Mathematical Methods in Physics-I
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Lecture - 19
Metric Space, Linearly Dependent-Independent Functions

Welcome back. So, in our last class I mentioned that today we will start something important which is called metric space or metric, but before that we need to know what is metric and how it is related to function and vector and all these things.

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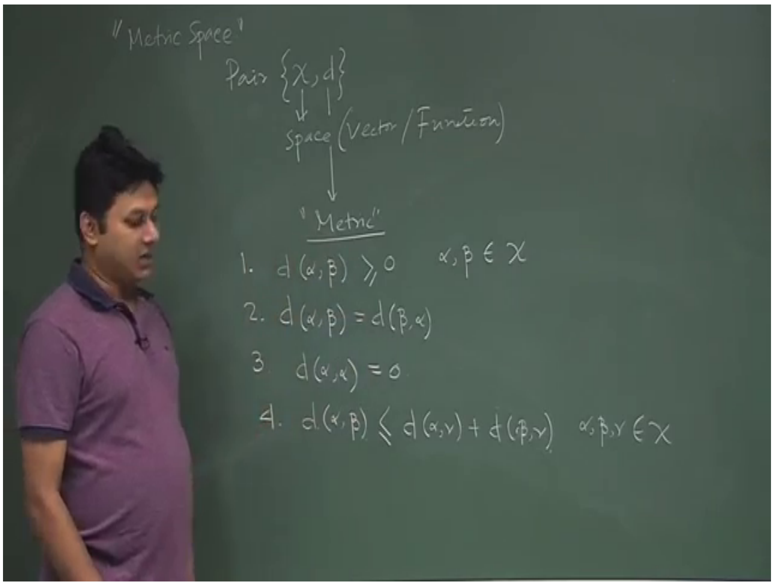

So, essentially metric is defined by a pair it is a pair say chi and d, but this chi is the space, it may be a vector space or the function space and d is something called metric. So, 2 thing one it is a pair and it defined over a space. So, metric whatever the d I mentioned here it defined over a space and we called as a metric, but I did not defined so far what is the meaning of d. So, now, d is called metric when they are following four properties d is following four properties what are they.

So, let me write d alpha beta it is always greater than equal to 0, when alpha beta belongs to the space whatever the space I am talking about. So, this is if this is a vector space alpha beta are 2 vectors, d is something which is related to this alpha beta in such a way that it should be greater than equal to 0 how it is related. So, far I do not know, but I am saying that d should be related over alpha beta in such a way that it is greater than 0 this is the first property. Second property that d alpha beta and d beta alpha are same; that means, if I interchange whatever the operation I like to make through d if I interchange alpha beta there will be no change; that means, alpha beta whatever the value, I have if I do the same operation with beta alpha I will have the same thing. D alpha alpha is equal to 0 please remember that in the first case I said it is greater than equal to 0.

Now, if there is a possibility that beta b equal to alpha this is arbitrary vectors, but I am not saying that alpha is not equal to beta there is a possibility that alpha may be equal to beta if that is the case then this quantity whatever the quantity I defined as metric should be 0 or vanish and d alpha beta this quantity should be less than equal to d of alpha gamma, plus d of beta gamma when alpha beta gamma is belongs to this quantity. I can take an additional quantity and I can say that whatever the metric I will have with alpha beta by operating these things and if I conjugate if I make a relationship this metric relationship with another element gamma with alpha gamma and along and that will be associated with beta also with this. And then take 2 metric and add this to these 2 things and I am getting something which usually always greater than equal to my original metric d. So, these are the definitions.

