Mathematical Methods in Physics-I

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Lecture - 15
Diagonalization of a Matrix

Student, welcome back to the next class; in the previous class we know very important concept that if a matrix is normal then they produce the orthogonal eigenvectors. Today we will go with a very important concept.

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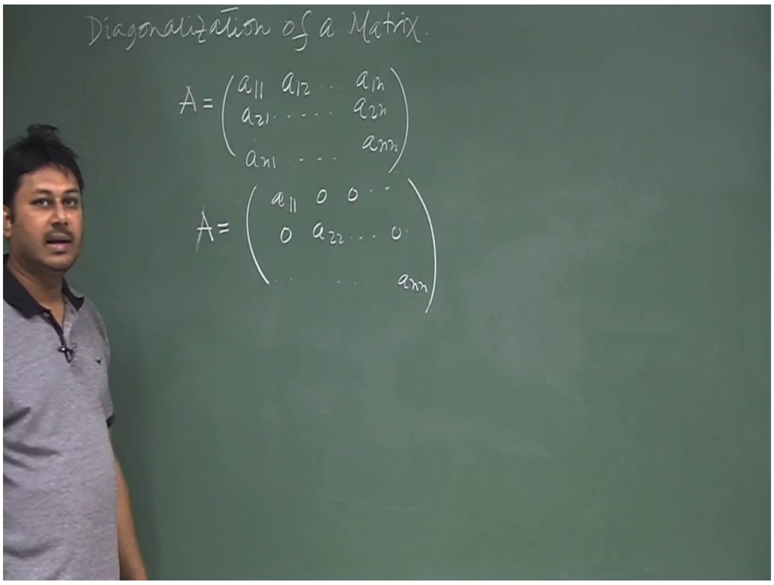

We will start a very important concept which is called diagonalization of a matrix which is equally important; diagonalization of a matrix. What is mean the diagonalization? So, a matrix is represented like this, these are the elements. This is not diagonalize, because all the elements here, if it is nonzero then this is not diagonalize. The diagonalize matrix is something where only we have the diagonal elements, other elements are 0. For example, this is; this is the diagonal matrix. This is general matrix and this is the diagonal matrix.

Now, the question is, it possible to diagonalize a matrix. So, let me first write the thing then.

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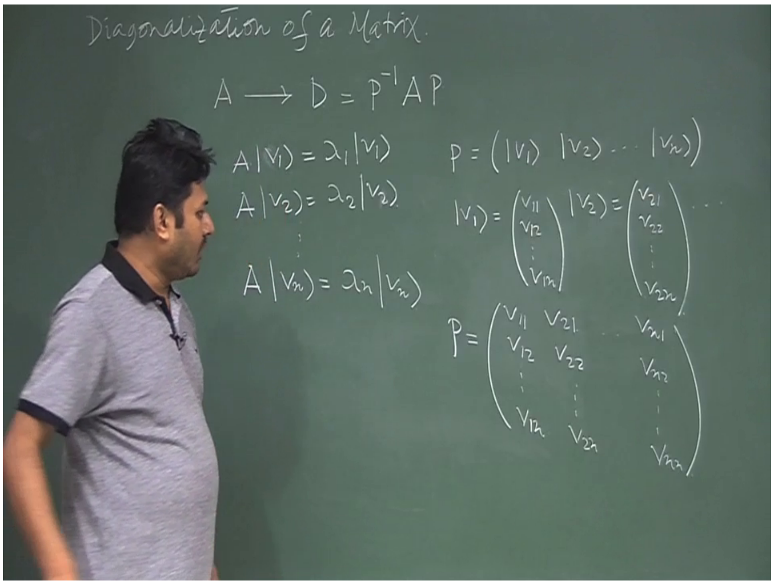

So, a matrix given matrix A can be diagonalized say D; a matrix A it is diagonalize to a matrix D with a transformations similarity transformation; now you should appreciate what is the meaning of similarity transformation with suitable P, so if I able to find A P. So, in last class we find that these A and D are called similar matrix because they are related with a similarity transformation which is this that we defined already in the previous class. Now this similar matrix has a property important this D matrix as a important property that it has to be diagonal; so in order to diagonal that I need to have a proper P to that.

So, what should be the P that is a important question here; what should be the P, how to generate a P? P is nothing but let me say v 1, v 2, v 3; v 1, v 2, v 3, v n these are the corresponding eigenvector of A, I know that. I will try to find out what is P. So, before that I know this is my eigenvectors.

