Mathematical Methods in Physics-I
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Lecture – 14
Normal Matrix

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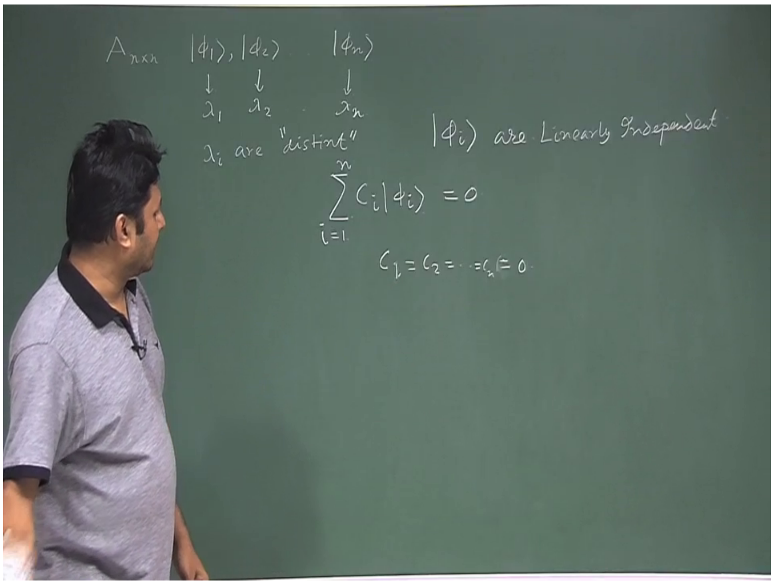

So welcome back student; in the last class, if you remember; we ended with a very important conclusion that if a matrix a which is n by n matrix as a set of eigenvectors phi 1, phi 2, phi n; this is a set of eigenvector of a and corresponding lambdas are lambda 1, lambda 2, lambda n when lambda 1, lambda 2, lambda n are the Eigenvalues of corresponding eigenvectors for the matrix a.

Then if lambda is are distinct if lambda is are distinct, then we have a relationship between this eigenvectors; what relationship we got that they are phi is are linearly independent to each other; that means, if this is true if they are distinct, then I can have a relationship like this and this relationship holds only when C is or c 1 is equal to c 2 is equal to c n is equal to 0 this is the standard definition of linearly independent.

So, now we will go further we will go further with this. So, now, we know that if a matrix has eigenvectors phi 1, phi 2, phi 3 with this thing Eigenvalues, then this eigenvectors are linearly independent. Now we go further because linearly independent eigenvectors can be orthogonal to each other and is there any special condition. So, that the linearly independent eigenvectors are orthogonal now we try to find it out. So, now, before that we need to know; what is normal matrix. So, this is normal matrix.

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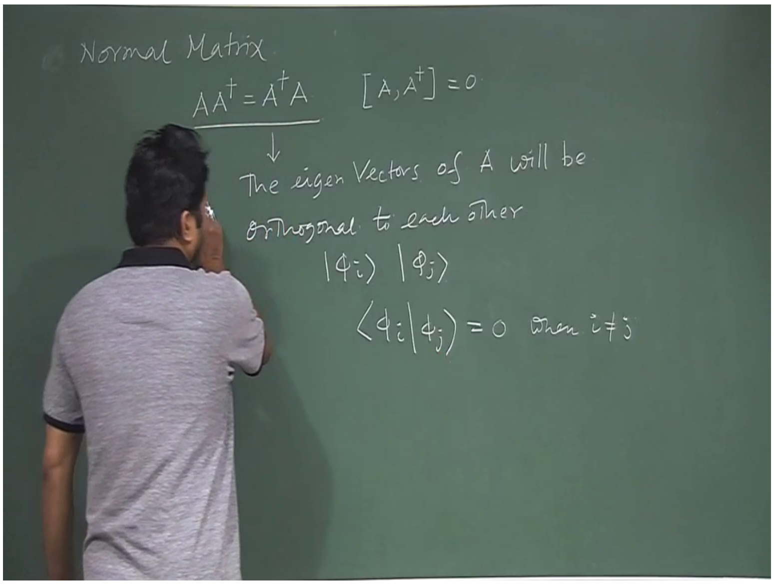

