Mathematical Methods in Physics-I
Prof. Samudra Roy
Department of Physics
Indian Institute of Technology, Kharagpur

Lecture - 17
Rank Of A Matrix

Welcome back student. So, in the last class we did something related to matrix; so diagonalization and all things are there. Today we will do something which is called Rank of a Matrix.

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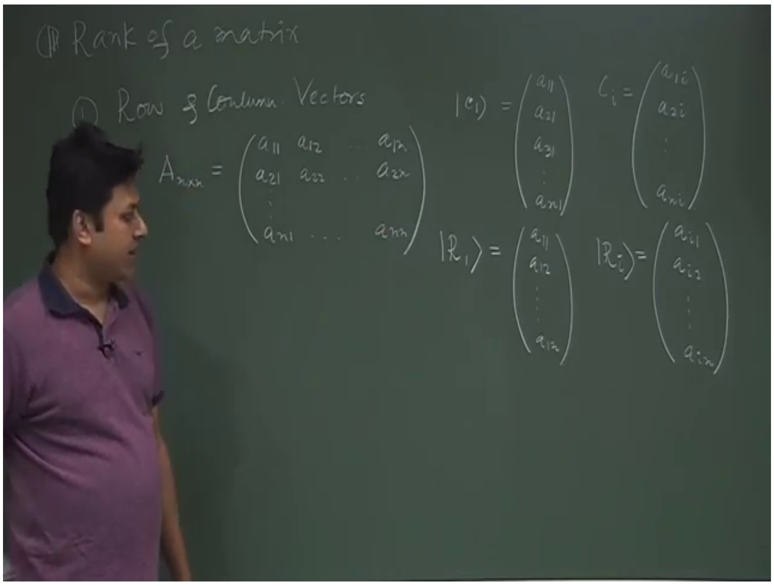

It is very important concept rank of a matrix, but before going to the definition and all this things, I would like to introduce few other things related to the rank of a matrix. So, first one is row and column vectors.

Now, what is this row and column vectors. Let us suppose a matrix is defined say n by n matrix is defined as a 11 a 12 a 1n a 21 a 22 a 2n a n1 a nn. Now these matrixes contain different rows and columns. If I take individually each row or each column they will form a matrix. So, that is nothing but the row and column matrix.

For example, if I take the column c 1 in our standard matrix notation is standard vector notation rather. So, this c 1 in n tuple form can be represented at a 11 a 21 a 23 a 31 a n1 like this. In the similar way I can write the nth. So, here a is represented by this 1 1 first one is for the row and second one is for the column this is the first row that is why this is one this is the second column that is 2, that is the standard way we define a matrix. So, the i-th column that means this one, if I write in this way in tuple form then it will be the column vector this is a column vector.

So, general column vector is nothing but a 1 i a 2 i and a n i. This is the i-th column vector. In the similar way, if I write the row then I can write the row in this way also this is the row vector this is the row vector defined by r, one is the first row; that means, this 1 i would like to write. And if I write it will be something like a 11 a1 2 a 1 n in n tuple form. In the similar way if I write the i-th row, it will be something like a i 1 a i 2 a i n.

So, I have a matrix and this matrix has this form and if I take the row individually then I will have a individual column vectors this column vectors are here and also I can write this rows in to vectors. And these are here these are column vector and these are called row vectors. It is important in understanding the rank of a matrix, but before that I need to define this row and column vectors in this way. After that, I will define another thing.

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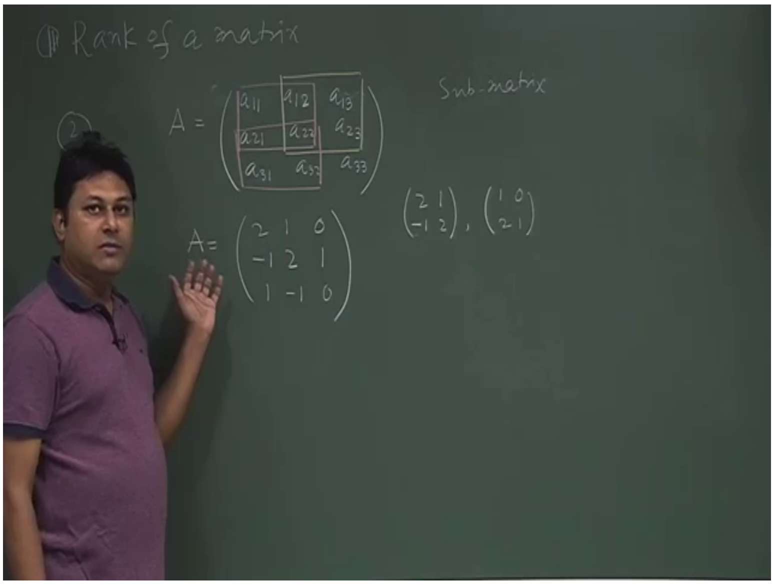