Now, if I form a P with putting this vectors like this column vector here this column vector here this column vector here then I can form a matrix p this is the p which can diagonalize so; that means, I will need to form a p which is made of the corresponding eigenvectors in this form. So, now, let me explain more; what is v_1 component wise? It is, say v 11, v 12, v 1n, what is v 2, say v 21, v 22, v 2n and so on. Now, I will put this v thing here. So, my P matrix will have the form v 11, v 12, v 1n these are the nothing but the corresponding vectors, eigenvectors element, v 21, v 22, v 2n; v n1, v n2, v nn; so, this is my P this ok.

Now, I have my P matrix which is formed by the matrix which is the nothing but the eigenvectors.

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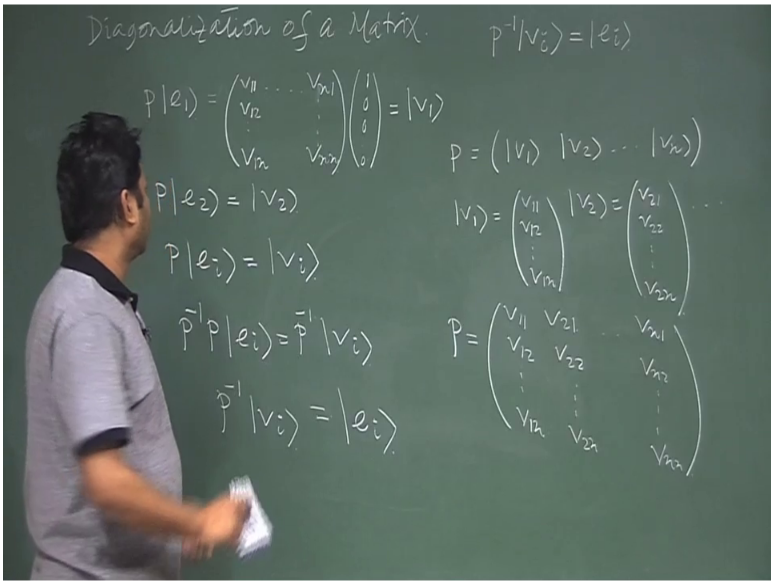

So, now few things, so P if I operate over e i what I am getting, let me write it clearly. My P is v 11, v 12, v 1n; v n1, v nn; I operate with e i. P is a matrix that I operate over e i. So, just I am try to find out what is the value of this things which is interesting. e is a natural basis so, I will have this. As a result what I am getting; this multiplied by this or the, this multiplied by this, only this column will retain other things will vanished. I will have v 1 in the similar way if I do e 2, I will have v 2 and so on. In general, P e i if I operate I will get a vector v i, which is the i-th eigenvector.

So, now another important thing what is P inverse P e i which is P inverse v i; P inverse v i is coming out to be e i which is also an important outcome. P inverse of v i is nothing but e i with this, I required that because I need to. So, let me write this outcome here, P inverse of v i is e i. I will going to use this expression to prove that they will going to diagonalize this, similarity transformation I will going to diagonalize.

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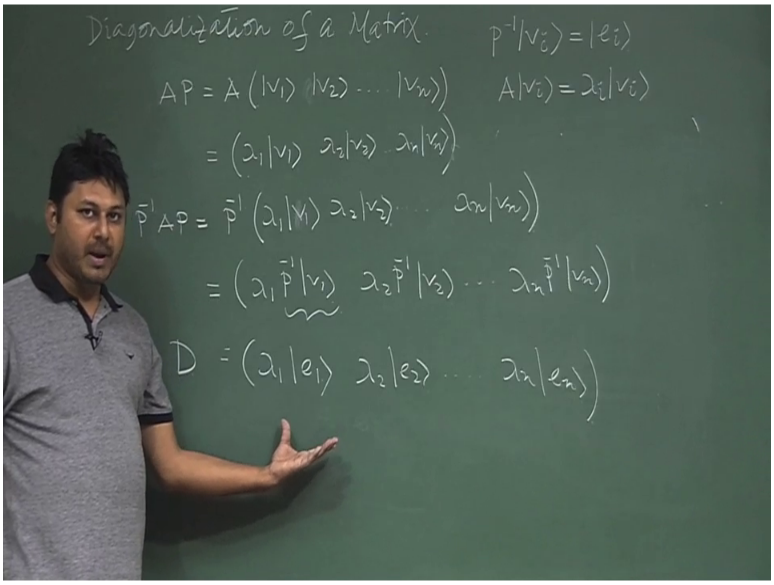

