Mathematical Methods in Physics-I
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Lecture – 18
Cayley – Hamilton Theorem, Function Space

So, welcome back students. So, in the last class we were learned a very important concept related to matrix, which is the rank of a matrix and rank of a matrix is nothing but a rank is nothing but the number of maximum number of linearly independent vector row or column vector inside a matrix. So, inside a matrix, you will find the row and column vector as I mention in the last class. And if they are linearly independent to each other all the row and column then the rank of a matrix is nothing but the order of a matrix. And if it is linearly dependent then it will be less than the order of matrix, you need to calculate how this is there and with that you calculate the rank of a matrix.

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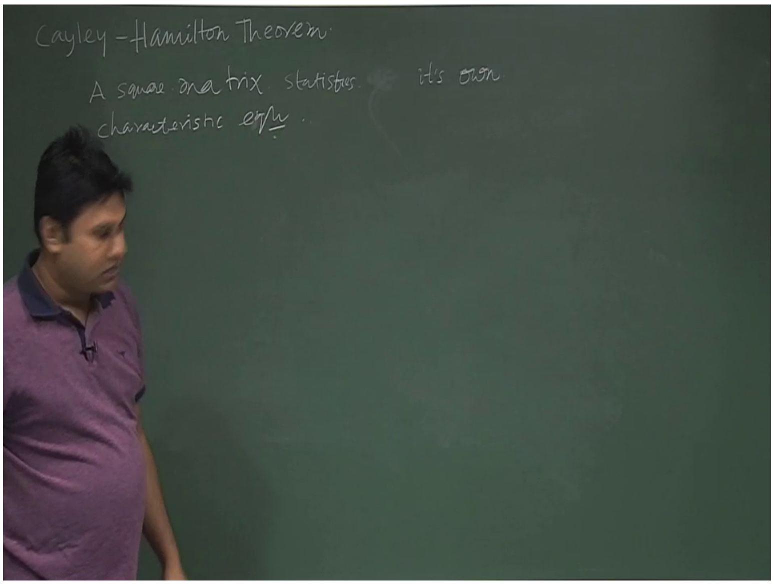

Now, today we will go another important thing, which is called Cayley-Hamilton theorem. I will not going to prove this, there is a tricky prove of that, but let me first state what is the theorem suggest. That a square matrix satisfy the characteristics its own characteristics equation, a very simple statement a square matrix satisfies its own I should not put the here, a square matrix satisfied its own characteristics equation.

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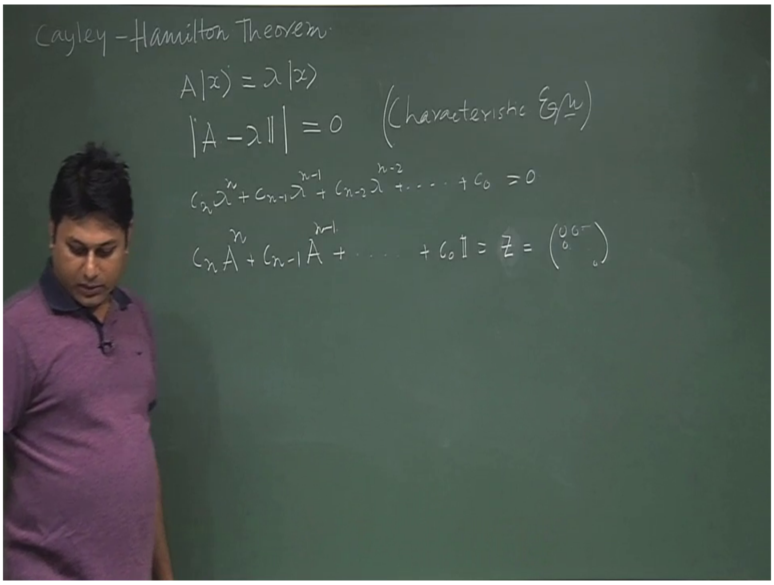