Again a matrix is defined say a 3 by 3 matrix is defined like this 12 13 a 21 a 22 a 23 a 31 a 32 a 33. So, now, we will going to define something called the sub matrixes. For example, these matrixes contain few square sub matrixes. So, sub matrix? What is sub matrix. So, this is a matrix, but you can find that these confirm individual matrix we called is sub matrix. In the similar way this can also form this can also form a sub matrix. In the similar way this can also form a sub matrix and so on.

So; that means, there are many sub matrix one can form from a given matrix. And these sub matrixes are important also in understanding the rank of a matrix. So, for example, if I have a matrix A say I have 2 1 0 minus 1 2 1 1 minus 1 0. Suppose this is a matrix I have now, if I write the sub matrix I can write one sub matrix like this, this is a square submatrix mind it I am just talking about the square sub matrix, but there are also some column matrix and row matrix that is associated with this matrix also their called the sub matrix, but right now I am not bothering about those matrix which are not square, but only those matrix which are square.

So, this is one submatrix, square submatrix then 1 0 2 1. First one, I am taking this, this, this, this element. So, I am having one submatrix like this second 1 I am having this and so on. You can generate many submatrix by decomposing this given matrix a. So, after having the knowledge of submatrix and the row and column vector, then we are in a position we are in a position to understand the rank of a matrix. So, let me define; what is the meaning of a rank of a matrix and what is the physical aspects of this things.

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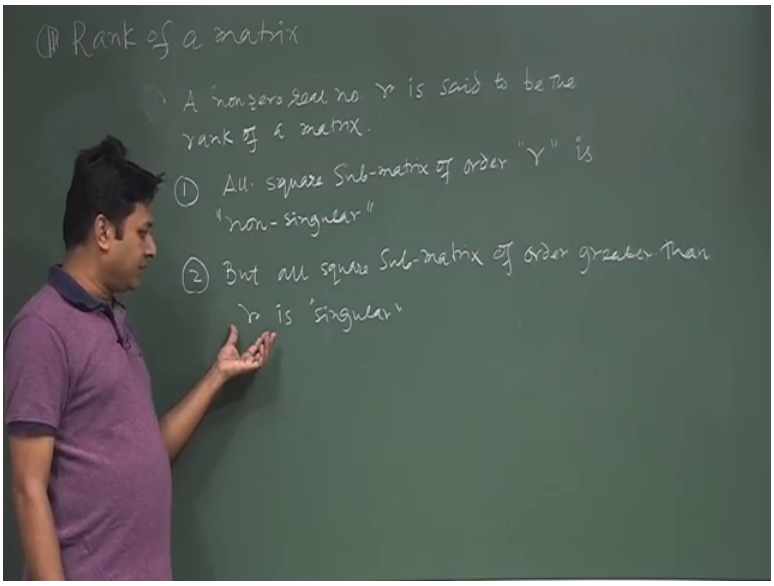

So, first thing is that, a nonzero real number r is said to be the rank of a matrix. So, a nonzero real number r, is said to be the rank of a matrix. It should follow few property. So, one is that all square submatrix of order r is nonsingular. And secondly, but all square submatrix of order greater than r is singular, what is the meaning of that? It is not the definition is something like that, but in order to understand that we need to find out what is the meaning of non singular and singular. So, I believe all of you are aware of this things nonsingular. The meaning of non singular means that the determinant of the matrix is not 0, and singular is the determinant of the matrix is 0, that is one definition other way you can say that there is no unique universe exist for singular matrix.

But in nonsingular matrix you can have the inverse of that, but we should, now remember only the thing that nonsingular means the determinant has to be non 0. So, what is the meaning that that all square submatrix. So, first I need to find out what is the submatrix and then find out a order r for which this is nonsingular. And also that the higher all higher order r, if I try to find out the determinant of that matrix formed which is the order of greater than r then the determinant of that matrix should be 0. So, let us do one example that might be easier to understand the rank of a matrix.

