Mathematical Methods in Physics-I
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Lecture - 16
Hermitian Matrix

Welcome back student. So, in last class we learn a very important concept of matrix which is the diagonalization of that. So, today I decided to do few other things, few identity of the matrix and few other small calculations to know. So, let us start with very simple thing.

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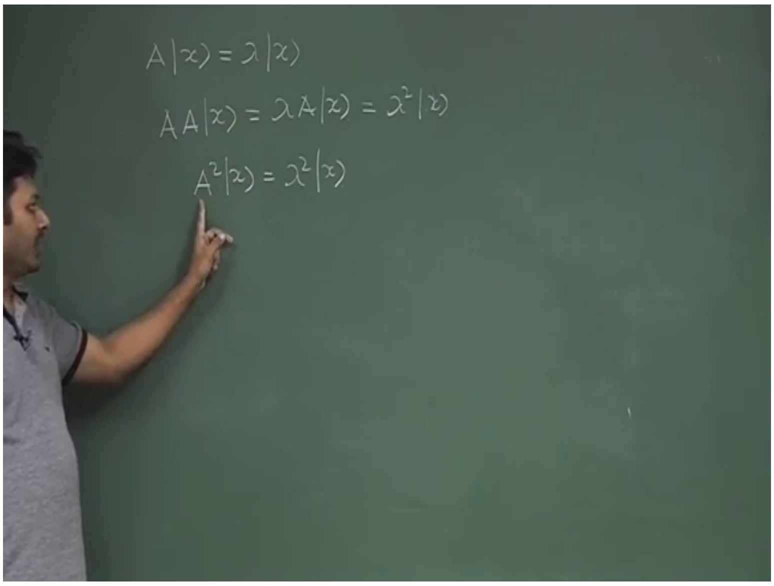

If a matrix this is quite simple, but still I believe this this is the eigenvalue equation from this eigenvalue equation we can have many information.

For example, if I operate a both the side. Then it will be lambda square a lambda square x. And this is nothing but A square. So, it readily shows you a very important outcome that A square if a is A matrix, has a eigenvalue lambda then A square is a eigenvalue lambda square and so on. So, you can readily with this very simple structure you can find what is the value of A square what is the eigenvalue of A square and this things.

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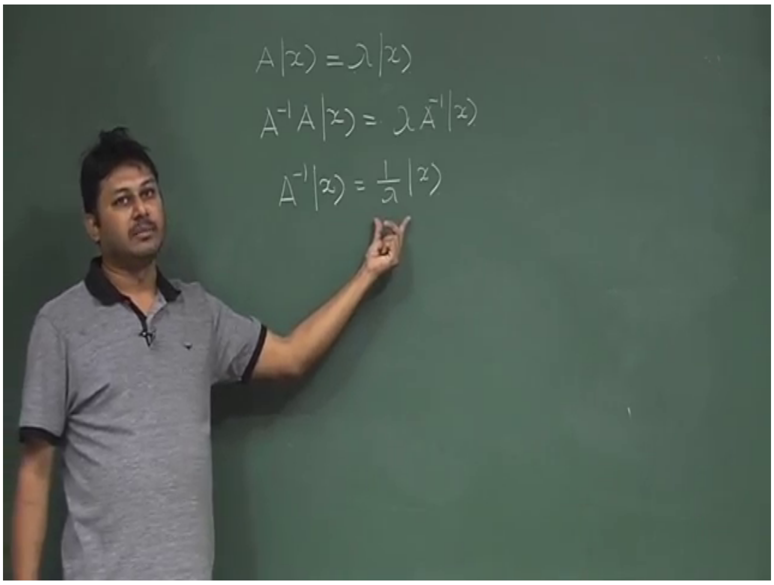

Fine another simple thing quick and simple same. Equation I am writing and for this equation if I operate say A inverse over x. And I have lambda A inverse over x A inverse A is 1. So, I can write this equation in slightly different way like this. So, if I know what is the eigenvalue of a matrix with this? I can also know what is the eigenvalue of A inverse which is just one by lambda with the simple calculation also you can find it out. This is the 2 point I wanted to mention in the last class, but anyway. So, you should remember these things.

Now, one important concept I like to introduce or important.