So, before that we need to find out; what is normal matrix, we need to know; what is normal matrix. If A A dagger is equal to A dagger A, then the matrix A is said to be a normal matrix, I believe you know what is the meaning of A dagger; A dagger is a transpose followed by complex conjugate of each element which I defined in my last class. Now if this is the case for normal matrix; that means, A A dagger come it to each other that is the another thing we know from that.

If a matrix is normal, then the corresponding eigenvectors; the Eigen vectors of A will be orthogonal to each other eigenvectors of A will be orthogonal to each other; that means, if phi i and phi j suppose; these are the 2 eigenvectors of matrix A which are normal, then they hold this relationship not critically this relationship; rather I should say this hold this relationship. When i is not equal to j; I do not know whether it will be 1; when i equal to j or not in that case this matrix has to be normalized, but in general they will be orthogonal; that means, the 2 matrix will be perpendicular to each other; if i is not equal to j; this 2 vectors will be perpendicular to each other; when i is not equal to j to prove that it is a tricky prove.

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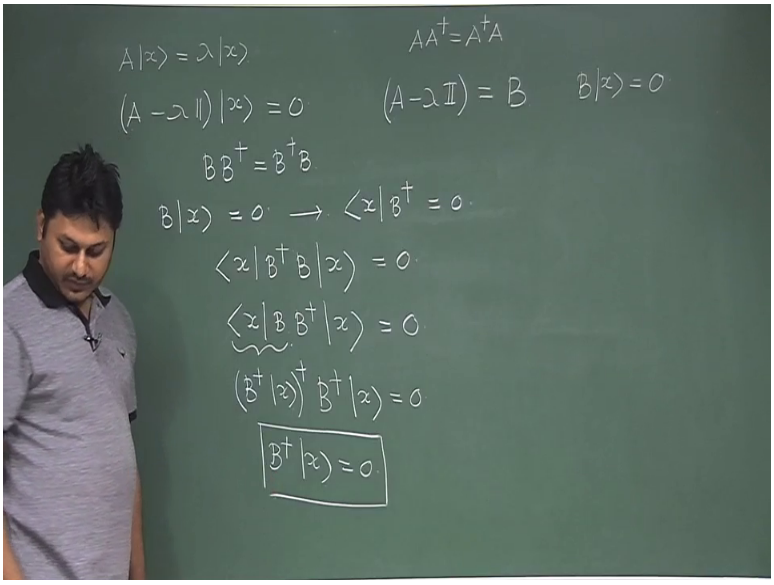

So, let us try to find how to prove. So, my condition is A A dagger is equal to A dagger A this is given.

Now, I write A x is equal to say lambda x; this is a Eigenvalue equation for A. Now I can write this equation as A minus over x is equal to 0 A minus lambda i is also A matrix; I called this matrix as B. This is another matrix I called B. So, from this equation, I can write here B of x is equal to 0.

Now, I can show that B B dagger is equal to B dagger B, if A followed this rule, you can you can do that quite easily you can check that quite easily, I am not going to do that I assume that this is; this hold the treatment is straight forward you just make A B dagger and then multiply B with the B dagger, you will find some elements and then B dagger and B you will also find some elements. So, both that right hand side and left hand side you will find it is equal.

So, now next is what I will do that B x; I know it is 0 if I make a dual conjugation of that; then it will be x B dagger which is equal to 0. Now with these 2 equation, I also write one equation; I just make these things operate over this from the left hand side and I can get this things this things; I will operate over this from the left hand side and I will get this things now note that B dagger B is equal to B B dagger. So, I will going to change that.