What is the meaning of that, say a matrix A the Eigen value equation is this det of this quantity is having equation this equal to 0 is called the characteristics equation or the secular equation, this is characteristic equation that we know I mean in the previous class we have already discussed this in a elaborate way. So, this is the characteristics equation. And now if I expand this things, what happened I will have a polynomial of lambda, I will have a polynomial of lambda A is given. So, it will be something like lambda to the power n plus say c n minus 1 lambda to the power n minus 1 plus c n minus 2 lambda to the power n minus 2 let us put in this way plus c 0 is equal to 0. I have A polynomial of order n if the matrix of order n, I should have this kind of polynomial. By the way this polynomial can be represent is lambda to the power I mean if I divide entire thing by c n then I will have lambda to the power n then sum coefficient lambda to the power n minus 1 and so on.

But the thing is that this same equation will be satisfied by the matrix itself. This 0 should be represented by the null matrix, so I write it is z here; note that z is A null matrix. So, first one is equation the scalar equation, but the second word is a matrix equation because A is a matrix, A to the power n and all these things are there; coefficient is remains same. C 0 is there, I since is the matrix equation I need to multiply I with the same order of A, and z is a null matrix of order A. So, it is a null matrix of order A, whatever be the order it will be something like this.

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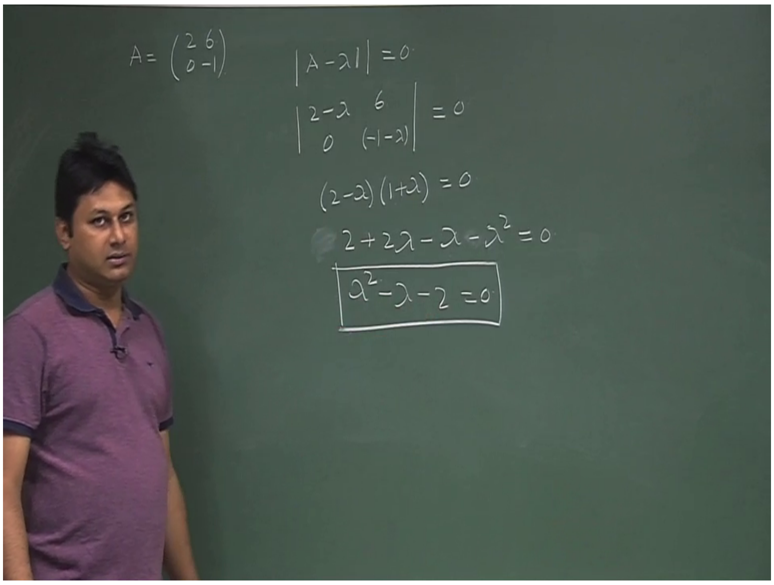

So, let us now check this thing. Whatever the definition I put let us check whether really it is happening or not. So, let us take one example. This is the old example we are taking, I have a matrix A. So, secular equation is 1 minus lambda I det is equal to 0. From here, I can write it is 2 minus lambda 6 0 minus 1 minus lambda is equal to 0. Then I have 2 minus lambda 1 plus lambda is equal to 0. So, here I can have 2 or 2 into 1, it is 2. Let me do it explicitly, 2 into 1, 2 into lambda 2 plus 2 lambda minus lambda and then minus lambda square. So, lambda square then with a negative sign, so it will lambda, so minus lambda minus 2 is equal to 0. So, this is my characteristics equation or secular equation where I can get the Eigen value of lambda, but the equation of secular equation can boils down to this particular form lambda square minus lambda minus 2. So, then the next thing is that if this is my equation then I can have I can put just A here and check whether this is satisfying or not.