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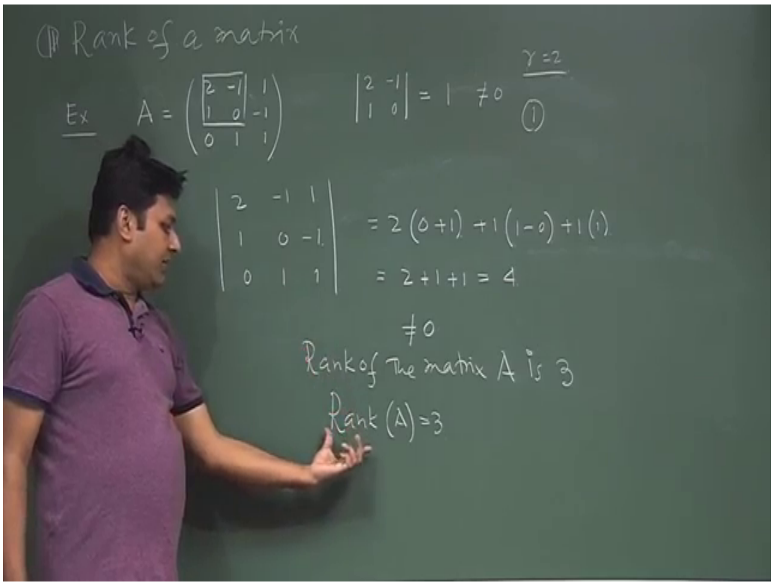

Example, say a matrix 2 1 0 minus 1 0 1 1 say minus 1 say one something like that.

So, this is a matrix given and we need to find out that; what is the maximum order of submatrix which is nonsingular. So, first please remember let me try with this matrix this submatrix. If I do if I take this submatrix and try to find out the determinant of this things I readily find that the determinant is one which is non 0. So, this is a submatrix of order 2. So, here r is order 2 if r is order 2. Here; that means, readily I can say that the rank of a matrix is 2 because it is nonsingular, but please hold on this is the first condition that it is satisfying by this.

At least one matrix should be non 0, if one matrix one submatrix is non 0 then readily I can say that the that the rank should be at least 2, but this matrix if I write this matrix in the whole matrix it is also a submatrix of the of the given matrix. So, it is not that I can only call this is a sub square matrix the wholes matrix is also some sort of submatrix of the given matrix. So; that means, I need to check whether these matrix itself is nonsingular or singular; that means, I need to find out the determinant that is all if I do that let me do and try to find out what is happening here. So, 2 0 plus 1 multiplied this and this it is 0, and then minus of minus 1 1 it is plus, then plus 1 this and this. So, 1 into 1; so I will have 1 minus 0.

And finally, plus 1 this 1, so 1 into 1 and 0; so it is again 1. So, if I do then I will find the it is 2, plus 1 plus 1. So, it is coming out to be 4 which is non 0, which is non zero; that means, first before doing the submatrix like order less than the order of the matrix, first we need to find out the determinant of the matrix itself and need to find out whether it is non 0 or not. If it is non 0 then readily I can say that the rank of the matrix A is 3 or rank a is 3.

So, matrix is given first I check the determinant whether the determinant is non 0 or not. Once you get the determinant non 0 then readily you can say the rank of a matrix 3 you do not need to do anything, but if the determinant is 0 then you need to go to the other sub matrixes which is the order less than r. So, here from the very beginning I find that the rank is 3. So, let us go to another example let us go back to another example.

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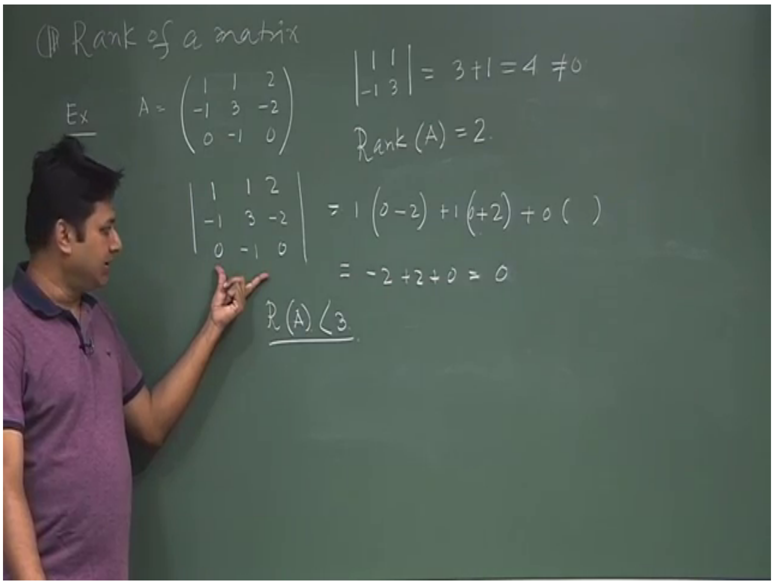