This is still hold if this is hold then because these things and this things are same as I mentioned here; next things what I will do? I will do a trick; this part I will write in a different way; this part I will write in a different way, I will write it as B dagger x whole dagger; this part I can write because if you make that whole thing into dagger. Then x will converted onto this B dagger will converted to B because there is another dagger you know that a of dagger dagger A dagger whole dagger its nothing, but A.

Using this property, I can write in this way and the rest part is same as this. So, this is a operator over a vector giving a vector which is dagger and this is the same vector which is un dagger. So, dagger vector multiplied by un dagger vector; that means, this is nothing, but a norm of a vector which is generated by operating B dagger over x. Now this is equal to 0 that eventually means that eventually means from that I conclude B dagger x is equal to 0 this is my first conclusion. Now, what is B dagger? So, I get one important thing what is B dagger. So, let me erased it.

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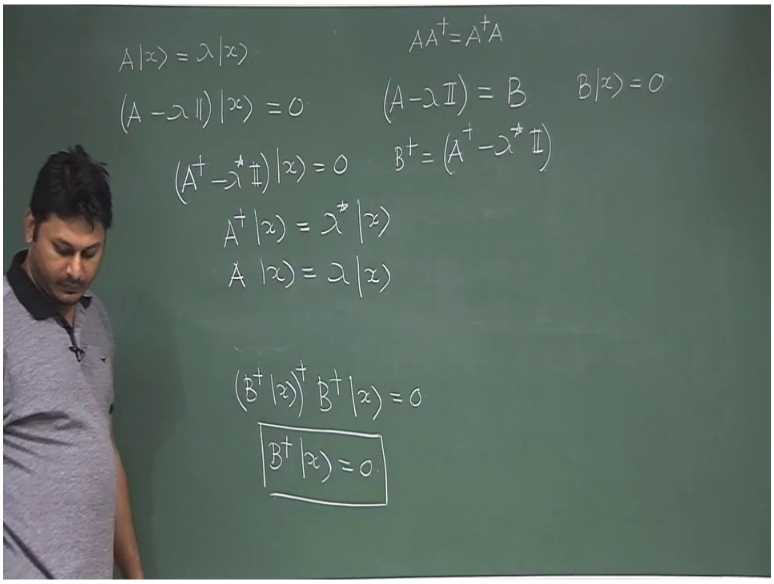

What is B dagger? B dagger is if I put B dagger here, it will be A dagger minus lambda star; I mind it, I also make a dual conjugation of this when I make a dual conjugation you know that whatever the constant is; here I need to put in principle I need to put the complex conjugate of that; i is the identity matrix if I even if I change the identity matrix to A dagger, it will not going to change anything. So, it will be remain same.

A dagger will be change because I am making a dagger. So, now, my equation is A dagger minus lambda star i that operates over x is equal to 0 or in other way I find another important conclusion that it is x star of x. So, what essentially I get try to find out a is a matrix I operate over x, I find a Eigenvalue of that; this is eigenvector. Now I want to find out; what is the eigenvector or Eigenvalue of A dagger operator; when A is a normal matrix, if I do that then you find that this is A dagger operate over x is nothing, but the complex conjugate. So, Eigenvalue is nothing, but the complex conjugate of the previous Eigenvalue and with the same eigenvectors. So, I will have these 2 equation in my hand; if I write side by side.

So, now I will let me write it in somewhere here because this conclusion is important.