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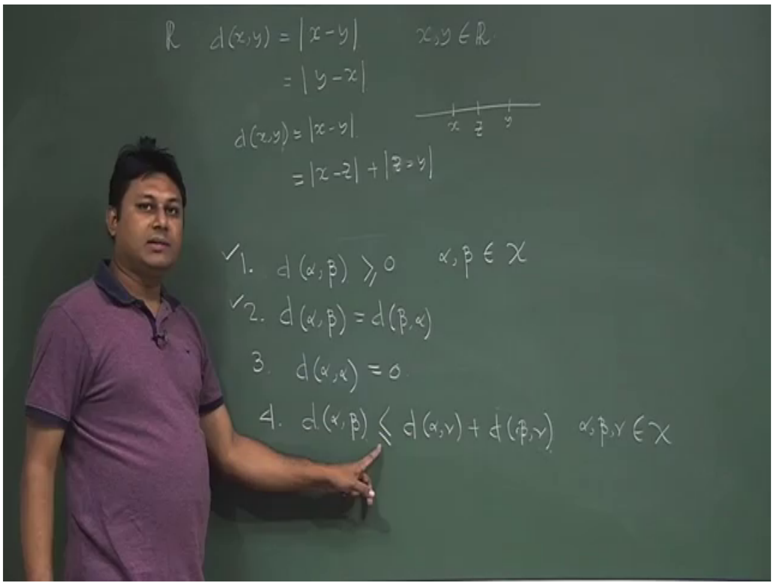

So, it may be confusing for you what is the meaning of these things you may not understand clearly, but let me say that in R 2 in R, I defined metric d x y I defined d x y is nothing, but mod of x minus y when x and y belongs to the real number whatever the real number you can take? If I take x and y if I take a mod of that it will be always greater than equal to 0 first thing is valid. Second thing is that if I interchange if I write x here and y here and x here it will not going to change anything the value should be same this is same as y minus x mod. So, second thing is also valid. Third thing I am saying that when the 2 things are equal then it will be 0. So, if I take x both the cases are y both the cases readily I find that the d which is defined like this that is the distance between 2 point in real axis here.

If it is x and if it is y distance within these 2 things not only distance, but mod of the distance the magnitude of that thing not the only the numeric value then when x is equal to y equal to y x is y then do these 2 things are same then I will have 0 no doubt about that, and in between if I put something here in this case z. So, I can readily say that these things will also follow and here since it is falling in the same line since it is falling in the same line. So, I can have d x y which is mod of x minus y which will be equal to these things, but here you look that it is never say that it will be always greater than equal to, but it is less than equal to this quantity so; that means, the equal to sign is there. So, it is valid still it is valid.

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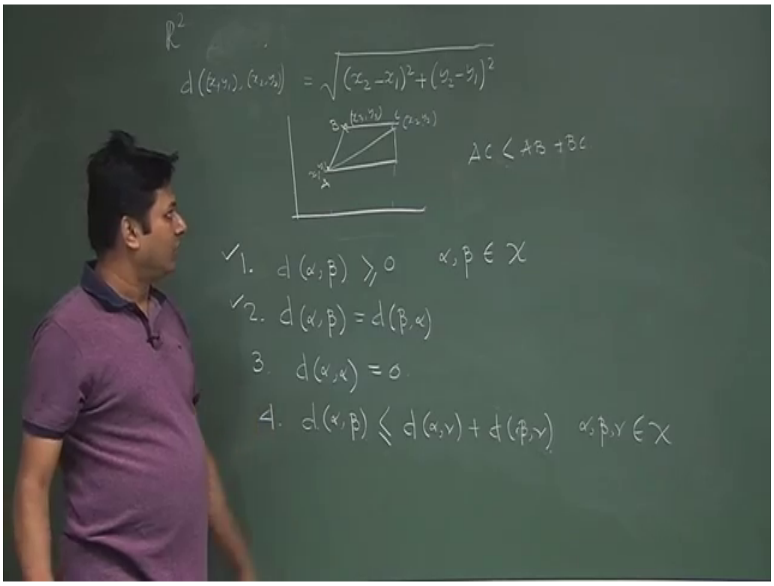