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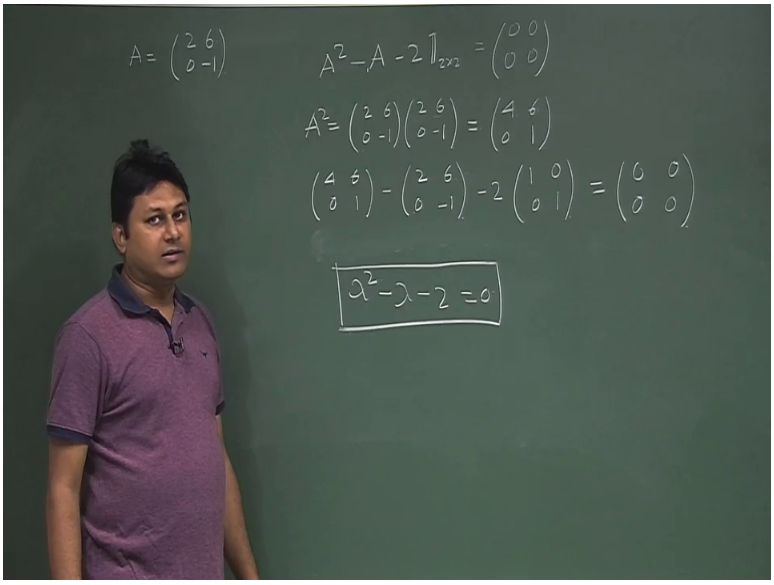

So, the next thing is according to the Cayley-Hamilton theory, if this is my characteristics equation then A square minus A minus 2, I here I will be a 2 by 2 identity matrix should be equal to 0 0 0 0. This is according to the Cayley-Hamilton theory this should satisfy. Now, we should check whether reality satisfying or not, we can calculate very easily. So, what will be my A square 2 6 0 minus 1, 2 6 0 minus 1. It is 2 into 2, it is 4; 2 into 6 – 12, 6, so it is seems to be 6; 2 into 0 minus, so it will be 0; 0 into 6 minus into it will be 1. I have A square this quantity. Then A is there I know and two is and both the things are know. Now, I can write readily write this things 4 6 0 1 minus A is 2 6 0 minus 1 minus 2 is 1 0 0 1 is it given is it equal to 0, let us check. The first thing is 4 minus 2 minus 2 4 minus 2 minus 2 so that means the first quantity element is 0. Second 6 minus 6 - 0, 6 minus 6 is 0, and the last one is 0. So, the weightage is 0.

So, next is this is 0, this is 0, 2 multiplied by 0, so it is also 0. And the last one 1 plus 1, it is 2; minus 2 into 1, it is minus 2. So, 2 minus 2, it is 0; so that means, this equation is true because this is following the same characteristics equation. So, A is following the same characteristics equation that is why it is satisfying this quantity. So, Cayley-Hamilton theory I can have this things in my hand that it is a very powerful thing and I can say that this is satisfying.

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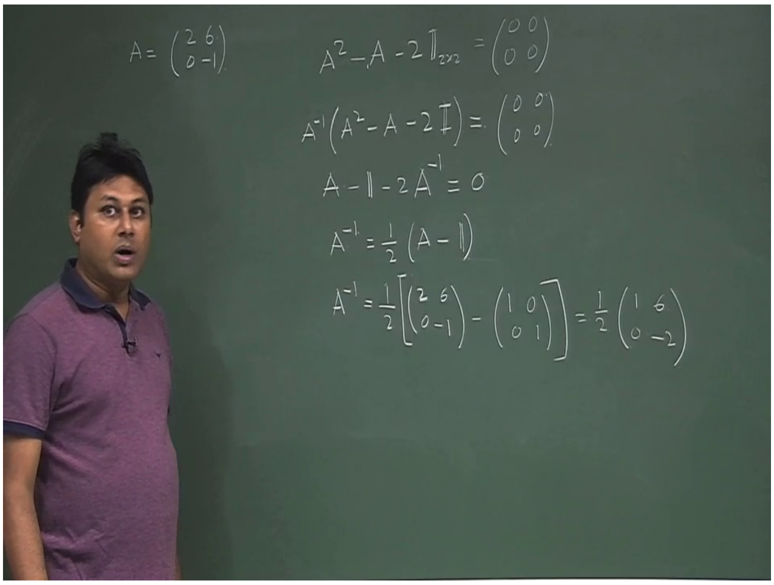

Next thing is that with Cayley-Hamilton theory, applying Cayley-Hamilton theory what extra we can do, what extra we can do. So, this is the equation I can have. So, we can do one thing what I can apply A inverse to the left hand side; in the right hand side, since it is 0, so I will have 0 0 0 0. So, when I apply A inverse here then it will be A inverse AA A inverse A will be 1. So, I will have a lesser order in the first multiplication I will have A here. When I multiply this A inverse I will have I here, because A inverse A is the identity matrix I and then minus 2 of I, which is equal to 0 0 something because say it is 0.