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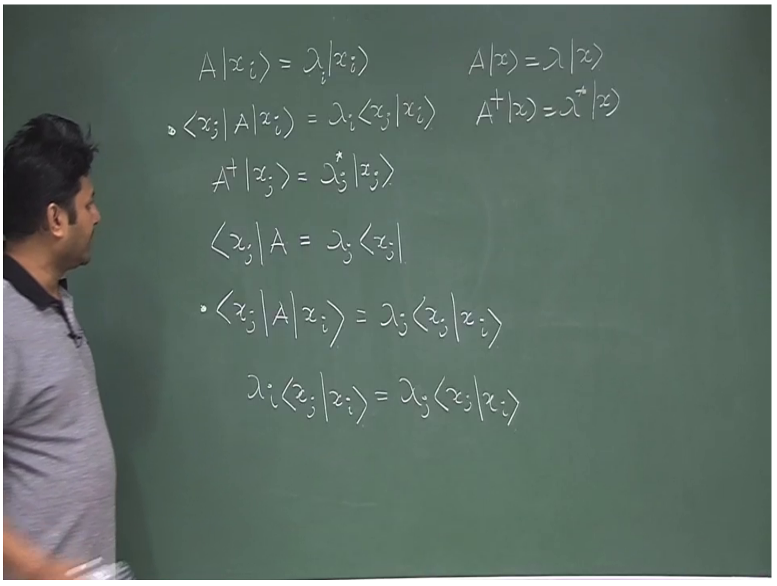

So, A x is lambda x A dagger x is lambda star x that I got. Now let us say A x i is equal to lambda x I; I put 1 Eigen; one of the eigenvector, this is the general form; I just change and say that ith eigenvector, I am try to calculate. So, here for that this is my Eigenvalue. So, I need to put a suffix for Eigenvalue there also. So, this is the Eigenvalue corresponding to the ith eigenvector. Now I operate this with j A x i and I will get lambda i x j x, I get this element this is j, this is I; I just operate x j and I will get something here for this also, I can say this is I operate over jth; I have this in my hand, I will again make a dual conjugation of this; if I do then my equation simply changed to something like this right.

Now, here one I think; I should note that when I make this is this; star will gone because A dagger x i is equal to this star is there. So, this star will be here. Now I am making a dual conjugation; when I making a dual conjugation, then this star will now remain non star so; that means, the complex conjugate again I will make a complex conjugate of that. So, this will be a non star thing and I will have this. Now in this case, I will operate for this equation with x i like this. So, if I note; then these 2 equations; these equations and these equations are same because left hand side; this left hand side are the same thing. So, right hand side should be equal. So, if I make right hand side equal, then let me write it here, then I have x i x j x i is equal to lambda j x j x i this equation; I have from this equation; I can readily find my desired result; now I put this things this side.

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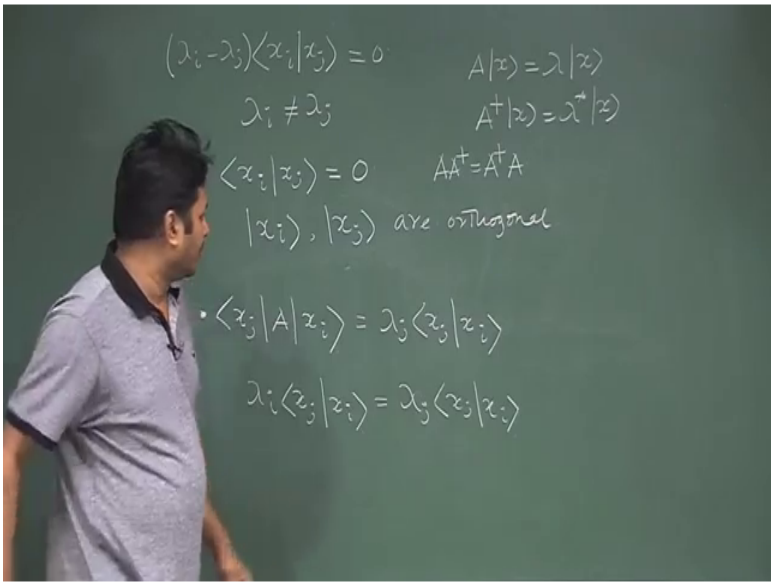

So, lambda i minus lambda j is x i x j is equal to 0 for the distinct Eigenvalues; this condition holds and the rest thing that I have is x i x j is equal to 0 which essentially means the inner product of the eigenvector of i and j ith eigenvector and j th eigenvector is 0; that means, x j are orthogonal. So, prove this; we can prove one very important conclusion that if a matrix is normal; that means, if I say A is equal to A dagger A; A dagger; let me write once again to make it familiar if a is coming to it a dagger.