So, now after having that, let us now let us now write what is AP now. AP is a A matrix multiplied by the P matrix; P matrix I write in this form v 1, v 2, v n. P is a matrix which I define like this, which is a combination of all the column matrix, that I have which is the Eigen corresponding eigenvector of A. I have this matrix in my hand now I operate A over P. If I operate this things, what happened A will operate over that, it will gives you lambda 1 v 1, lambda 2 v 2 to lambda n v n, it will give in this because A is a Eigen, v are the eigenvectors, so, v i is lambda i v i, no problem in that. I will have another vector another matrix form; this is also a matrix form with this condition.

Now, if I operate P inverse over AP because this I defined as my diagonal matrix D. If I operate these things, then what happened P inverse will be operated over lambda 1 v 1, lambda 2 v 2 so on. Let me erase this because we are not using this anymore, because we know how these things are forming. I got this; now P is operated over this lambda is a constant, if that is the case I can write lambda i P operated over v i, lambda 2 P inverse operated over v 2, lambda n P inverse operated over v n like this. Now this quantity I already figure out earlier, this is the same quantity, I will just replace this. So, it will be lambda i e i, lambda 2 e 2, lambda n e n. This is my according to my notation this is D matrix, I can write this D matrix in this compact form, now I will expand this form and you will readily understand what is going on.

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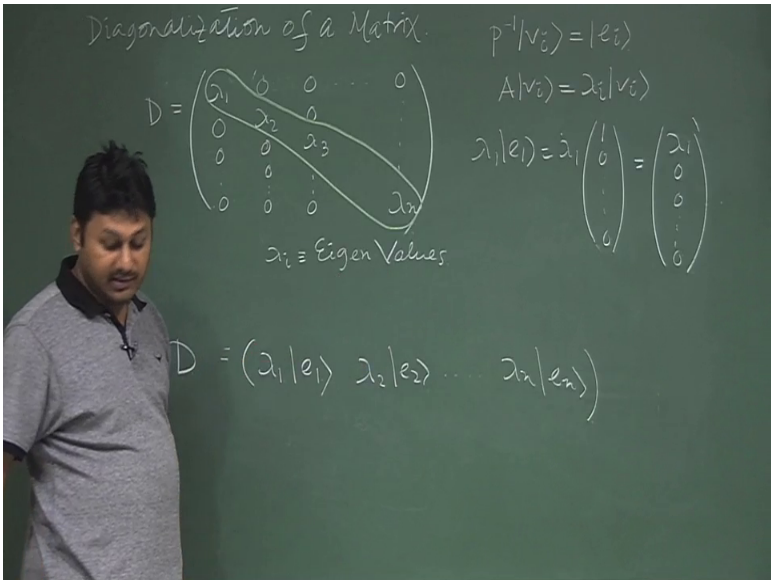

So, I get D matrix like this I just replace e i to the matrix form. What will be lambda i multiplied by e i, lambda i, let me write it here first lambda i e i is how much? Lambda i is a constant multiplied by 1 0 0 0. So, it will be simply lambda 1 0 0 0.

So, my first thing is 0 0 0. What about second thing? In the second case the first term will be 0 e 2 is here, instead of having 1 here I have 1 here, so, lambda 2 will be placing here. So, it will be lambda 2 0 0 0; 0 0 0 in case of and so on. So, you can see that this D matrix is now diagonalized. These D matrix is now diagonalize, not only that we get a very important information out of that; this diagonal elements, whatever the diagonal elements I have here, this diagonal elements are the corresponding Eigen values.

So, diagonal elements is lambda i are the Eigen values. Essentially, what we are doing, we diagonalize, we find a recipe to diagonalize a matrix, not only that if I able to diagonalize the matrix, then in the with the similarity transformation that we have; we need to find AP and also AP which contain the inverse of that. It is not necessarily that always AP inverse is exists. There is a possibility that it will not going to exist. So, it should be a nonsingular matrix.

So, if I able to figure out my P properly with the eigenvectors then I can diagonalize and this diagonal element will be the corresponding Eigen value. This is the recipe now we will going to use this recipe to figure out, try to diagonalize something one matrix. So, let me try to do that quickly.