I am doing one mistakes 2 I, it is 2I, so when I multiply the A inverse it will be A inverse here. So, I multiplied by A inverse, so it will be A inverse. So, finally, I have this equation. So, this is a very powerful equation in the sense that without doing anything I can calculate the A inverse, which is some it is not always very easy to find out a inverse of a matrix, but using the Cayley-Hamilton theory, I can have inverse something like this a very straightforward form. For this at least for this given matrix I will have this.

So, once I have this then if I apply if I put the value of A and I then readily I can have the A inverse. So, let us check whether I can get it or not. So, A inverse According to the Cayley-Hamilton theory is half. So, A is 2 6 0 minus 1 minus 1 0 0 1. What I am getting here half 2 minus 1, it is 1; 6 minus 0, it is 6; 0 minus 0, it is 0; 1 minus 1 seems to be minus 2. So, I am getting A inverse something like this. So, I do not know whether this is the correct thing or not, we need to again cross verify by multiplying A and check that whether it is giving the identity or not.

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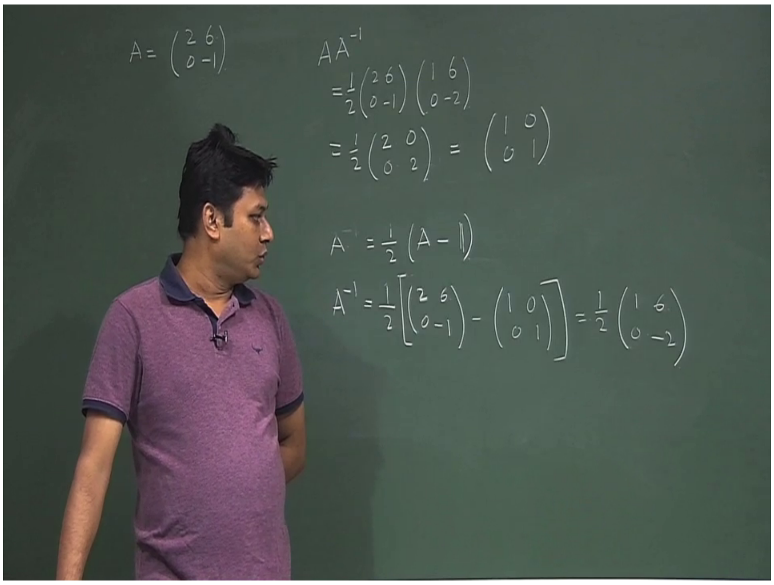

So, finally, I am getting something 1 6 0 minus 2 divided by 2. So, A A inverse is 2 6 0 minus 1 multiplied by A inverse A inverse is it this should I can take half outside. So, one 6 0 minus 2, it is something like this. So, if I calculate half 2 into 1 – 2, 6 into 0, so I will have here 2, 2 into 6 – 12, 6 into minus 2 minus 12, so cancel out I will have 0; 0 into 1 minus 1 into 0, I will 0; 0 into 6 minus 1 into minus 2 - 2. So, 2, 2, I can take two outsides. So, this two will cancel out. So, finally, I will have which is this. So, that means, I can find out the A inverse correctly because when I calculate A A inverse I can get the identity matrix in my hand.

So, with that I can say that Cayley-Hamilton theory is indeed a very powerful theorem through which by applying which you can calculate the inverse of a matrix very easily. If it is a 3 by 3 matrix then it is very difficult sometimes it is very difficult to find out the inverse by the standard method. But if you apply the Cayley-Hamilton theory it seems to be simpler because you need to just multiply the matrix by itself, and you can just do some simple algebra and you can get the inverse readily. So, with that I would like to conclude this particular topic this matrix and all this things.