So, this is a special matrix call normal matrix in normal matrix, I will always generate a set of eigenvectors which are orthogonal to each other which is very important because we always want orthogonal set of vectors because orthogonal set of vectors is a potential vector which can be used as a basis. So, this vectors whatever the vector I get is since it is a orthogonal to each other I can make it orthonormal by making the by dividing it with the norm of this vector and it will be a potentially a basis it can work as a potential basis. So, this is important; they are linearly independent; no doubt about that on top of that another extra information we find that they are orthogonal.

So, now let us try to find some example; a simple example; say example 1.

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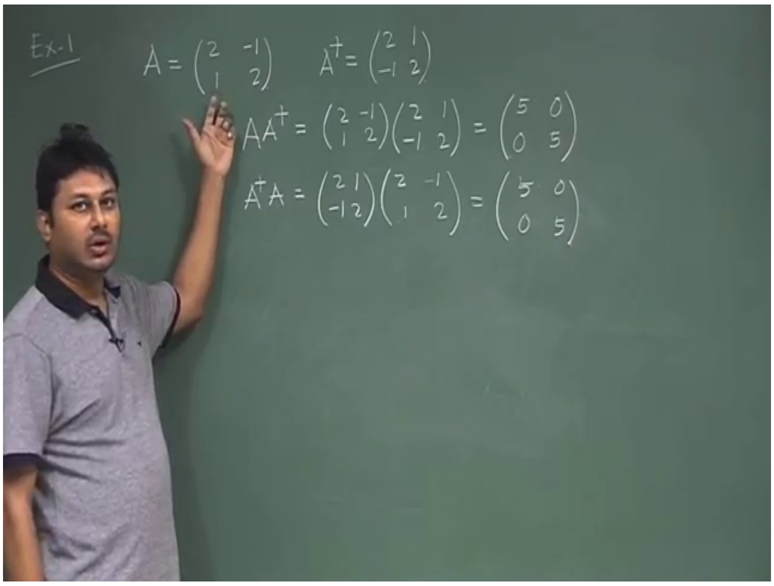

Say I put a matrix A just check whether say 2 1 minus 1 2; this is A matrix which is given to me is this a normal matrix the question is this A normal matrix; first, we need to find out whether this is a really normal matrix or not. So, A dagger will be 2 minus i 1 2 this and now I need to find out whether they are normal or not. So, I will first find this quantity A A dagger 2 minus 1 1 2 2 minus 1 here 1 2 here what I get 2 into 2 4 1 into 1 1. So, 2 plus 1 is 5; 2 into 1 1 into 2 with a negative sign. So, it will be 0 1 into 2 2 into 1 with a negative sign. So, it will also be 0 1 into 1 2 into 2 it is 5, some diagonal matrix with diagonal element 5; no problem with that.

Now, I need to figure out what is the value of this when I reverse the ordering is this the same thing 2 1 minus 1. So, I need to write this one here 2 into 2 1 into 1 5 here, fine seems to be here equal 2 into minus 1 1 into 2 it is 0, fine, 1 into 2 2 into 1 0 again fine, 2 into 1 into 1 minus 1 2 into 2 so; that means, it is again 5. So, these things and these things are same. So, I cross verify that this is matrix which is normal. So, another example just I change in. So, this is a normal matrix.

So, now I will just change say this is this a normal matrix it is it is simple to check that this is also a normal matrix because if I make A dagger of that.