Say a is 1 minus 1 0 say 1 3 minus 1. And 2 minus 2 0, it is something this this is a matrix I have.

So, first I need to check whether the determinant of this matrix is 0 or not. If the determinant is non 0 then readily I can say this is a 3 by 3 matrix. So, readily I can say that the rank of this matrix is 3. Now, first the determinant 1 1 2 minus 1 3 minus 2 0 minus 1 0; if I do, then I will find that one 3 into 0 zero minus 2. This into this then minus 1, then I am I am going to this. So, this cut then this. So, minus 1 should be plus 1. So, I, will cut this row and this column. Then I will have I will have. So, let me check once again I will have this I will have this cut and this cut; so 0 into 0. So, 0 minus 2.

So, it will be plus 2 and then plus 0 into something which is 0. So, please check I take this one this into this this into this here saying to 0 0, then minus 1 into minus 2 which is 2 minus 2 then I take with a negative sign. So, it will come to a positive. So, I will multiply 1 into 0 and minus 1 into 2 1 into 0 0 minus 1 into 2 is minus 2, but there should be a negative sign. So, this negative sign make it plus and finally, 0 into something which is since the 0 is sitting here I am doing this way.

Anyway you can do the normal way also by doing this. So, here I can see it is minus 2 plus 2 plus 0 which is 0; that means, this matrix is singular, but my definition suggest that the matrix should have at least 1 nonsingular matrix submatrix with order r, but here the order r means order 3. So, third order matrix is 0; that means, the rank should be the rank of the matrix; obviously, less than three; obviously, less than 3. So, next we need to find out whether any of the sub matrixes are nonsingular or not. So, let us try the first 1.

So, first one the submatrix is 1 1 minus 1 3 minus 1 3. So, readily I can find that it is 3 plus 1 which is 4 which is not equal to 0; that means, the submatrix is a nonsingular matrix of order 2. So, readily I can say rank of a given matrix is equal to 2. So, this is a standard way to find out what is the rank of a matrix. So, the recipe is something like that a matrix A is given to you when the matrix A is given to you if it is a 3 by 3 matrix then first you need to find out the determinant of this matrix if the determinant is 0. Then readily you can understand the rank of this matrix should be less than 3, if the determinant is not equal to 0. The rank of matrix is equal to the order of the matrix in this case it is 3, and then you gradually you need to go to the to the submatrix is an find out how the rank is defined.

Let me take another example.

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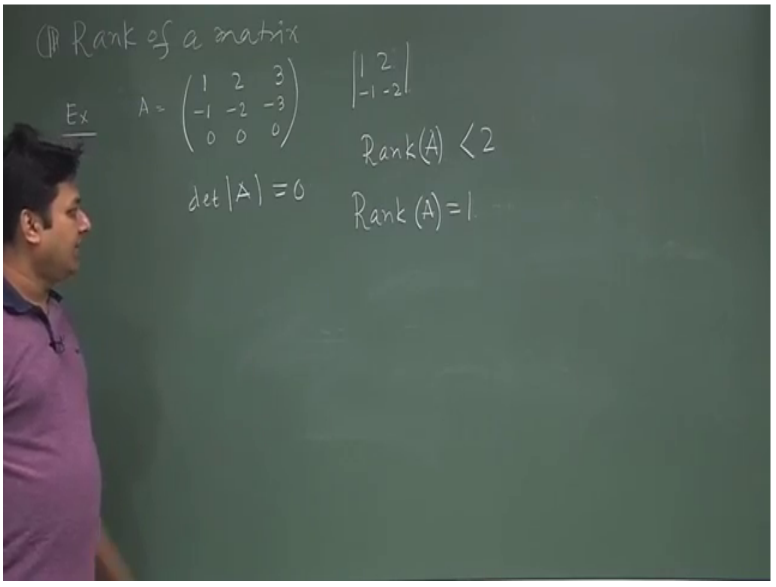

Let me take another example. Say 1 minus 1 0 2 minus 2 0 and 3 minus 3 0. If I take this matrix what happened. So, first the determinant: so determinant I can readily understand that the determinant will going to vanish, if I take this row. So, determinant is 0. So, det of this matrix is 0; that means, the rank of this matrix should be less than 3, that is for sure now I will going to take the sub matrixes, if I take this sub matrixes what will happen 2 into minus 1 minus 2 minus 2 into 2 this plus 2. So, it will going to vanish.