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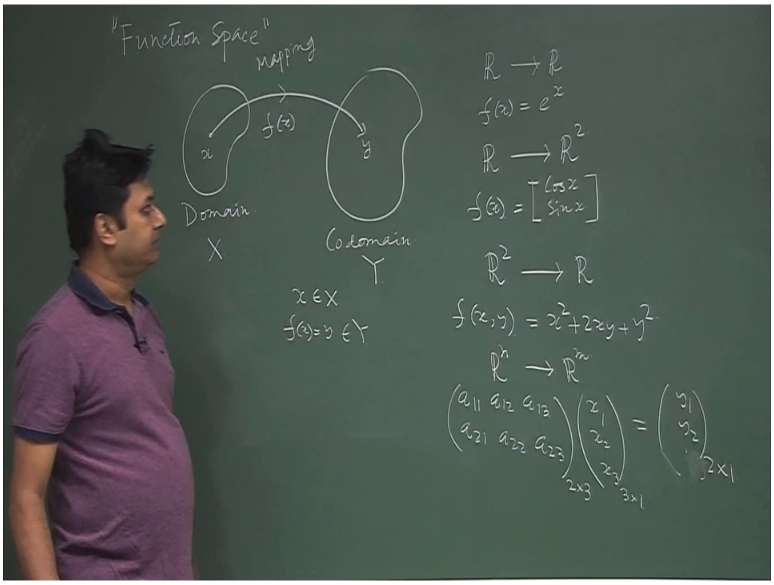

Now we will go to some another important topic which is function space. So, let us go straight way function space. Like a vector space, now I can have a new name function space. So, it is entirely in the new thing function space, but before going to the function space and all these things, so let us try to find out what is the meaning of function and all these things. So, we know that function is nothing but some mapping. So, I have some it is called domain, it is called codomain. I call it X; and I call it Y. One element X is here, I apply a mapping this is mapping we call it is mapping as f x; and as a result I am getting another quantity Y which is in codomain. So, function is nothing but mapping from one set of elements which I called domain. So, here x belongs to X, and f x which is equal to y belongs to Y - big Y so that means, I am putting some kind of mapping, so that I can get this element and I will get another element in the domain. So, this is in the codomain. This is the domain, and this is the codomain.

Now, there are different kind of transformation, for example, this is if I say this is this mapping is real to real quantities, so R to R, so real number to real number mapping. So, for example, if I take function as e to the power x means this is example where the real quantity is going to real quantity. So, I can take any x here and I put the operation on a mapping such that I can find e to the power x. So, e to the power x is my rule and based on rule I can find every x, I can find one y here in the codomain. It is not necessarily it will go to R-to-R real to real, for example, I can go like R to R 2. So, this is a real quantity and R 2 is also a real quantity, but 2 means it is definition that dimension is different.

For example, f x is cos x sin x. So, I can have my x here in domain, I can operate f x. And my operation and my rule is suggest that whatever the x I will take I will take the cos x and sin x and put this value. If I put this value, so this is nothing, but a vector quantity is a column vector like thing. So, I can go from one dimension to higher dimension also, so it is not necessarily. In the similar way, I can go R 2 to R how instead of one input I need to put two input here as a result I can get say this. I will take one x, I will take one y so that means, two elements I am taking; and as a result, what I am getting is a single element through the mapping. I can generalize also. So, R in to R m I can take.

For example, I can take a 2 by 3 matrix. I multiply this with x 1, x 2, x 3, x 1, x 2, x 3 so that means, 3 of order 3, 3 value I am taking. And as a result I am getting something here which is y 1, y 2, y 1 and y 2, there will be no y 3, because 2 into 3 and 3 into 1. When I multiplied that I will get 2 into 1 that means, I start R 3 and I return I go to R 2. So, from R 3 to R 2, so you can change the dimension and you can transform or map the thing in this way. So, you are getting something. So, the point is that function is something which is nothing but a rule or mapping; this rule and mapping allow us to put this value from here to here through this function or through this rule f x. So, this is a very, very basic thing of function. Now, I will go back to this function space, what is the meaning of function space.