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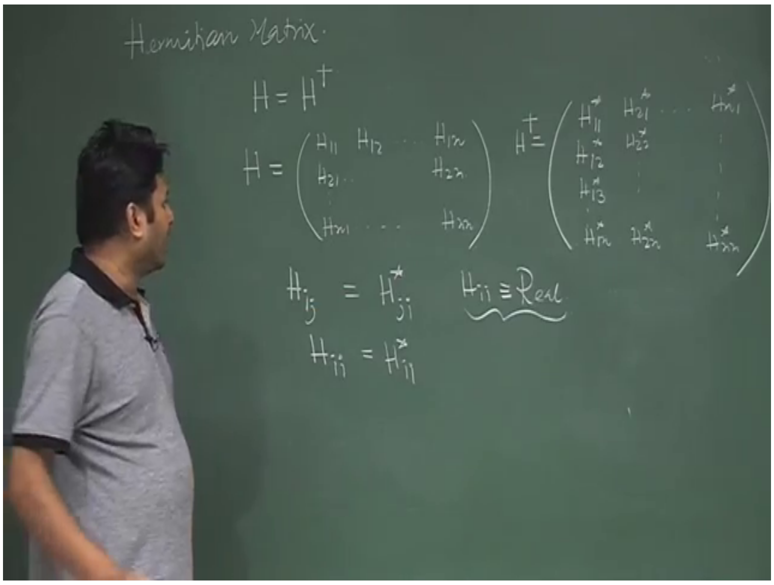

It is Hermitian matrix. It is Hermitian matrix. What is Hermitian matrix? A matrix H if it is equal to H dagger then this matrix is called the Hermitian matrix H is equal to H dagger. So, what is H by the way let me write it. So, component wise it is H 1 1 H 1 2 H 1 n H 2 1 H 2 n H n 1 H n n.

This is my given matrix H. What is H dagger? I know what is the dagger matrix dagger matrix is transpose followed by the complex conjugate. So, I need to first transpose, it is matrix and then make a complex conjugate. So, if I do that then I will find H 1 1 star this H 1 2 star H 1 3 star everything will be star H 1 n star H 2 1 star H 2 2 star H 2 n star H n 1 star H n n star.

So, the structure of the Hermitian matrix is this. Now if the general element of H is i j, and the general element of the Hermitian matrix is I mean in terms of H it is j i right because if i i is here, I make a transpose when I make a transpose j will be replace by i i will be replace by j. And with a distinct and a my condition is that H is equal to H dagger. So, H i j has to be equal to H distinct. If this is true what happened when I put i i both the cases H i i is equal to H star i i, i i is nothing but the diagonal elements and my Hermitian condition suggests that diagonal element both the cases is same as a complex conjugate. This is the same thing which is complex conjugate. So, we know that if the same thing. And his complex conjugate are same this is a quantity whose complex conjugate is the same as the that that quantity then it has to be real. So, H i i has to be real so; that means, in Hermitian matrix the diagonal element has to be the real quantity that is one important outcome that we find.

Now, the next most important thing for Hermitian matrix why it is; so important you can readily understand that.

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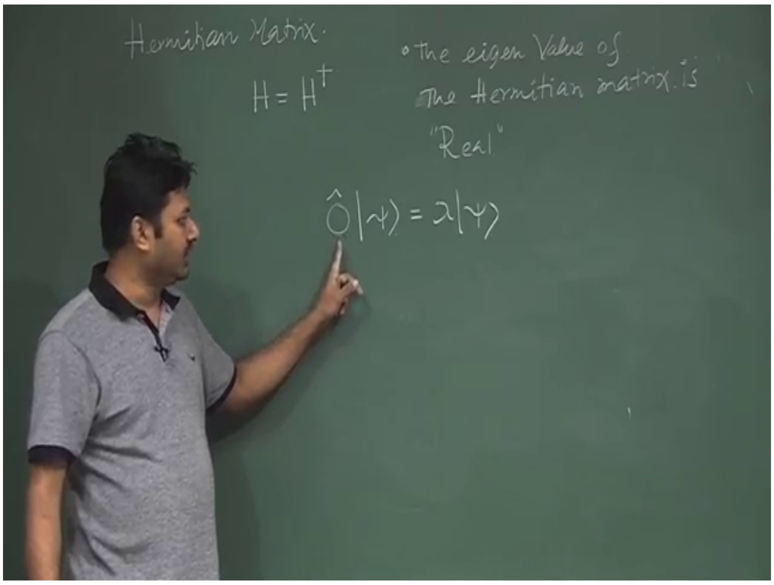

Let me write down first the eigenvalue of the Hermitian matrix is real. So, let me first explain very briefly why it is important in the context of quantum mechanics because in quantum mechanics. When we use some operator this operator in matrix notation used by some matrix when this operator for example, this is my operator say o i operate over that psi this is a state.