If I take this one it is also going to vanish if I take this one. So, any whatever the submatrix you imagine, if you take that and try to find out the determinant of that matrix you will find that it is 0 any 2 by 2 submatrix. So; that means, here rank of a is even less than 2. So, what is the rank of this matrix now the only element here only element here is one. So, all the submatrix if I take the individual one as a submatrix this is a, one by one matrix or this is a scalar if it is one then the rank of a matrix here is one.

Now, why the rank is reducing like this now you need to understand is in more physical way. So, let us try to find out in terms of row and column matrixes. Now, I know that how a matrix can be calculated in terms of it is rank. So, in order to find out the rank how, you we should calculate the thing. And now I like to introduce the row and column concept.

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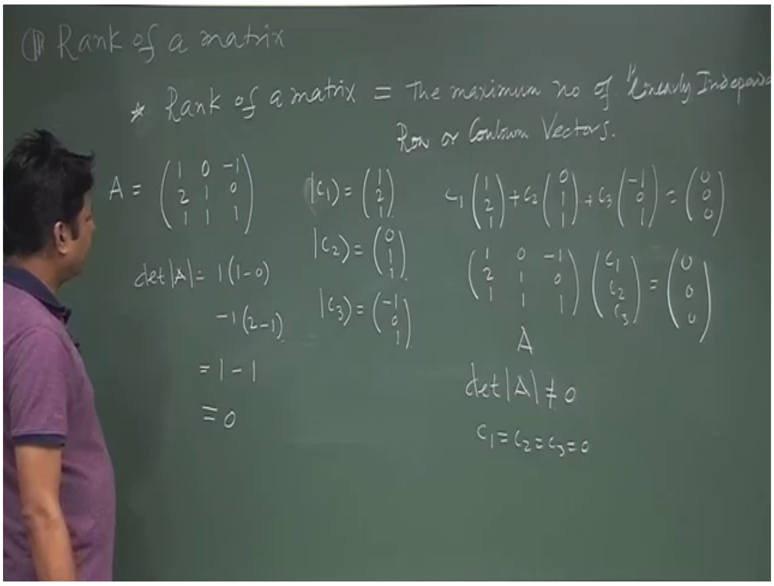

So, another definition which is more important definition: so rank of a matrix is equal to the maximum number of row or column vectors which is a very important definition that the rank of a matrix is nothing but the maximum number of, let me write the now linearly independent row or column vectors.

So, the maximum number of linearly independent row and column vectors is nothing but the rank of a matrix. So, let us go back to some example; so if I write1 2 1 0 1 1 minus 1 0 1. Here the row and column matrix if I first I need to calculate the determinant. So, what is what is the row matrix here. So, so say column matrix here is 1 2 1 column matrix 2 here is 0 1 1 column matrix 3 here is minus 1 0 1.

This 3 matrix I know. Now if I calculate and try to find out whether this 3 matrix are linearly independent or not; then essentially what we are doing we are taking the determinant of this quantity. Because c 1 of this matrix plus c 2 of this matrix plus c 3 of this matrix has to be like this, this equation I can write in more. So, this into this, this into this, this into this; first equation this into this, this into this, this into this; second equation and third equation there are 3 equations are hidden in this this form. So, I can write in a compact way and it is right hand side. So, this is nothing but a set of homogeneous equation.

So, now in order to find that whether I can have a non trivial solution or not I need to figure out whether the determinant of this quantity is 0 or nonzero. If the determinant of this things is 0 say this is a matrix A if det of a is 0, then we can say if the det of a is 0 then we can say that I can not have any non trivial solution. So, trivial solution will be c 1 is equal to c 2 is equal to c 3 is sorry. If the det is not equal to 0 then I will have this quantity that for non trivial solution determinant not equal to 0 ensures that you have a trivial solution and this trivial solution is this.