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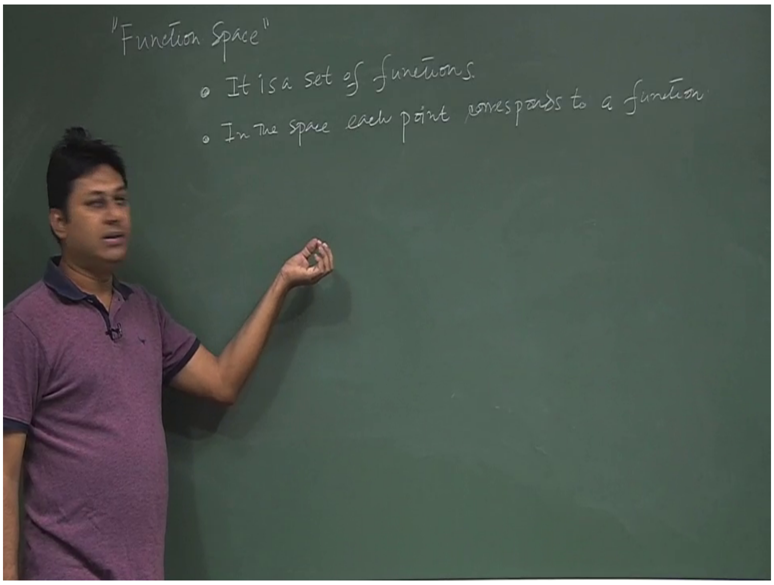

So, first of all, it is the set of functions, I have a set of functions. And in the function space in the space each point corresponds to a function. So, what is the meaning that first of all I have a set of functions, I do not know what is the number of functions it maybe infinity. So, I have a set of function. And I can define a space where all the all the point in the space corresponds to each functions, one functions is represented by the point in that space we called is a function space.

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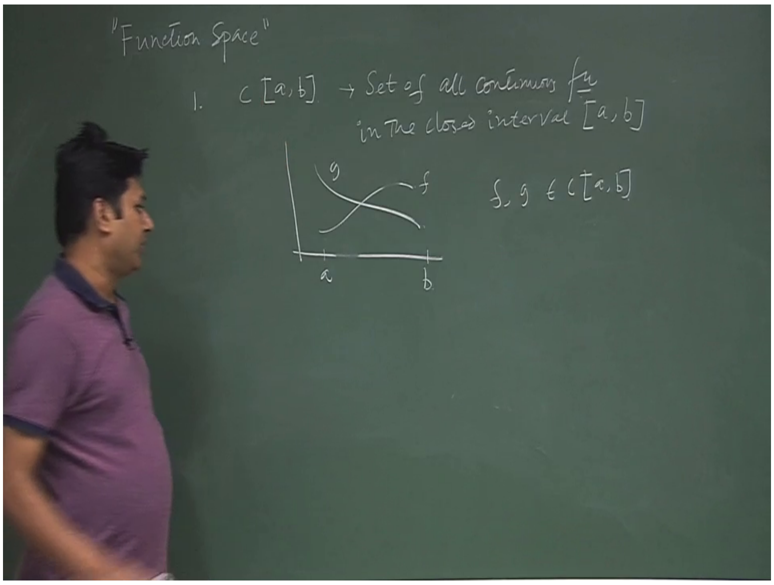

So, if this is the case then, we should have some kind of examples which are important. For example, in all c a, b this is there set of all continuous function in the closed interval a b. So, all the function you can imagine different kind of function in fact, infinite number of functions can be possible. So, if they are continuous in nature, they can fall on the set; and a and b for example, if I write this is say my a, this is my b. So, if this is a function, whatever the function maybe this is continuous in a and b. So, this particular function f belongs to this continuous function, and they form a function space. Another function g this is also continuous they can also form a function, there is a this both are f and g both belongs to c in the closed interval a, b because they are continuous. If there is a discontinuity like this, this function h is not a function of this continuous space or the space of continuous function.