If this psi is the Eigen state I am getting some eigenvalue lambda and the state back this is a same structural notation or same structure that we are using in the linear vector space in the context of matrix operation. Now, if this operator is the real operator; so the corresponding eigenvalue has to be real. So, that is why in quantum mechanics we choose the operator which is Hermitian in nature, if it is Hermitian then I always get the real eigenvalue.

However, I did not prove that so far. So, let me try to prove that.

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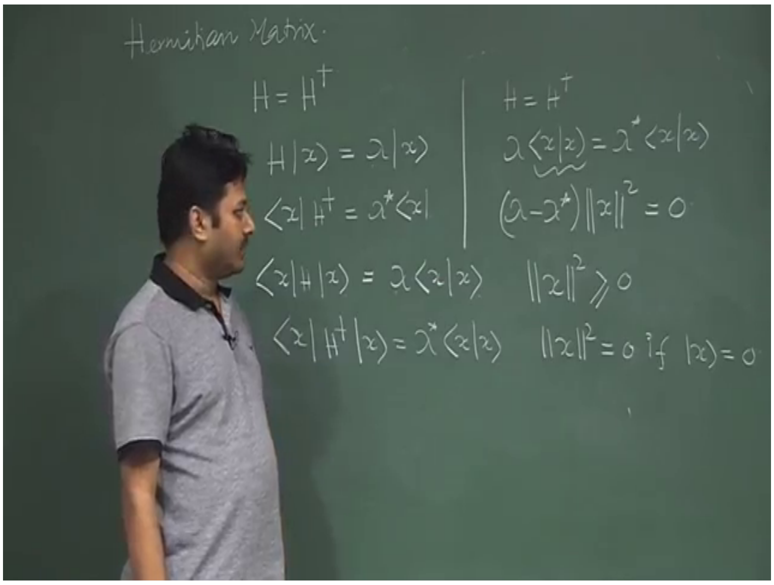

So, let see how it one can prove that. So, H x say this is the eigenvalue equation. If I make a dual conjugation of that, I need to write it like this. I have one equation this I have another equation this this 2 equation in my hand. So x from the first equation if I do, I can have this equation from here to here using this equation; however, we have x H dagger this is this quantity, but H and H dagger at same thing.

So, eventually since H is equal to H dagger I have lambda x this things is equal to lambda star this quantity fine. What is this quantity x the vector inner product with the same vector? So, this is a norm. So, I can write this equation like this. So, these equations suggest either this quantity is 0 or this quantity is 0 either this quantity is 0 or this quantity is 0.

Now, this is a norm. So, we know that from our knowledge from we know that this quantity will always greater than equal to 0, and is 0 if and only if the vector is a null vector. That means, for any arbitrary vector which is not a null vector I will have a general equation where lambda is equal to lambda star.

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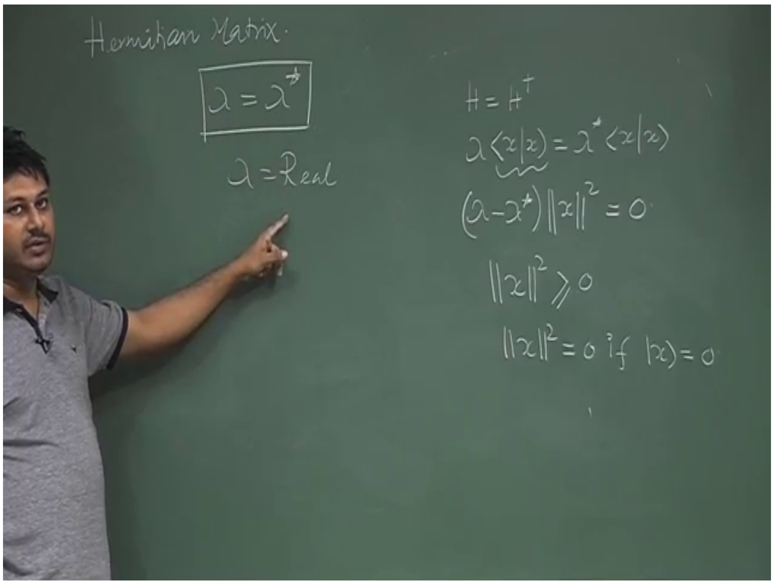

So, from this logic I can say that lambda must be equal to lambda star. Lambda equal to lambda star is nothing but lambda is equal to real.