Since it is following in the same line you can also define in R 2 in R 2 the matrix like this way. So, here I need to write say x 1 y 1 to quantity and x 2 y 2 another point if I define these things in to this way, x 2 minus x 1 square plus y 2 minus y 1 square then this quantity also can be considered as a metric this is nothing, but the distance between 2 point in Euclidean plane. So, I have one point here x 1 y 1, I have one point here x 2 y 2. So, in Euclidean plane this is the distance between these 2 and if I calculate the distance between these 2 by applying the Pythagoras theorem then readily I can have this equation in my hand. So, this is the distance now I am saying that this is nothing, but the metric. So, distance between 2 point is nothing, but the metric of this particular space say it is a 2 dimensional vector space. So, 2 dimensional vector space also I can redefine in this way.

This is always followed this is always followed if I interchange no problem with that this thing is also followed if this point collapses to this point. So, distance between 2 points which are same is 0 and this is nothing, but another point if I put any another point here which is x 3y 3. So, it follows the standard triangular rule that if this triangle is there I have a distance this A B C. So, AC this is the standard triangular in equality, AC will be always less than equal to AB plus BC where AC is a distance this is a quantity this is not a direction in vector they are same, but here I am taking the root over of that; that means, AB is nothing, but the distance. So, length is always these things. So, it is followed by this.

So that means, I can define my metric in R 3 R 2 R 1 R n according to my choice only thing is that they need to follow this four criteria which I have written here now in function space also I can define something like metric.

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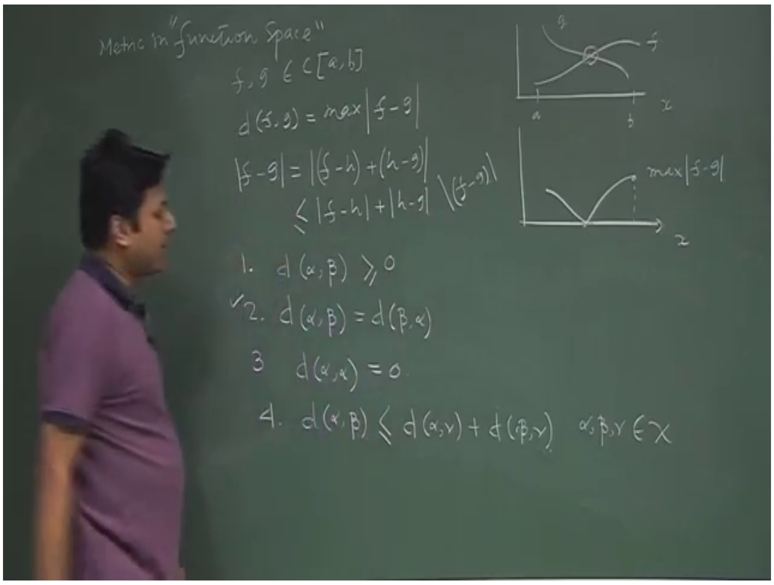

So, in metric in function space, so metric in function space in function space how I define a metric for example, 2 functions f and g which is which belongs to the continuous function say it of continuous function in the interval a b these are the 2 functions are given, then I define the metric d as of f g as maxima of f minus g. So, I have a function f, I have a function g, f minus g will give me another function take the mod of that function then the maxima if this is if I define my g d in such a way that I can find that this is following all the four rules.

Quickly let us check quickly. So, 2 functions a and b one function say this is my f and this is my g. So, if I draw if I draw if minus g then I will have something negative here and then at this point 2 function cross over. So, I will have a 0 here and then it will be something like this. So, it is f minus g I am plotting; now what the definition suggest I need to plot the mod of that thing and maxima. So, mod of that thing if I plot then what happens these negative things will flip and go to the upper side because I am plotting the mod of these things. So, it will flip and go to the upper side fine when it flip and go to the upper side then I will have this functional shape which is never negative may be it is 0.