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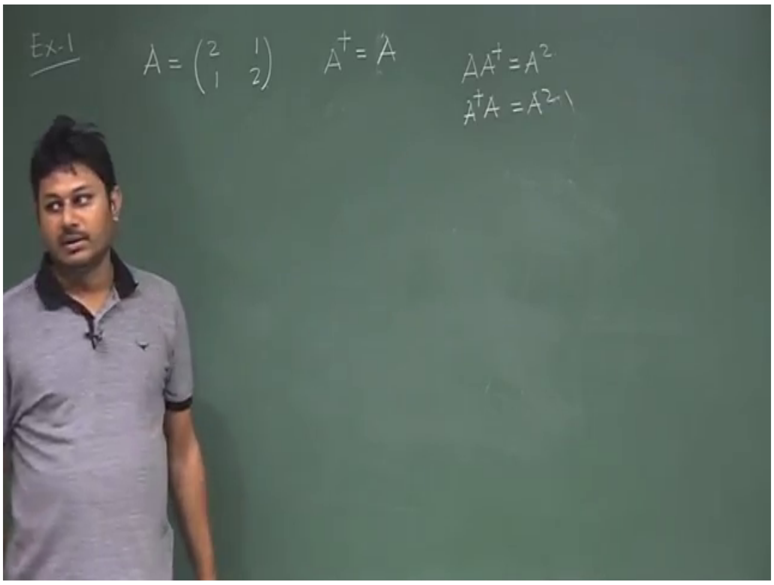

It will be simply the same matrix; I will have if I make A dagger of that. So, it will be a same matrix 2 1 1 2. So, there is no change so; that means, A A dagger is nothing, but a square and A dagger A is also a square. So, no need to verify further that whether it is a normal matrix or not; now the question is if it is a normal matrix, then what should be the eigenvectors of this first second thing; this eigenvector should be this eigenvector should be orthogonal to each other. So, we need to show that also that these are orthogonal A; they produce the orthogonal eigenvectors.

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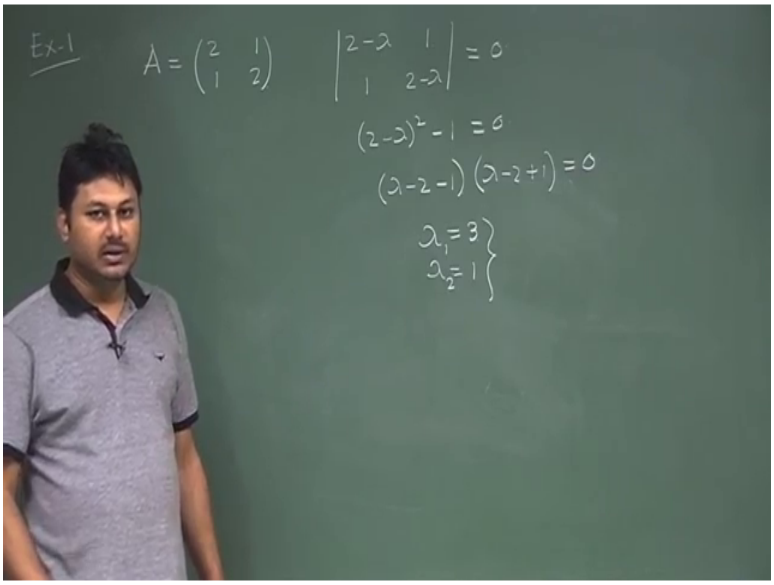

So, we need to check. So, the checking is same lastly we did few problems. So, first I need to find out the characteristics or secular equation which is this equal to 0. So, 2 minus lambda square minus 1 is equal to 0. So, I find lambda minus 2 minus 1 lambda minus 2 plus 1 is equal to 0 A square plus A square minus B square formula simple.