A quantity which is equal to the complex quantity is only possible when the complex part when the imaginary part is 0. And here lambda is equal to lambda star that eventually means lambda is equal to a real quantity. So, now very briefly, I mean we need to show some example very briefly.

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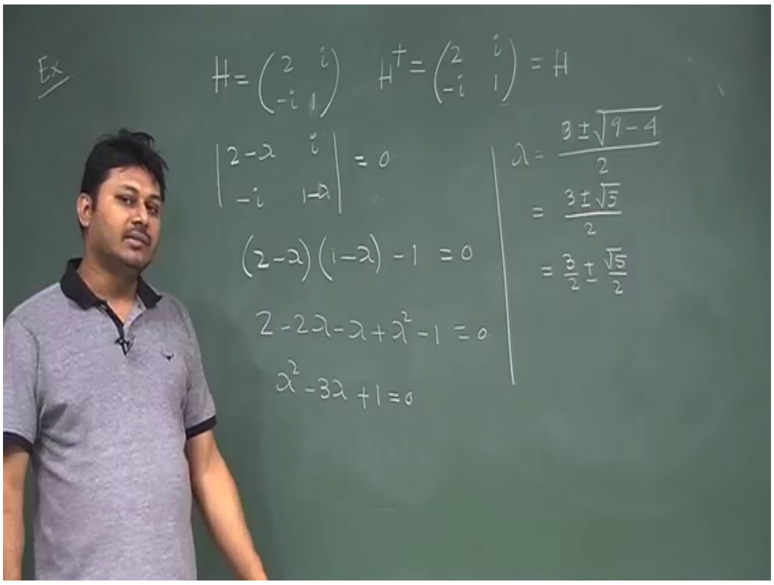

So, say H a matrix 2 1 i minus i is this matrice Hermitian, first we need to check it. So, when I make a H dagger then the complex conjugate the diagonal element is real. So, no problem with that that is one condition to form a Hermitian matrix and this quantity should be the complex conjugate to each other that is another quantity. So, another property of Hermitian matrix that is need to be followed.

So, here we find that it is really followed. So, this if i root a then I will have the same matrix. So, this is nothing but H once I have that, then I can find out what is the eigenvalues of that really the eigenvalues are real or not. So, let find the eigenvalue. So, characteristic equation or secular equation is this sorry 1 minus lambda this is my equation. So, 2 minus lambda 1 minus lambda minus 1 is equal to 0, i minus, i minus i it is 1. So, there will be a minus 1 because it is a cross multiplication.

So, this characteristics equation from that characteristic equation, this is a quadratic equation of lambda. So, I can find out what is the value of lambda it should be real. So, let us check just cross verify whatever the theory is given is. So, minus 2 lambda then minus lambda then plus lambda square then minus 1 is equal to 0 explicitly I am writing that. So, lambda square minus 3 lambda then plus 1 is equal to 0. So, this gives me let me once again check everything is correct or not my Hermitian matrix is this try to find out what is the value of lambda this is my characteristic equation 2 minus lambda, i minus i 1 minus lambda. So, determinant of these things 2 minus lambda multiplied by 1 minus lambda. And then minus this 2 multiplication, I have this is equal to 0.

Then I simplify. So, it is 2 minus 3 x plus lambda square minus 1. So, this minus 1 gives me one. So, I will get this. So, my lambda will be minus b, plus minus root over of B square which is 9 minus 4 a C which is 4 by 2 a 2. It is. Finally, I will have 3 plus minus root over of 5 divided by 2 or 3 by 2 plus minus root over of 5 by 2, whatever but the thing is that this quantity is real. So, just a simple check that any arbitrary Hermitian matrix I take and after taking any arbitrary Hermitian matrix, I try to find out what is the value of is Eigen what is the eigenvalue and when I find the eigenvalue it comes out to be the real as expected.

You can do you can do the same thing for different other Hermitian matrixes and you can check by yourself, that whether every time the this as an exercise you can take this an exercise that every time you are getting the eigenvectors eigenvalues, which are real or not. So, we have now a very brief idea about what is Hermitian matrix.