But never negative and then I say it is maxima I am trying to find out the maximum; that means, at this point this is maxima. So, this is say maxima of f minus g now this is my metric now you can readily find that this quantity is always greater than equal to 0 it at most 0, but it should be always greater than 0 because I am taking the mod of that. If I interchange f and g there will be no problem because whatever you do that if f minus g and g minus f will be same because you are taking the mod sign. So, mod sign will take care of if both the functions are same alpha alpha so; that means, instead of putting h if I put g then what happened the function will be sub subtracted from the same function. So, I will have entirely something 0. So, maxima of 0 will be maxima will be 0. So, this quantity will again be 0.

And again if I put some another quantity say another function h. So, I can write this quantity say maxima of f minus g that will be always be greater than of maxima. So, how I let me do that in more critical more general way. So, f minus g mod I write it as f minus h plus h minus g right. So, mod of a plus b should be less than equal to this quantity, I can write from here to here.

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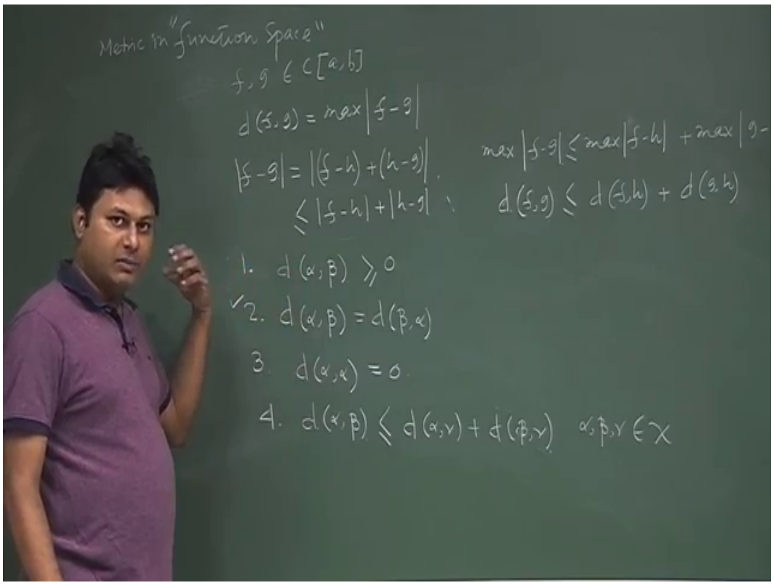

If I write here then I can take from this equation I can take the maxima maximal not going to change f minus g is equal to less than maxima of f minus h plus maxima of g minus h, this is my definition this is f g my definition, it is this quantity and my definition it is this quantity.

So, these things and these things are same so that means, in function space also we can define something which is behave like a metric and these are the 4 properties that should follow by this metric. So, you can define by your own way, but the only condition is that this 4 quantity should be satisfied by this metric whatever you define. Now we will go to more important things after having a brief idea of what is the metric and how it works in function of vector space, now we go back to something important which is linearly dependent and independent function.

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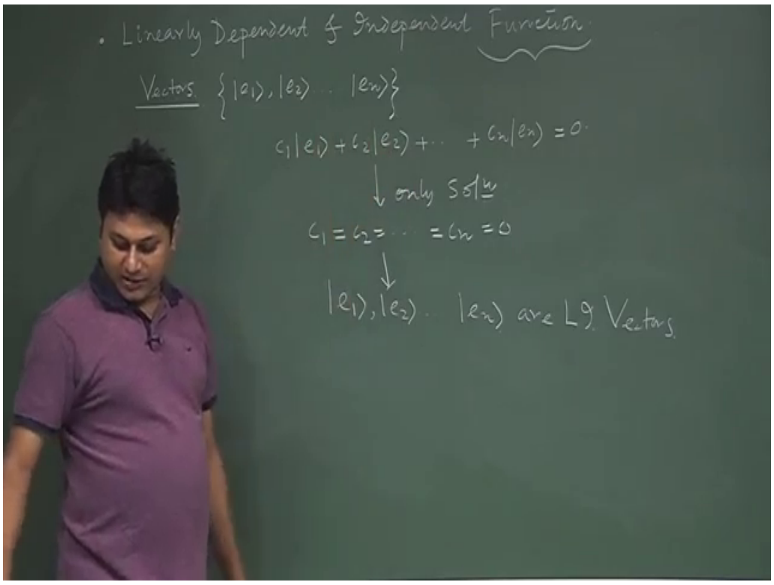