This essentially means that the matrix that the vectors are linearly independent to each other. So; that means, if I take the determinant of these things and find that let me try to find out the determinant of this matrix A. So, what is the determinant. So, one into one into 1 minus 0 0 is there; so minus1 2 into1 2 minus 1. So, I have I have, 1 minus I have 1 minus 1. So, here let me try to do once again. So, ensure that I am not taking any mistake doing any mistake.

So, 1, this into this, this into this, this is 1 0 is not there. Minus 1 if I calculate then 2 into one is 2 and minus 1 is minus 1 with a negative sign I can have 1; so 1 minus 1. So, I find that it is 0 so; that means, there should be some trivial solution associated with that so; that means, the rank of a matrix is not equal to the order of this or less than that. So, now, I will take another example, which will be easier.

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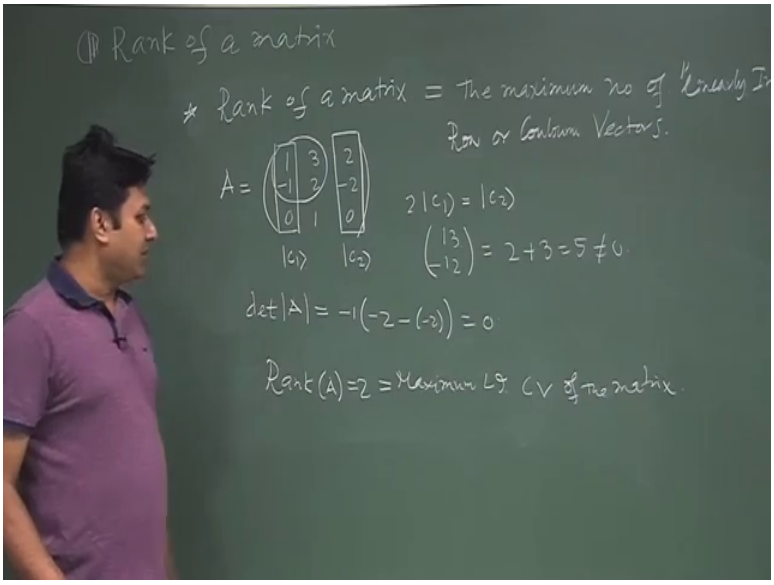

So, a say 1 minus 1 0 3 2 1 and 2 minus 2 0.

Now, if I look please look one thing here, that this column this column vector and this column vector are linearly independent linearly dependent to each other; that means, if I write this is c 1. And if I write this is c 2. Then I can write c 1 and c 2 in this way c 1, multiplied by 2 is nothing but c 2; that means, this 2 are linearly independent are not linearly independent to each other they are linearly dependent.

When they are linearly dependent; that means, linearly independent number of rows number of column vectors are not 3, but 2. So, the matrix has to be the rank of thus this matrix has to be of the order of 2, not the 3 I can check also with this I can check also. So, det of a if I calculate det of a first I will check if it is 0 or not. Then readily I find that one multiplied by 2 into 0, which is 0 minus it will be plus 2 into one it is 2 then minus of 3 this one this one or I can I can start from here also.

So, only just check whether I. So, let me do this it will be easier because there are 2 zeroes are only be sitting here. So, I need to find out minus 1 into in this column. So, it will be minus 2 minus of minus 2. So, it is 0 if you use this. And calculate the determinant you will find the determinant of a is readily 0 so; that means, this matrix has rank less than 3. So, now, you need to find out the submatrix. So, this is the submatrix and this submatrix readily suggest 1 3 minus1 2. This submatrix readily suggest it is 2 plus 3 5 which is not equal to 0; that means, rank of this matrix A is equal to 2.

What is this number 2 this 2 is the maximum this is the maximum linearly independent in this case column vector of the matrix? So; that means, once the matrix is there you can inspect and find out whether it is linearly dependent or not if the 2, 2 row or column matrix are linearly dependent then you can say the matrix is not 3 the rank of the matrix is not 3, but less than that and with that you can calculate with this submatrix formula you can also cross verify and calculate the rank of a matrix.

So, the take violation is that rank of a matrix is the maximum number of linearly independent column or row vector that is existing in a given matrix. So, with that note today we will like to conclude this class in the next class. So, we will also calculate something related to matrix, which is important and which is called the Cayley Hamilton theory.

So, with that let us conclude here. So, see you in the next class, with Cayley Hamilton theory.

Thank you.