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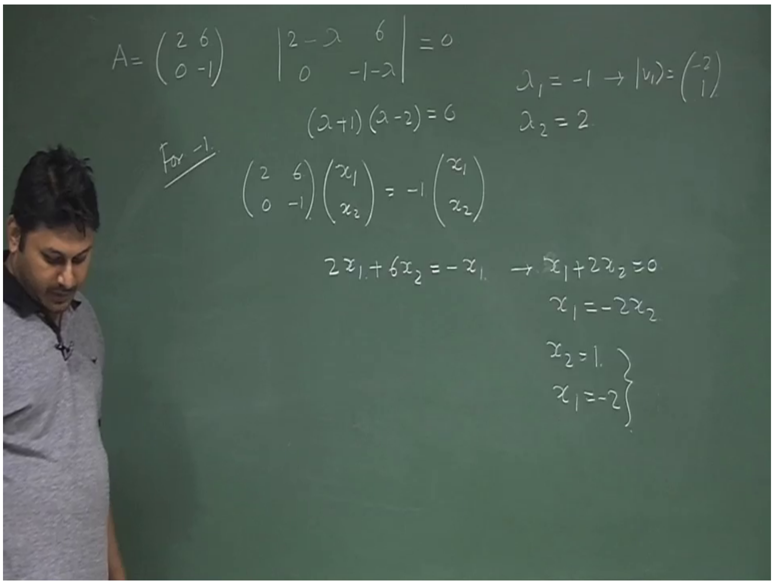

So, I need to choose a suitable matrix. Let me choose a suitable matrix say A is 2 6 0 minus 1. This is the matrix I take. I need to diagonalize this matrix. First thing is that I need to find out the Eigen values and eigenvectors. So, 2 minus lambda 6 0 minus 1 minus lambda is equal to 0. I have a straight forward equation lambda plus 1 by lambda minus 2 is equal to 0 this. So, my Eigen values are lambda 1 is minus 1, lambda 2 is 2.

Again you cross verify 1 and 2, if I add it will be minus 1. Here also I have the same quantity, so trace is matching. If I multiply it will be minus 2, the determinant of this matrix is also minus 2. Now I need to find out the eigenvectors out of that. So, 2 6 0 minus 1, x 1 x 2 say I find it for, I want to find it for say minus 1; minus one x 1 x 2. What expression I am getting? 2x 1 plus 6x 2 is equal to minus of x 1. If I take it here, then from that I will have 3 of, right way it will be 3 and then 6 it will cut. x 1 plus 2x 2 is equal to 0, x 1 is equal to minus of 2x 2.

So, my first eigenvector for this if I put x 2 is equal to 1, then x 1is equal to minus of 2. My first eigenvector say v 1 is coming out to be minus 2, 1; coming up to be minus 2, 1.

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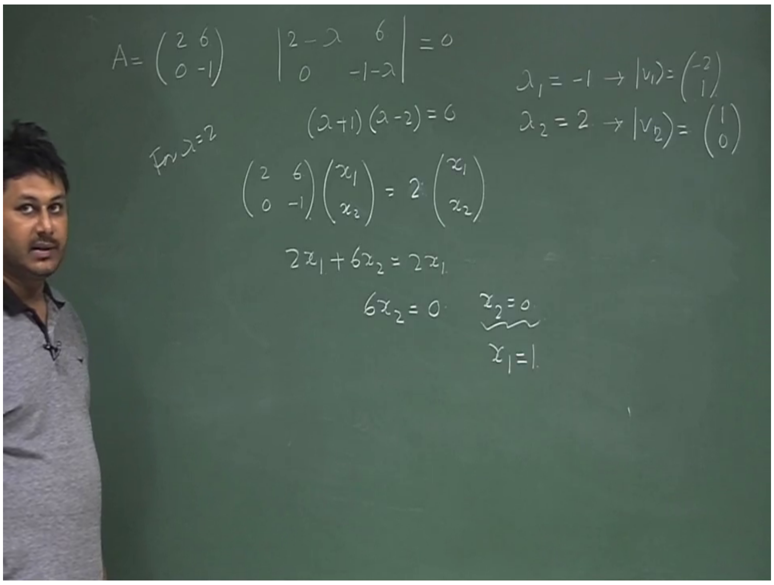

Now I will do the same thing for other; for lambda is equal to 2 for lambda is equal to 2, I need to replace this with 2. Now, I need to calculate once again. So, 2 x 1 plus 6x 2 is equal to 2 of x 1. Here, we find x 1, x1 is cancel out. So, I find one 6 x 2 is equal to 0, so that means; one equation I get x 2 equal to 0. If x 2 equal to 0 in the second expression if I do, then I will also get the same thing where x 2 is equal to 0 equation will come. So that means, I can put x 1 arbitrarily say I put it is 1, now, this equation, this eigenvector v 2 is something like 1, 0 it is like 1, 0.