So, far we are dealing with something linearly in dependent and independent vectors, now I am saying that I am doing the same thing, but here in state of vector I am saying that linearly dependent and independent functions. By definition both the things are same the definition is exactly same what is the definition so for vector if you remember the vector. So, let me start with the vector e 1 e 2 e n is a set of n vectors this is for vectors.

They are linearly independent if I have this equation has a trivial solution only solution c 1 is equal to c 2 c n is equal to 0. If we have the only solution that these are the 0 then I can say that then my conclusion was it is just a recapitulation of what is linearly dependent independent vectors are linearly independent vectors. This same thing I just rewrite once again now I will say I will define what happened in case of linearly dependent independent functions.

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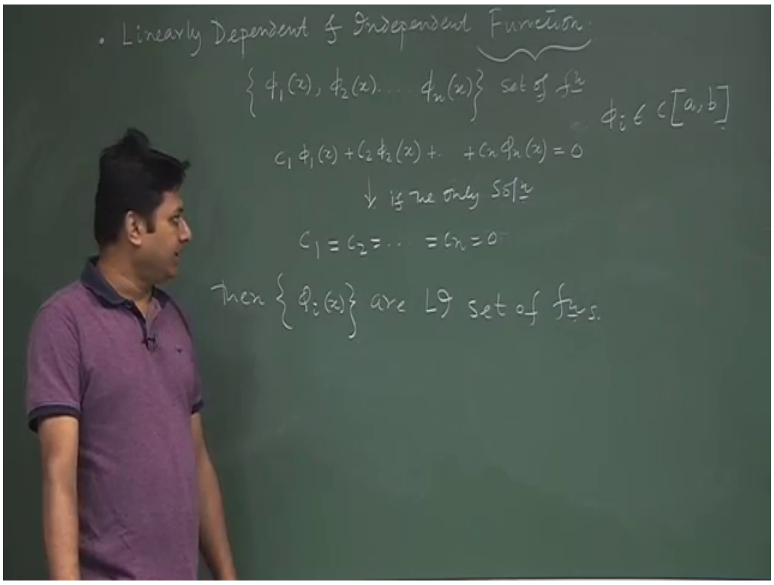

So, let phi 1 x, phi 2 x, phi n x, a set of functions all this phi is belongs to the continuous set of continuous function this is a sub subset of continuous functions and the subset of continuous function means all are continuous in the regime a and b. So, this are the functions set of functions I have. Now, if this is 0 then only if the only solution c 1 is equal to c 2 is equal to c n is equal to 0, I say this is equal to 0 and I am saying that the only solution for that is equal to 0 then we can say the set of functions phi i x are linearly independent set of functions.

So, the by definition this is exactly the same that we have in the vector space, in the vector space I have few vectors set of vectors and then I make a linear combination of this set of vectors with c 1 c 2 c 3 the coefficient here I am doing the same thing I have some functions in my hand the set of function I am making a linear combination of this set of functions, when I make a linear combination of this set of functions and I am saying that the linear combination give me something 0. Then I am saying that when they are 0 then the only possible solutions and only solution is all the c is are 0, then readily I can say that these functions are linearly independent set of functions by definition.

The linearly independent set of functions are linearly set of independent vectors are exactly the same, but the question is here how to find out a set of function is linearly independent or not that is a more important question. If a set of function is given to you how you figure out whether they are linearly independent or not.