So, there are many functions infinite number of functions which are continuous in a b interval and all this function which are continuous in a b interval are in the set of this function continuous set of continuous function in a, b and they form a function space. Also there are different other kind of functions, which are important in different cases.

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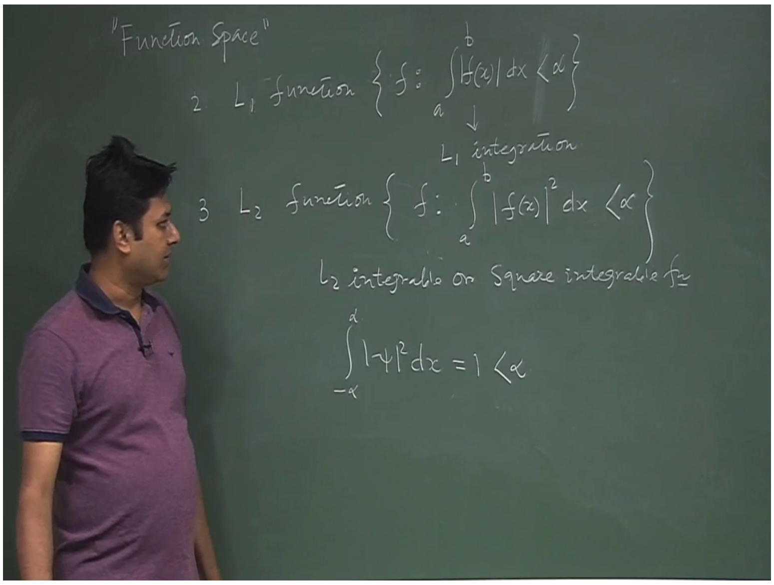

For example, say L 1 type function L 1 function. What is L 1 function? By definition it is something that this function suggest whatever the function that is belong to L, if I do in the closed interval if I do this calculation, if I take this value this integration, this integration should be finite. So, I will have a function f x, I take a mod of this function then I integrate over a to b. And if this set is this whatever the value I have if it is finite then I can say that this L 1 this particular function fall to L 1 type of function. This integration is called L 1 integration.

In the similar way, I can have L 2 function. Here the rule is in the interval a and b, the square of this function should be finite. I take the mod of this function, make a square of that in a and b if this function if I integrate and if this function is finite then this kind of functions belongs to L 2 functions. So, it is called L 2 integrable or square integrable square integrable functions. This square integrable function has a huge implication in quantum mechanics, you should always correlate this things to quantum mechanics. Because in quantum mechanics we know that the psi function mod of psi minus infinity to infinity say dx in 1D this quantity gives us the total probability. If this is normalized then this value has to be 1, which is less than infinity or finite. If the normalized quantity is not, there I need to put a value n here. Again this n should be less than equal to less than infinity that means, it is a finite quantity.

So, in quantum mechanics, you remember that in the interval this psi function should be continuous which is here in already discussed. And not only that the mod of this square dx if I do this calculation which gives the probability then that value should be equal to something which is finite. So, this is directly the consequence of whatever the L 2 function by definition is mean. So, whatever the wave function you are taking is followed by this L 2 functions is the set of L 2 functions.

With this very brief notation and very brief discussion with this function and all all the different kind of thing related to L 1, L 2 and continuous function I would like to conclude here. In the next class, we will start the more important thing related to function and we need to find out this function, how these functions space are formed that is one. Second thing how different kind of functions are behave like a vector because this is like a vector space. So, in the vector space, I find linearly dependent independent vector that is the first thing we find in the vector space. In functions space, would it be possible to find out linearly independent dependent functions, we will check that.

But before that we will do something called matrix space. So, matrix is important concept that I would like to introduce you, you should know that this kind of things are call matrix because in higher studies when you go to the tensor analysis and all these things gravitation this matrix concept of matrix maybe importance. Very briefly I will try to describe these things, but mainly I would like to concentrate on this different kind of linearly independent or linearly dependent function, how you calculate that how you find whether the linear in the functions are set of functional linearly dependent or independent etcetera. So, with that let me conclude this class. So, see you in the next class where we start with the matrix space.

Thank you.