So, now after having these two eigenvectors here I can form my P matrix. Let me now form my P matrix. What is the mechanism or what is the recipe to form the P matrix.

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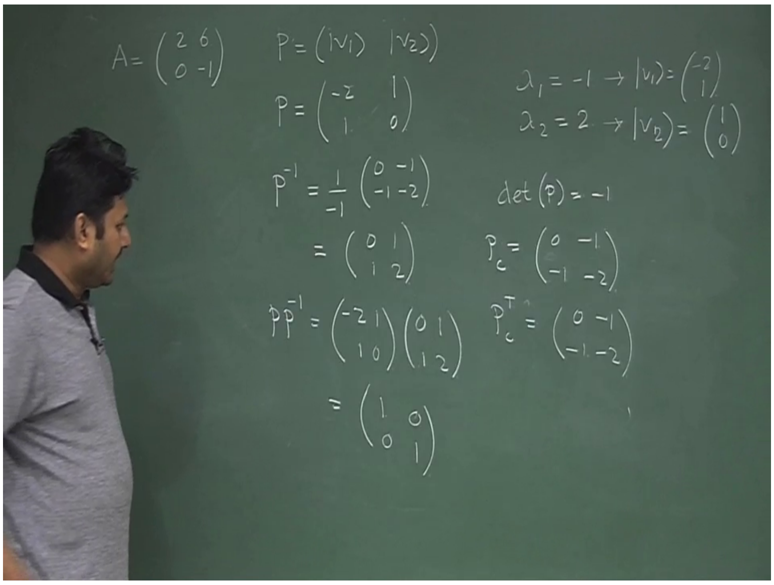

P matrix should be like v 1 v 2. Here, my P matrix v 1 which is minus 2 1 and v 2 is 1 0. So, I can form my P matrix ready. Next thing is to form the P inverse; inverse you know in order to find inverse, what it you need to make a ad joint and then divided by the det of this things. First, I need to find out det of P, I believe all of you know how to calculate the inverse of a matrix, I am not going to spend time on that. So, det of P is equal to minus 1 and P transpose if I make a P co vector matrix is something like this; it will be 2 for 0, 1 it will be minus 1, when I do this will cut and this value will put here, this value will be negative sign will be here, for this value I will cut this and will be negative sign put here and for this value I will put this here.

So, my value is this; now I need to make a transpose of this. Transpose of this it will be 0 minus 1, minus 1 minus 2, the same thing. If I do then P inverse is come out to be 1 by det of this things which is minus 1, multiplied by 0; minus 1, minus 1, minus 2 and my P inverse supposed to be 0 1 2. After having the P inverse, the next important thing is that please check whether whatever the P inverse you are getting or I am getting here, is the correct thing or not. So, first I will going to check that P inverse is minus 2 1 0 and P inverse is 0 1 1 2; minus 2 into 0, 1 into 1, I have 1 here, minus 2 into 1, 1 into 1, 2 into 1 this is 0, 1 into 0, so it is 0, 1 into 1, so it is coming. P inverse; whatever the P inverse I get is the correct one.

So, I now get P inverse, once I get the P inverse then my job is almost done. P I get, P inverse I get, my job is almost done. Next, what I will do that, I will just plag it this to find out what is.

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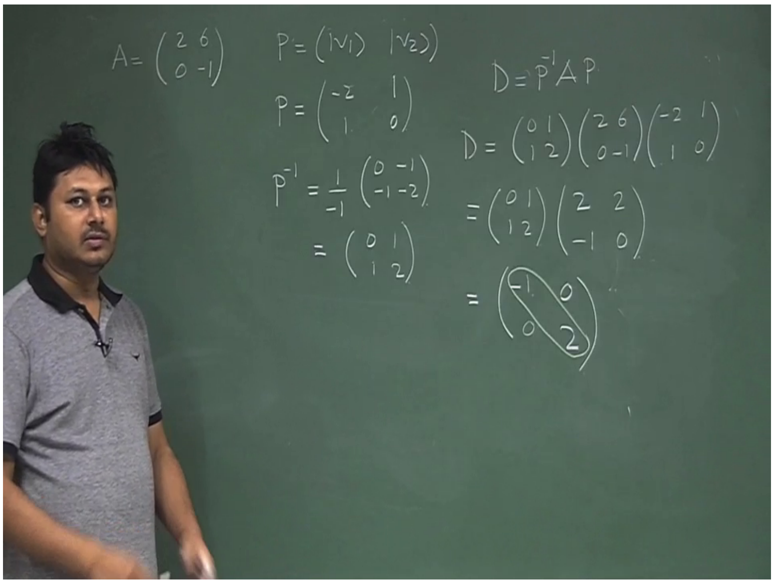