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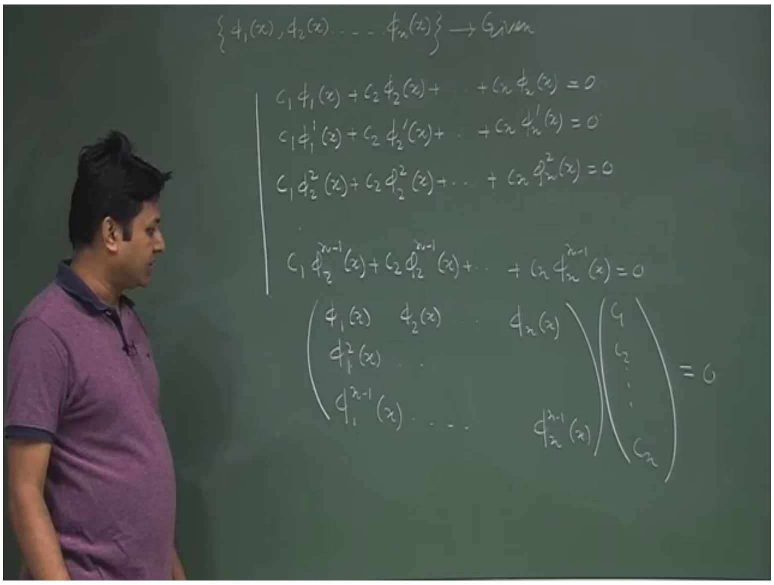

So, let us take phi 1 x, phi 2 x, phi 1 x is given I need to find out whether these are the linearly set of independent functions or not. So, I start with this equation phi 1 x plus phi 2 x this is my equation 1. So, I start with this equation next what I will do I will take a derivative of this equation both the side.

So, c 1 phi 1 x derivative means the standard derivative there is no other things first derivative. I have another equation 2 in my hand I will do again I will do one derivative, this suggests that the order of derivative this is a second derivative this is the first time I am making a derivative this is the second derivative and so on if I do up to n minus 1 then I will be ended with this equation. So, this is the set of equations n number of equations in my hand, if I write this n number of equation in matrix form then how it look like? This is not in fact, a homogeneous equation this equation I already shown in previous just previous lecture that I can write in this. So, see this multiplied by c 1, this multiplied by c 2 and so on and I can write this. So, now, I have a set of equations with this form homogeneous form. So, in order to have a nontrivial solution what I will get. So, we know that from this if I write this in this way.

So, for trivial for homogeneous equation for trivial solution I write this entire thing in a metric form. So, say let me write in once again.

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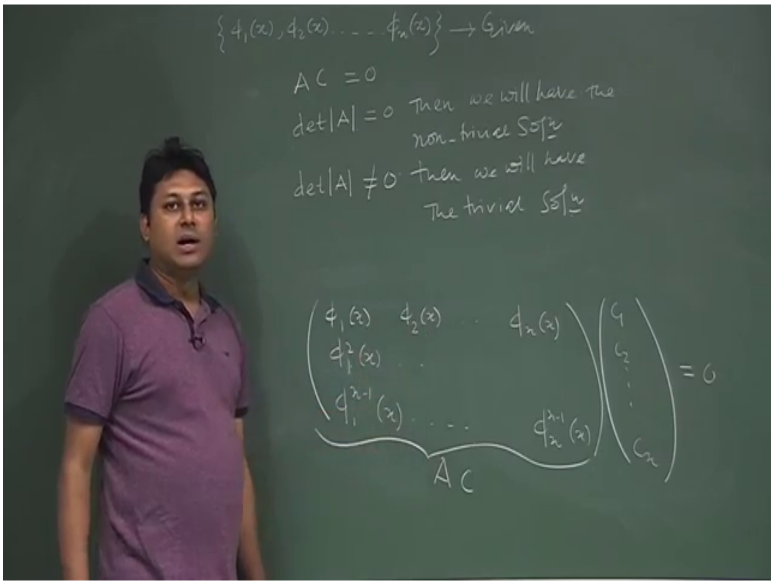