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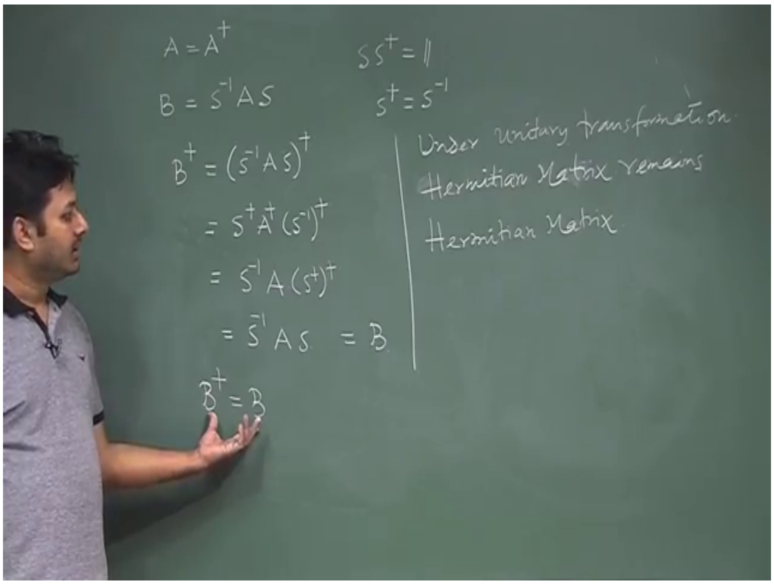

Next few thing few things Hermitian say a is a Hermitian matrix and B is a similarity transformation like this. This is the similarity transformation with the fact that S is unitary matrix.

So, a dagger is equal to 1. So, what I am trying to find out that I have a matrix which is Hermitian. Now I want to transform this Hermitian matrix in another coordinate system or another basis where by making a unitary transformation because this unitary transformation this unitary transformation if you remember then the unitary transformation is required, when I transform from a orthogonal basis a set of orthogonal basis to another basis which is still orthogonal to each other. If this is the case then try to find out what is the value is this B is still Hermitian A is Hermitian, I know is this a Hermitian.

So, here I can write S dagger is equal to S inverse is the extra property that I have because of this unitary property S is a unitary matrix. So, B dagger is S inverse A S whole dagger now I know that if 2 multiplication are dagger like this. So, it is something like AB whole dagger is B dagger a dagger or AB C whole dagger is C dagger B dagger a dagger, if I do that then it will be S dagger A dagger and this S inverse whole dagger.

Now, S inverse is already dagger. So, it is S dagger, I can replace to inverse A dagger is a I already know S inverse is S dagger. So, S is S dagger. Finally, I have S inverse A, S this thing is nothing but B because this is B. So; that means, B dagger is equal to B. So, what is the meaning of that the meaning is the meaning is if I have a Hermitian matrix in one coordinate system under one basis I make a basis transformation.

When I make a basis transformation my new basis is also orthogonal or orthonormal to each other in order to ensure the orthonormal transformation, I have a matrix i i have a transformation matrix which is unitary in nature. So, S dagger is equal to 1. So, under unitary transformation; so the conclusion is under unitary transformation Hermitian under unitary transformation, Hermitian matrix remain remains Hermitian matrix.

So, you make a unitary transformation a Hermitian matrix is giving to you after making unitary transformation you have a new matrix say B. So, B is also Hermitian that is the that is the conclusion in this entire this things. So, this is a small thing I wanted to mention that what is the property apart from that, there are few lay under similarity transformation I have few other properties. For example, A and B are A and B are related by the similarity transformation.

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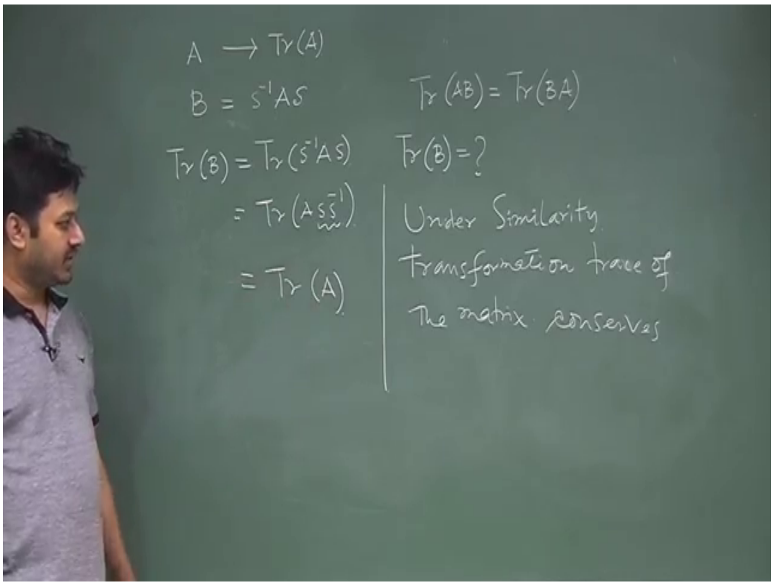