So, from here I have lambda equal to 3 1 Eigenvalue and lambda equal to 1; another Eigenvalue say this is my 1 and this is my 2; these are the 2 Eigenvalues that I derive readily; these are distinct note that these are distinct also. Also we can cross verify that the multiplication of that I show lastly that if I add this 2 things 3 and 1 every time when you find out the Eigenvalues, then readily you can check whether this Eigenvalues are correct or not this is a 2 by 2 matrix simple, but if it is a 3 by 3 or 4 by 4 matrix, then you are confused with the fact that whether the Eigenvalues that you are getting is the correct or not, but the simple thing is that add these 2 if we add these 2 3 and 1, it will be 4 and the trace of matrix will be same. So, here trace of the matrix is 4. So, it is correct also another if you multiply this 2; in this case, if you multiply these 2 then it will be same as the determinant. So, the determinant is 2 into 2 minus one. So, determinant is 3 and if I multiply lambda 1 and lambda 2 it will be 3.

So with this, I can find the corresponding eigenvectors which is a next important thing and need to show that they are forming.

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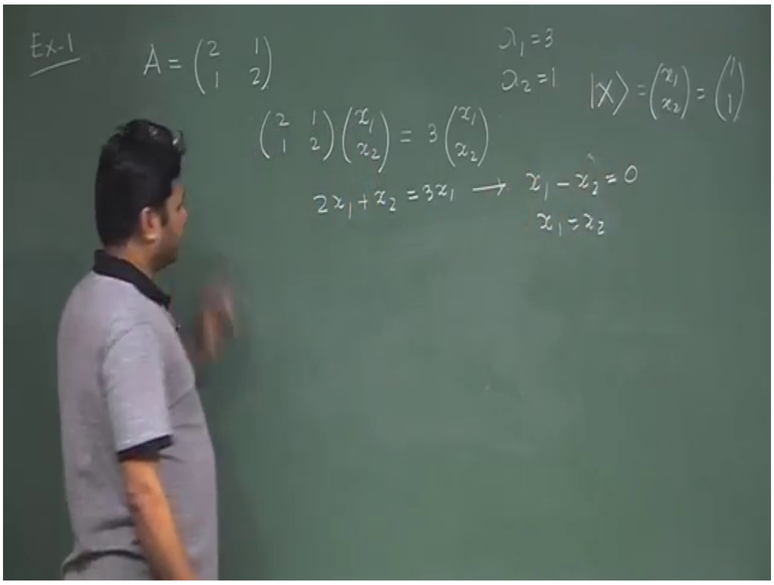

So, let me write it here lambda 1 is equal to 3 lambda 2 is equal to 1 and let us find. So, for lambda say equal to lambda 1 is equal to 3, I need to find out the eigenvectors. So, I write it here. So, 2 1 1 2 x 1 x 2 is my corresponding component of the eigenvectors is equal to 3, I know the Eigenvalues then this. So, from that I can have one equation 2 x 1 plus x 2 is equal to 3 x 1; the next equation will also be the similar kind of expression, it will not going to give any extra information. So, from the first equation, I can have a relationship between these 2. So, what is the relationship the relationship is if it is coming this side. So, x 1 minus x 2 is equal to 0 or x 1 is equal to x 2.

Now, from that I can write another thing from this x 1 is equal to x 2. So, I can make a eigenvector I write this my first eigenvector say x x 1 not x 1 say y make a big x big x which whose component is x 1 and x 2 the eigenvectors is 1 1, I find out this is for the Eigenvalue 3. Now I will do the same thing for Eigenvalue 1 because there are 2 Eigenvalues are there. So, second Eigenvalue is one. So, I need to put one here.