So, this is A and this is C. So, A in to C is equal to 0 this is a matrix form equation and in order to find the nontrivial solutions of A, I should have date of this things when this is equal to 0 then I will have a nontrivial solution. Then we will have the nontrivial solution, but if date of A is not equal to 0 this quantity is not equal to 0, then we will have if this is not equal to 0 then we will have the trivial solution; that means, trivial solution means all the cs are 0 the only solution. So, what I need to find out that I need to find out determinant of this quantity, if the determinant of this quantity is not equal to 0 readily I can say that the functions are linearly independent we there is a special name of determinant of this matrix.

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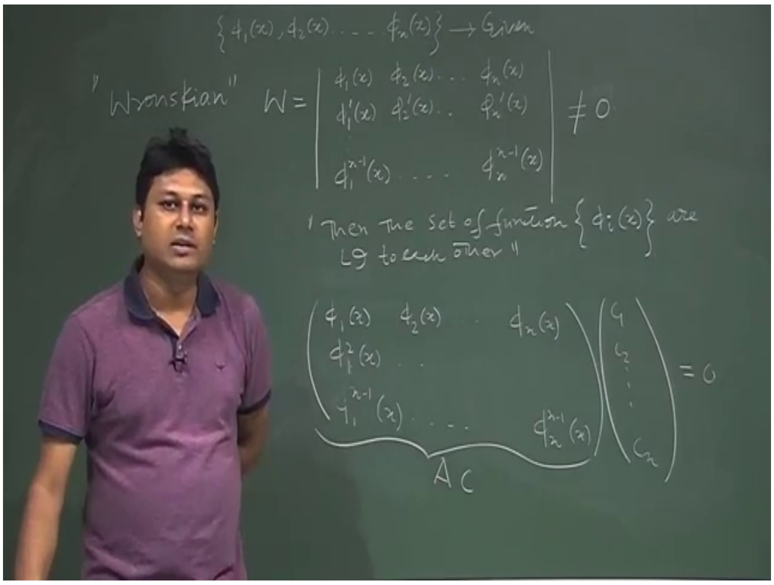

Whatever the matrix I form this is called Wronskian define is a w. So, w is this quantity this is called the Wronskian. So, first if the state of function is giving to us then what we will do will form the Wronskian of this set of functions. So, this is the Wronskian ,first I write the function then the corresponding derivative first derivative and so on the n th order derivative if the number of function is n, I need to go to n number of derivatives.

So, after doing the n number derivative if I put this function in whatever the result I have if we put these things in this way. So, determinant of this quantity is called the Wronskian. If this Wronskain is not equal to 0 I can calculate that easily whether this is not equal to 0 or not then the set of vectors set of functions here rather assumed set is a vector anyway they can the set of functions are linearly independent to each other so; that means, if a set of function is given to us. So, what we need to do is essentially find out the Wronskian, calculate the determinant and if by calculating the determinant I find out the Wronskian, if the Wronskian is equal to 0 then readily I am saying that this functions are somehow dependent to each other.

If it is not equal to 0 then I say that is these functions are linearly independent. So, with that I would like to conclude. So, I will start in the next class with some real examples to understand how a set of function be linearly independent and through which you can find it is a linearly independent or not apart from that I will also like to mention another way to find out whether they are linearly independent or not this is some sort of tricky, but still we can find and this is called the process is almost the same this process that process is called the Gramian method or Gramian determinant method, we will also learn that in the next class.

So, with that let me conclude here in the next class we will start with this Wronskian process and try to find out if a set of function is giving to us whether how to find out whether they are linearly dependent or not.

So, with that see you in the next class.

Thanks.