So, my recipe is D is equal to similarity transformation of A like this. I find my P inverse, I find my P, I will just plag it there and try to find out. My D matrix which should be the diagonal matrix should be P inverse which I get is 0 1 1 2, A which is my original matrix 2 6 0 minus 1 and finally, P which is, minus 2 1 1 0. I need to calculate that, this will give me the diagonal matrix if I am right. 0 1 1 2 this is 2 into minus 2 it is minus 4, 6 minus 4 it is 2, 2 into 1, 2; 6 into 0 so, it is 2. 0 into minus 2 0, 1, it is minus 1 0 into 1 into so, it is 0 and then 0 into 2 this into this it is minus 1, 0 into 2 this is 0, 1 into 2 this into 2 into minus 1 it is again 0, 1 into 2 into. So, it is a diagonal matrix; first conclusion.

Second thing is that please look to this elements which are the eigenvectors that I figure; out one is minus 1 and another is 2. So, I find a recipe which is very important that using which you can figure out, how a matrix can be diagonalized the same recipe can be applied for any dimensional matrix as I mention in my last class every time I am taking it 2 by 2 matrix, because it is easier to calculate in the blackboard, but when you do that sit down in your room and try to calculate the, you can do that for 3 by 3 matrix also. The recipe will be exactly the same. So, once again I remind what actually I do here.

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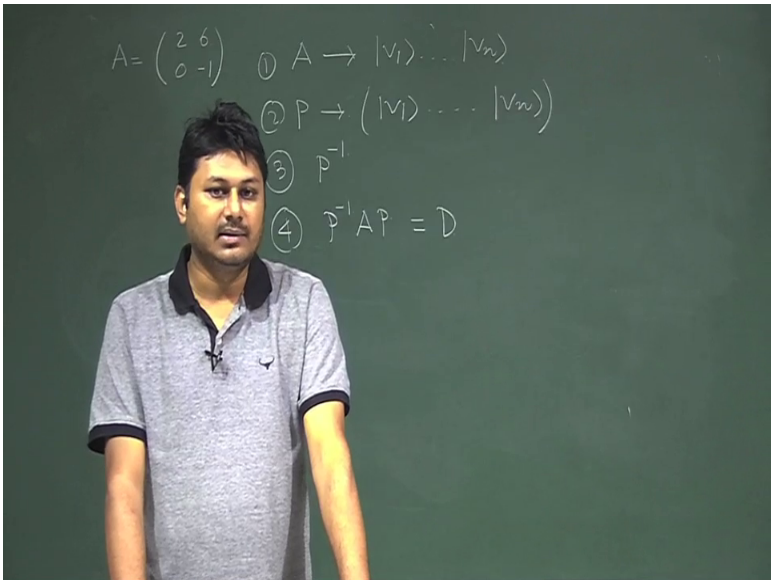

That if a matrix is giving to you first you calculate the corresponding eigenvectors and then from this eigenvectors you can form a matrix P next step this is a step 1, this is a step 2. You find you can construct P; step 3: you need to find out P inverse. And step 4; you make a similarity transformation of this P inverse AP, which gives you the diagonal matrix. where the diagonal elements is nothing but the Eigen values of A, this is related to a similarity transform and P is a matrix, that you figure out with this recipe.

So, with this note today, I will like to conclude my class here. So, one this class you learned a very important thing that, how a matrix can be diagonalized. As I mentioned that, first in order to do that you need to just figure out your eigenvectors and from that eigenvectors you need to form a matrix called P. And with the similarity transformation of the P you can diagonalize your matrix.

So, with this note, let us conclude this thing. In the next class we will carry forward with more on the matrix things some specific matrix and similarity transformation. And all this things and also try to understand one important concept is called rank of a matrix or another important thing which is Cayley Hamilton theory if time permit.

With that let us conclude this class. See you in the next class.