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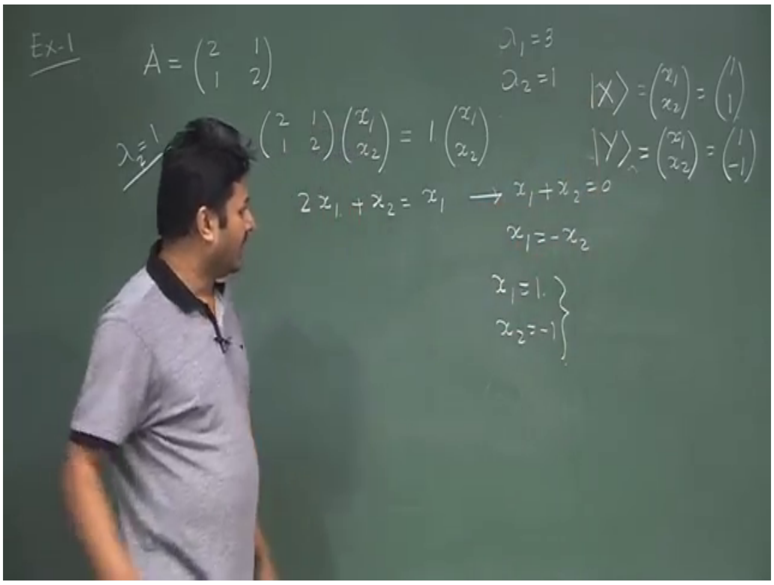

If I do that then I will find 2 x 1 plus x 2 is equal to x 1 from this; I can have a relation that x 1 plus x 2 is equal to 0 or x 1 is equal to minus of x 2; that means, my second there the relation if I put x 1 is equal to 1, my x 2 will be minus of 1. So, my second eigenvector which I write as a big y is equal to x 1 x 2 which is now 1 and minus 1; I write 1 and minus 1 2 eigenvectors I get B x and B y.

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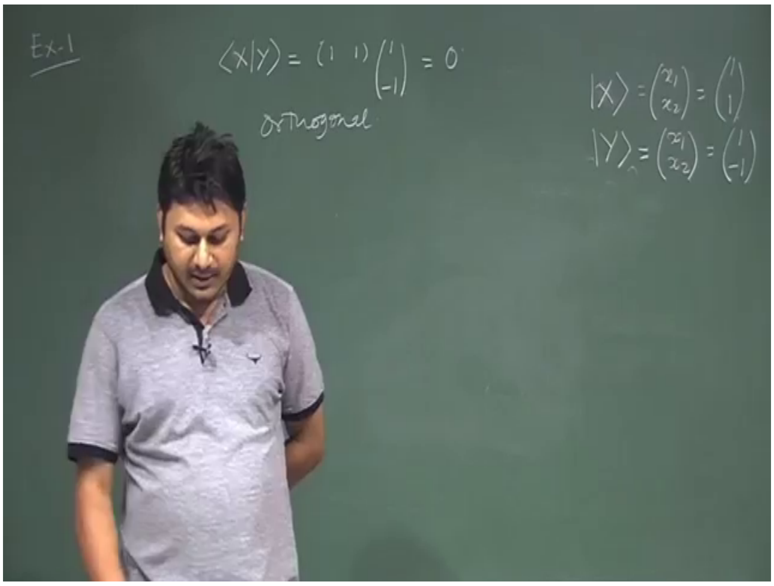

Now, I need to show that quickly; I will show that they are orthogonal to each other. So, it is simple if I put big x big y it has to be 0. So, big x is big x is 1 1; this is 1 minus 1.

So, result is 1 into 1 plus 1 into minus 1 is 0. So, with that I proof that they are orthogonal. So, with this we conclude that if a matrix is normal, then we have the additional property of that matrix and the additional properties they produce the orthogonal eigenvectors which is very important they produce the orthogonal eigenvectors for the condition that the matrix has to be normal to the matrix has to be the normal. So, they will commute with their dagger.

So, with this today I will like to conclude the class. In the next class, we start a very important concept which is the diagonalization of the matrix. So, with that let us conclude. So, see you in the next class where we find how a matrix what is the diagonalization and how a matrix can be diagonalized with having all this knowledge whatever we have so far in our class.

Thank you and see you in the next class.