrix of n by n, all this set of matrix with real coefficients, for example a matrix if I write 2 by 2 matrix say M is 2 by 2 matrix whose element is represented by M11 M12 M21 M22 like this. So, they can form I mean this is a structure 2 by 2 structure and based on the value of M, I can form infinite number of matrix and all these are elements of vector space. They are forming a vector space. Why they are forming a vector space? It is because all the properties that we have discussed few minutes ago can be followed by this. These can be obeyed by this structure of matrix. Also, it is not necessarily 2 by 2. It will be any dimension say m by n also in general identity elements.

You can readily find that if I add two matrix. These two matrix will give me another matrix which is of 2 by 2 order. If I add a matrix say in this form, this is also an element of 2 by 2 matrix and this behave like an identity any matrix and not this one. This one rather it will behave like a matrix which if I add any matrix with that, then as a result I will return back the same matrix, same 2 by 2. On the other hand, I wrote the one which is the identity matrix. If I multiply these two things, then again I return back the same thing. So, this zero matrix or null matrix behave like an additive identity where as this matrix which I wrote earlier will behave like an identity for multiplication.

So, the point is that it is not necessarily true that only the vectors can form vector space. Here vectors means which have a sign and direction matrices are also another kind of elements that can form the vector space. So, I will like to conclude this class here.

So, in the next class, what I will discuss is very important that when the vectors space is formed, so the concept of vector space you know that a vector is something which, vector space is something which can form over a real or imaginary field. If it is formed over a real field, we call a real vector space. If it is in a complex field rather I should not say the imaginary. I should say the complex, then you should form a complex vector field, a complex vector space.

So, in the next class, what I will do that I will start from this concept of vector space and find something related to its spanning as how the concept when I say vector space. That means, some kind of space is there. So, how this space is defined, what is the dimension of this vector, how to define the dimension of this vector and what is linearly independent dependent vectors, all this important concept will come together because now the association of these things scalar and vector I introduced. So, you know that a vector can be associated with the scalar also which looks to be very trivial, but will go to further and further and starting from a linear combination of the vector, we will try to find out how these things is important in defining a dimension of the space and also, I will be going to define something called inner product and all these things which is also very interesting.

So, with that let me conclude this class. So, in the next class, we will start from here and we would like to learn how the space is defined, what is the dimensionality and all these things.

Thank you for your kind attention. So, see you in the next class.