So, a is a matrix given B is given by S inverse a S. Now I need to find out what is the trace of these things. So, I know trace of AB is equal to trace of B A. So, here trace of a is this after I have a matrix A whose trace is given by this. I make a similarity transformation with this and try to find out what is the trace of B. So, my goal is to find out what is the trace of matrix B.

So, if I do that trace operation over B, then I need to do the trace operation both the side. I write it like this; this is nothing but unit; so trace A. So, trace of B become trace a so; that means, what is the conclusion with this small thing that under similarity transformation trace of the matrix conserved trace of the matrix conserved. So, the conclusion is under similarity transformation trace of the matrix conserves.

Next thing is again a very short thing that is a matrix is given the corresponding similarity transformation is this is what about the eigenvalue or eigenvector of B.

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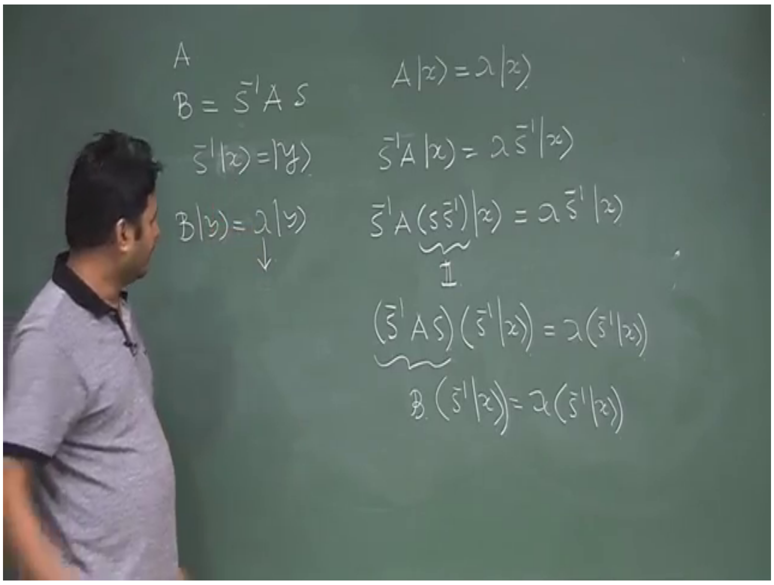

So, a has a eigenvector like x with the eigenvalue lambda, suppose if this is the case then what will be the Eigen value of B. So, B if I operate over x, I am operating a let me let me do in that way try to. So, a is here S is related to that. So, I write this thing in this way S inverse a x is lambda S inverse x this things.

Now, what I do that S inverse A, here in this place I put 1 S, S inverse I can do that because S, S inverse is unity this is unity. So, lambda S inverse will remain this this quantity. I can now write S inverse A S S inverse x lambda S inverse x this quantity S inverse A S is nothing but B because B is related to this.

So, B S inverse x x equal to lambda S inverse x S inverse x is also another matrix another vector. So, if I say S inverse A S inverse x is y. Then B y is lambda y. This is nothing but a eigenvalue equation eigenvalue x is lambda, but eigenvectors are different. So, what is the meaning of that I have a vector, I have a in my hand lambda is the eigenvalue of a x is a eigenvector of that.

I make a similarity transformation and try to find out if B is my new matrix under similarity transformation what is the value what is the eigenvalue of that, I find the eigenvalue will remain same that of A, but the eigenvector is changed an eigenvector will be something like this. This is again a small thing. With that I would like to conclude this class because in the next thing will be a very important topic which is rank of a matrix, but I do not have that much time to explain that in this class. So, in the next class we will start a very important concept of rank of a matrix. And also like to learn what is the meaning of Cayley Hamilton theory, and how the using Cayley Hamilton theory. I can find the inverse of this is the another way to find inverse and other physical meaning of rank which is also very important in terms of physics.

So, with that let me conclude my class here. So, see you in the class, where we try to cover rank of a matrix and another thing is the Cayley Hamilton theory if time permits. So, see you in the next class.